FOUNDATIONS OF GLOBAL NON-LINEAR ANALYSIS

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W. A. BENJAMIN, INC.

New York 1968 Amsterdam

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Library of Congress Catalog Card Number 68-26678
Manufactured in the United States of America

The manuscript was put into production on March 27, 1968;
this volume was published on June 7, 1968

W. A. BENJAMIN, INC.
New York, New York 10016

INTRODUCTION

This paper is an expanded version of notes for lectures delivered at the Mathematics Institutes of Bonn University and the University of Geneva during the summer of 1966. While it includes some new results, the main emphasis has been on exposition and on developing in detail a particular perspective toward non-linear analysis. As explained in section 1, it is my contention that non-linear analysis has developed beyond the stage of being a collection of isolated problems to be treated by ingenious ad hoc techniques pulled from an esoteric bag of tricks. It can now be treated as a relatively mature mathematical subject, which is to say it has a definite categorical framework. If, as is perhaps the case, I have tended to overstress the functorial aspects of the theory, I hope this will be excused as the natural result of trying to emphasize my basic thesis. I should perhaps also apologize for a failure to pay sufficient attention to the problem of crediting various mathematicians with the development of the ideas that arise along the way. My only excuse here is that I mean these notes to be read either after or along with James Eells review article, "A setting for global analysis" which has recently appeared in the Bulletin of the American Mathematical Society, and the reader will find there a careful and more than adequate treatment of these bibliographical questions which I have tended to slight.

As far as pre-requisites go, I have tried to make these notes immediately accessible to anyone familiar with Lang's "Introduction to Differentiable Manifolds" and Chapter IV of Annals Study no. 57, "Seminar on the Atiyah-Singer Index Theorem", although actually very little is needed from the latter source and I have tried to recall what is in section 2. Otherwise the goal has been to make these notes self-contained. An exception is Chapter 18 on the index problem for non-linear elliptic differential operators. This section is a very condensed report of research in progress and will probably be rather tough sledding for anyone not reasonably familiar with the Atiyah-Singer index theorem. It was put in only to suggest the flavor of the kind of partially analytic, partially topological question which I suspect may prove to be one fruitful and interesting direction in which global non-linear analysis may move in the near future.

I would like to thank Karen Uhlenbeck and Stephen Greenfield for suggesting numerous improvements incorporated in the final version of this paper.

During the preparation of this paper the author was partially supported by National Science Foundation Grants 4598 and 6085 and by a fellowship from the Alfred P. Sloan Foundation.

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1. WHAT IS GLOBAL ANALYSIS?

It has become a more or less accepted principle that a mathematical specialty is defined by the category it studies. For example, linear algebra is the study of vector spaces and linear maps, topology the study of spaces and continuous maps, homotopy theory the study of spaces and homotopy classes of maps, differential topology the study of differentiable manifolds and differentiable maps, and so on. Thus to answer the question "What is global analysis?" we should specify a particular category, and this we shall proceed to do.

First we consider global linear analysis. From an abstract point of view linear analysis is simply the study of topological vector spaces and continuous linear maps. In concrete linear analysis the topological vector spaces in question are classically spaces of real, complex, or more generally vector valued functions on $\mathbb{R}^n$ or on some domain in an $\mathbb{R}^n$, and the linear maps are differential (or more generally integro-differential) operators. This might be called local linear analysis. To pass to global linear analysis we replace the domain in $\mathbb{R}^n$ by an arbitrary differentiable manifold M, and rather than merely taking topological vector spaces of vector valued functions on M we more generally consider topological vector spaces of cross sections of differentiable vector bundles over M. The continuous linear maps are again defined by linear differential or integro-differential operators. This is the general setting for such results as the Hodge theory of harmonic forms, the Atiyah-Singer Index Theorem, and the Atiyah-Bott Fixed Point Formula. Roughly speaking the questions here concern relating analytical invariants of the operators involved with topological invariants of M and the given vector bundles.

Finally we must say what non-linear analysis is. Often one sees in the literature the statement that non-linear analysis is simply all of analysis which is not linear. This is of course not a definition at all since it begs the question (what is analysis?), and it could be treated as just a bad joke were it not apparently taken so seriously. A corollary of this non-definition is that non-linear analysis can never be a real subject of study, but only a collection of isolated problems and their solutions. This point of view has had a pernicious effect on the growth of the whole field.

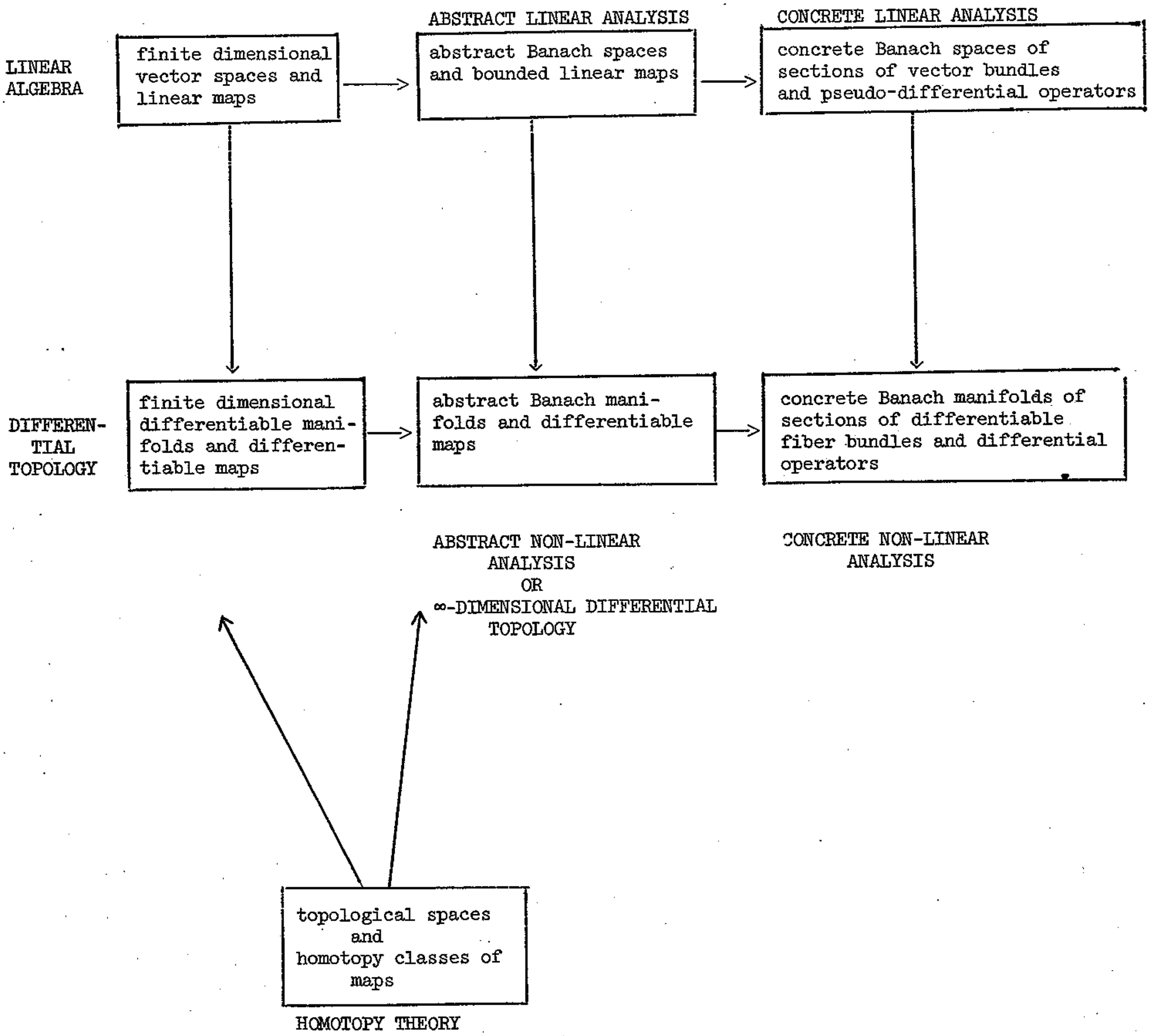
In fact there is a unifying technique which runs all through what is usually called non-linear analysis and which gives an important clue to what it is about, namely the idea of "linearization", i.e. replacing a non-linear map "locally" by an approximating linear map. If one looks behind this, one sees that what is usually the case is that the sets on which the map is defined and into which it maps have natural infinite dimensional manifold structures with respect to which the map is differentiable and that the linearization of the map near a given point is just its differential at that point. Thus what abstract non-linear analysis turns out to be is the study of infinite dimensional manifolds and differentiable maps, i.e. infinite dimensional differential topology in another guise.

It should be admitted that this has to be taken with a grain of salt. There are subjects which are often put under the general heading of non-linear analysis, such as Leray-Schauder fixed point theory, where the maps are not differentiable. Basically however the principle is correct, "non-linear" means "differentiable", not just "continuous". Thus non-linear analysis lies in an intermediate position between linear analysis and topology, and topics such as Leray-Schauder theory should properly be considered as belonging to topology (indeed to homotopy theory).

This same principle, that to pass from linear to non-linear means to replace linear spaces and maps by differentiable manifolds and maps suggests the proper definition of global non-linear analysis. Instead of differentiable vector bundles over a differentiable manifold M we consider more generally differentiable fiber bundles over M. Instead of topological vector spaces of sections of the vector bundle we consider differentiable manifolds of sections of the fiber bundle, and for maps we again take differential operators (now non-linear) which define differentiable maps between such manifolds of sections.

This seems to be the proper framework for a large variety of diverse subjects. Perhaps most obvious are the theory of non-linear differential operators and the calculus of variations, in particular Morse Theory and Lusternik-Schnirelman Theory. But also as Abraham has shown it is the natural setting for general transversality theorems. For a survey of the present status and recent history of the field we refer the reader to the excellent review article by James Eells, Jr. which will appear shortly in the Bulletin of the American Mathematical Society under the title "A setting for global analysis".

Before stating the precise scope and purpose of the present paper let us summarize what has been said above in a diagram.


ABSTRACT LINEAR ANALYSIS
CONCRETE LINEAR ANALYSIS
LINEAR ALGEBRA
finite dimensional vector spaces and linear maps
abstract Banach spaces and bounded linear maps
concrete Banach spaces of sections of vector bundles and pseudo-differential operators
DIFFEREN-TIAL TOPOLOGY
finite dimensional differentiable mani-folds and differen-tiable maps
abstract Banach mani-folds and differentiable maps
concrete Banach manifolds of sections of differentiable fiber bundles and differential operators
ABSTRACT NON-LINEAR ANALYSIS OR ∞-DIMENSIONAL DIFFERENTIAL TOPOLOGY
CONCRETE NON-LINEAR ANALYSIS
topological spaces and homotopy classes of maps
HOMOTOPY THEORY

The Banach spaces of sections of a vector bundle that are actually considered by analysts have proliferated greatly in the past several years. Aside from the classical C^k, the Holder spaces C^{k+a}, and the Sobolev spaces L^p_k there are many more exotic spaces that arise through various interpolation schemes and by taking "boundary values". In the first part of this paper we have abstracted four basic axioms (B §1) - (B §4) that these spaces all seem to possess, and develop their more elementary consequences. Our real goal however is the passage from Banach spaces of sections of a vector bundle to Banach manifolds of sections of a differentiable fiber bundle. The literature on this subject is in a somewhat unsatisfactory state. The conditions under which this transition goes through in general have not been clear, and the various techniques for putting on the differentiable structure have always seemed a little artificial. We shall show that if a Banach space section functor satisfies one additional axiom (B §5) then it extends in a very natural way to a Banach manifold section functor on fiber bundles.

The reader is expected to be familiar with the basic facts concerning differentiable Banach manifolds, as found for example in S. Lang's Introduction to Differentiable Manifolds. We also assume a knowledge of the elementary theory of differentiable vector bundles. Aside from this we have attempted to make the discussion relatively self-contained. Sections 2 and 3 discuss jet bundles and differential operators. These topics are basic to what follows. For a fuller discussion including complete proofs the reader is referred to Chapter IV of Seminar on the Atiyah-Singer Index Theorem, Annals of Math. Study No. 57, referred to henceforth as S.A.S.I.T.

2. JET BUNDLES

Given a paracompact C^∞ manifold M let $VB(M)$ denote the category of C^∞ vector bundles over M, with C^∞ vector bundle homomorphisms as the morphisms. Given an object ξ of $VB(M)$ let $S(\xi)$ denote the vector space of <u>all</u> sections of ξ and $C^k(\xi)$ $(k = 0,1,\ldots^\infty)$ the vector subspaces of C^k-sections. Both S and all the C^k are functors; if $f \varepsilon \operatorname{Hom}(\xi,\eta)$ then we have the linear map $f_*:S(\xi) \longrightarrow S(\eta)$ defined by $(f_*s)(x) = f(s(x))$, and f_* maps $C^k(\xi)$ into $C^k(\eta)$ for all k.

Recall VB is itself a (contravariant) functor; if $\varphi:M \longrightarrow N$ is a C^∞ map we have the induced bundle construction. $\varphi^*:VB(N) \longrightarrow VB(M)$. For each fixed such φ the map φ^* is itself a (covariant) functor; i.e. given a morphism $f:\xi \longrightarrow \eta$ of $VB(N)$ we have an induced morphism $\varphi^*(f):\varphi^*(\xi) \longrightarrow \varphi^*(\eta)$ defined by $\varphi^*(f)e = fe$. There is also a linear map $\varphi^*_\xi: S(\xi) \longrightarrow S(\varphi^*\xi)$ which maps $C^k(\xi)$ into $C^k(\varphi^*\xi)$ for all k, namely $s \longmapsto s \circ \varphi$. Regarding S and $S\varphi^*$ as functors from $VB(N)$ to vector spaces, φ^*_ξ is a natural transformation, i.e. we have commutativity in the diagram

$$\begin{tikzcd} S(\xi) \arrow[r, "f_*"] \arrow[d, "\varphi^*_\xi"'] & S(\eta) \arrow[d, "\varphi^*_\eta"] \\ S(\varphi^*(\xi)) \arrow[r, "(\varphi^*(f))_*"'] & S(\varphi^*(\eta)) \end{tikzcd}$$

We shall be interested in the above mainly when φ is a diffeomorphism of M onto an open submanifold of N, in which case if we regard φ as an identification (so M is an open submanifold of N and φ is inclusion) then all the above induced maps are restrictions.

Given a C^∞ vector bundle ξ over M and $p \varepsilon M$ define a subspace $Z^k_p(\xi)$ of $C^\infty(\xi)$ as follows. Let $x_1,\ldots,x_n$ be local coordinates at p, $s_1,\ldots,s_\ell \varepsilon C^\infty(\xi)$ a local basis of sections of ξ near p. Then if $s = f_1s_1 +\ldots+ f_\ell s_\ell$ near p $(f_i \varepsilon C^\infty(M))$ then $s \varepsilon Z^k_p(\xi)$ if and only if $(D^\alpha f_i)(p) = 0$ $i=1,\ldots\ell$ and $|\alpha| \leq k$ where $\alpha = (\alpha_1,\ldots,\alpha_n)$ are n-tuples of non-negative integers, $|\alpha| = \alpha_1+\ldots+\alpha_n$ and $D^\alpha = \partial^{|\alpha|}/\partial x_1^{\alpha_1}\ldots\partial x_n^{\alpha_n}$. Let $J^k(\xi)_p = C^\infty(\xi)/Z^k_p(\xi)$ and for $s \varepsilon C^\infty(\xi)$ let $j_k(s)(p)$ be its class in $J^k(\xi)_p$. Then it is possible in a unique way to make the $J^k(\xi)_p$ the fibers of a C^∞ vector bundle $J^k(\xi)$ over M so that $j_k(s) \varepsilon C^\infty(J^k(\xi))$ for all $s \varepsilon C^\infty(\xi)$. The linear map $j_k : C^\infty(\xi) \longrightarrow C^\infty(J^k(\xi))$ is called the k-jet extension map.

$J^k : VB(M) \longrightarrow VB(M)$ is functorial; given $f \varepsilon \operatorname{Hom}(\xi, \eta)$ there is a functorial induced map $J^k(f) \varepsilon \operatorname{Hom}(J^k(\xi), J^k(\eta))$ characterized by $J^k(f)_* j_k(s) = j_k(f_* s)$ for all $s \varepsilon C^\infty(\xi)$.

Also if $\varphi : \mathcal{O} \longrightarrow M$ is a diffeomorphism of $\mathcal{O}$ onto an open sub-manifold of M then there is a functorial induced vector bundle isomorphism $J^k(\varphi^*) : J^k(\varphi^*\xi) \approx \varphi^* J^k(\xi)$ defined by $J^k(\varphi^*) j_k(s) = j_k(s \circ \varphi^{-1})$ for $s \varepsilon C^\infty(\varphi^*\xi)$. In particular when $\mathcal{O} \subseteq M$ and φ is the inclusion $i : \mathcal{O} \longrightarrow M$ then we get an isomorphism $J^k(i) : J^k(\xi|\mathcal{O}) \approx J^k(\xi)|\mathcal{O}$ given by $j_k(s)(p) \longmapsto j_k(s|\mathcal{O})(p)$.

In view of the latter, in order to see the structure of $J^k(\xi)$ it suffices to consider the case where ξ is a trivial bundle $\xi = \mathcal{O} \times V$ and $\mathcal{O}$ is open in $\mathbb{R}^n$. In this case it is clear that

$$J^k(\xi) \approx \mathcal{O} \times \bigoplus_{j=0}^{k} L_s^j(\mathbb{R}^n, V) \approx \mathcal{O} \times \bigoplus_{|\alpha| \leq k} V$$

where $L_s^j(\mathbb{R}^n, V)$ is the space of symmetric j-linear maps of $\mathbb{R}^n$ into V and where the indices α as above denote n-tuples $\alpha = (\alpha_1, \ldots, \alpha_n)$ of non-negative integers and $|\alpha| = \alpha_1 + \ldots + \alpha_n$. The natural isomorphism $L_s^j(\mathbb{R}^n, V) \approx \bigoplus_{|\alpha|=j} V$ is given by $T \longmapsto \{T(e_1^{\alpha_1}, \ldots, e_n^{\alpha_n})\}_{|\alpha|=j}$ where $e_1, \ldots, e_n$ is the standard basis for $\mathbb{R}^n$ and $e_i^{\alpha_i}$ is the α_i-tuple $(e_i, \ldots, e_i)$. An element $s \varepsilon C^\infty(\xi)$ is a C^∞ map of $\mathcal{O}$ into V. Its k-jet extension $j_k(s) \varepsilon C^\infty(J^k(\xi))$ is given by a k+1-tuple of maps (the j^{th} being a C^∞ map of $\mathcal{O}$ into $L_s^{j-1}(\mathbb{R}^n, V)$) namely $j_k(s) = \{s, ds, d^2s, .., d^k s\}$). Since $d^{|\alpha|} s(e_1^{\alpha_1}, \ldots, e_n^{\alpha_n}) = D^\alpha s$, in the other representation of $J^k(\xi)$ as $\bigoplus_{|\alpha| \leq k} V$, we have $j_k(s) = \{D^\alpha s\}_{|\alpha| \leq k}$.

As a consequence of the above we note the following for future reference. The k-jet extension map $j_k : C^\infty(\xi) \longrightarrow C^\infty(J^k(\xi))$ has a natural extension to a linear injection $j_k : C^k(\xi) \longrightarrow C^0(J^k(\xi))$.

3. DIFFERENTIAL OPERATORS

If ξ and η are C^∞ vector bundles over M then a linear map $D: C^\infty(\xi) \longrightarrow C^\infty(\eta)$ is called a k-th order linear differential operator from (sections of) ξ to (sections of) η if for $s \in C^\infty(\xi)$ and $p \in M$, $j_k(s)(p) = 0$ implies $(Ds)(p) = 0$. The set of all such is a vector space $\mathrm{Diff}_k(\xi,\eta)$. If $T \in \mathrm{Hom}(J^k(\xi),\eta)$ then $s \longmapsto T_*(j_k(s))$ is clearly an element $D_T \in \mathrm{Diff}_k(\xi,\eta)$. Moreover the map $T \longmapsto D_T$ is a linear isomorphism of $\mathrm{Hom}(J^k(\xi), \eta)$ with $\mathrm{Diff}_k(\xi,\eta)$. It follows that $D \in \mathrm{Diff}_k(\xi,\eta)$ extends to a linear map $D: C^k(\xi) \longrightarrow C^0(\eta)$.

Let u be a strictly positive smooth measure on M, i.e. u is a Radon measure on M and relative to local coordinates u is absolutely continuous with respect to Lebesque measure and has a strictly positive C^∞ Radon-Nikodym derivative. Let ξ and η be Riemannian vector bundles over M and let $< , >_{\xi_x}$ denote the inner product in ξ_x. For $s_1, s_2 \in C^0(\xi)$ having compact support, let $< s_1, s_2 >_\xi = \int < s_1(x), s_2(x) >_{\xi_x} du(x)$, and similarly for η. Given a $D \in \mathrm{Diff}_k(\xi,\eta)$ there is a unique $D^* \in \mathrm{Diff}_k(\eta,\xi)$ called the formal adjoint of D such that $< Ds_1, s_2 >_\eta = < s_1, D^*s_2 >_\xi$ whenever $s_1 \in C^k(\xi)$, $s_2 \in C^k(\eta)$ and at least one of s_1 and s_2 has its support compact and disjoint from ∂M.

Given vector bundles ξ,η,ζ over M and $D_1 \in \mathrm{Diff}_k(\eta,\zeta)$, $D_2 \in \mathrm{Diff}\,(\xi,\eta)$, then $D_1D_2 \in \mathrm{Diff}_{k+\ell}(\xi,\zeta)$. In particular since $j_\ell \in \mathrm{Diff}_\ell(\xi,J^\ell(\xi))$ and $j_k \in \mathrm{Diff}_k(J^\ell(\xi),J^kJ^\ell(\xi))$ there exists a unique $T \in \mathrm{Hom}(J^{k+\ell}(\xi),J^kJ^\ell(\xi))$ such that $T_*j_{k+\ell} = j_kj_\ell$. This T is easily seen to be injective and so identifies $J^{k+\ell}(\xi)$ naturally with a sub-bundle of $J^k(J^\ell(\xi))$. From this it follows easily that every $D \in \mathrm{Diff}_{k+\ell}(\xi,\eta)$ can be written as a composite $D = D_1D_2$ where $D_2 \in \mathrm{Diff}_k(\xi,J^k(\xi))$ and $D_1 \in \mathrm{Diff}_\ell(J^k(\xi),\eta)$.

If $D \in \mathrm{Diff}_k(\xi,\eta)$ then for each cotangent vector at a point $x \in M$, say (v,x), there is a linear map $\sigma_k(D)(v,x): \xi_x \longrightarrow \eta_x$, called the (k-th order) symbol of D at (v,x), which can be defined as follows: choose $f \in C^\infty(M)$ such that $f(x) = 0$ and $df_x = v$; then for $e \in \xi_x$ $\sigma_k(D)(v,x)e = \frac{1}{k!} D(f^ks)(x)$, where s is any element of $C^\infty(\xi)$ such that $s(x) = e$. It can be shown that $\sigma_k(D)$ is linear in D, and is zero for all $(v,x) \in T^*(M)$ if and only if

$D \in \mathrm{Diff}_{k-1}(\xi,\eta)$. Also $\sigma_k(D^*)(v,x) = \sigma_k(D)(v,x)^*$ and if $\tilde{D} \in \mathrm{Diff}_\ell(\eta,\zeta)$ then $\sigma_{\ell+k}(\tilde{D}D)(v,x) = \sigma_\ell(\tilde{D})(v,x)\sigma_k(D)(v,x)$.

Suppose now $M = \mathbb{R}^n$, $\xi = \mathbb{R}^n \times V$ and $\eta = \mathbb{R}^n \times W$. Then $J^k(\xi) = \mathbb{R}^n \times \bigoplus_{|\alpha| \le k} V$ so an element $T \in \mathrm{Hom}(J^k(\xi),\eta)$ is given by an indexed set $\{T_\alpha\}_{|\alpha| \le k}$ of C^∞ maps of $\mathbb{R}^n$ into the vector space $L(V,W)$ of linear maps of V into W. Now a C^∞ section s of $\xi(\eta)$ is a C^∞ map of $\mathbb{R}^n$ into $V(W)$ and clearly $(D_T s)(x) = \sum_{|\alpha| \le k} T_\alpha(x)(D^\alpha s)(x)$. If (v,p) is a cotangent vector of $\mathbb{R}^n$ at p then $v = \sum_{i=1}^n v_i(dx_i)_p$ where $(v_1,\dots,v_n) \in \mathbb{R}^n$ and it is easily seen that

$$\sigma_k(D_T)(v,p) = \sum_{|\alpha|=k} v^\alpha T_\alpha(p)$$

where $v^\alpha = v_1^{\alpha_1} v_2^{\alpha_2} \dots v_n^{\alpha_n}$. Since we can always choose local coordinate systems in any M and trivialize bundles locally the above shows what differential operators look like locally and also how to compute their symbols locally.

4. BANACH SPACE VALUED SECTION FUNCTORS

Let $\mathcal{M}$ be a function which associates to each C^∞ vector bundle ξ over a compact n-dimensional C^∞ manifold (possibly with boundary) a complete, normable, topological vector space $\mathcal{M}(\xi)$ which includes $C^\infty(\xi)$ and which is a subspace of the vector space $S(\xi)$ of all sections of ξ. Actually for certain examples $\mathcal{M}(\xi)$ will be a little more general. Namely let $\eta(\xi)$ be the subspace of $S(\xi)$ consisting of sections which are zero almost everywhere (recall that since the base of ξ is differentiable there is a natural notion of sets of measure zero, namely those carried into sets of Lebesque measure zero by all charts). In many cases $\mathcal{M}(\xi)$ will not be T_1 and in fact $\eta(\xi)$ will be the closure of the origin and it is really $\mathcal{M}(\xi)/\eta(\xi)$ which is the complete normable space in question. Having remarked this we shall as usual ignore the distinction between sections and classes of sections which are equal almost everywhere.

Our first axiom is a functoriality assumption.

Axiom (B§1). For each compact C^∞ n-dimensional manifold M, $\mathcal{M}$ is a functor from VB(M) to the category of Banach spaces. To be precise if ξ,η are C^∞ vector bundles over M and $f \in \mathrm{Hom}(\xi,\eta)$ then we assume that $f_*:S(\xi) \longrightarrow S(\eta)$ maps $\mathcal{M}(\xi)$ into $\mathcal{M}(\eta)$ and defines thereby a continuous linear map $\mathcal{M}(f): \mathcal{M}(\xi) \longrightarrow \mathcal{M}(\eta)$.

If ψ is a C^∞ scalar valued function on the base space of ξ, then multiplication by ψ defines an element $T_\psi \in \mathrm{Hom}(\xi,\xi)$. If $\psi_1,\ldots,\psi_q$ is a C^∞ partition of unity for the base of ξ then $\mathcal{M}(T_{\psi_1}),\ldots,\mathcal{M}(T_{\psi_q})$ gives a partition of the identity map of $\mathcal{M}(\xi)$ into continuous linear maps. Moreover if $s \in \mathcal{M}(\xi)$ then $\mathcal{M}(T_\psi)s\ (x) = \psi(x)s(x)$ so support $(\mathcal{M}(T_\psi)s) \subseteq$ support ψ and $\mathcal{M}(T_\psi)s = s$ wherever $\psi = 1$. As an immediate consequence we get the following:

4.1 <u>Localization Theorem</u>: Let ξ be a C^∞ vector bundle over a compact n-dimensional C^∞ manifold M, $s \in S(\xi)$ and suppose that for each $p \in M$ there is a neighborhood U of p in M and $s_p \in \mathcal{M}(\xi)$ such that $s|U = s_p|U$. Then $s \in \mathcal{M}(\xi)$.

The category VB(M) is additive and Whitney sum is the bi-product. Moreover $\mathcal{M}$ is clearly an additive functor. Hence we have the:

4.2 <u>Direct Sum Theorem</u>: If ξ and η are C^∞ vector bundles over M then

$$\mathcal{M}(\xi \oplus \eta) = \mathcal{M}(\xi) \oplus \mathcal{M}(\eta)$$

The second basic property our section functors $\mathcal{M}$ will be assumed to possess is functoriality with respect to diffeomorphisms of the base space, often referred to as "coordinate invariance".

Axiom (B §2). Let M and N be compact n-dimensional C^∞ manifolds and let $\varphi : M \longrightarrow N$ be a diffeomorphism of M into N. If ξ is a vector bundle over N then $s \longmapsto s \circ \varphi$ is a continuous linear map of $\mathcal{M}(\xi)$ into $\mathcal{M}(\varphi^*\xi)$.

<u>Remark</u>: In all natural examples this map is onto. If $M \subseteq N$ and φ is inclusion then (B §2) just says that restriction is continuous from $\mathcal{M}(\xi)$ to $\mathcal{M}(\xi|M)$ and in this case onto-ness expresses the possibility of extending $s \in \mathcal{M}(\xi|M)$ to an element of $\mathcal{M}(\xi)$. In fact usually restriction has a continuous, linear right inverse.

4.3 <u>"Mayer-Vietoris" Theorem</u>: Let M be a compact n-dimensional C^∞ manifold and let $M_1, \ldots, M_r$ be compact n-dimensional C^∞ submanifolds whose interiors cover M. Given a C^∞ vector bundle ξ over M define $\tilde{\mathcal{M}}(\xi) = \{(s_1, \ldots, s_r) \in \bigoplus_{i=1}^{r} \mathcal{M}(\xi|M_i) \mid s_i|M_j = s_j|M_i\}$ and define $F : \mathcal{M}(\xi) \longrightarrow \tilde{\mathcal{M}}(\xi)$ by $F(s) = (s|M_1, \ldots, s|M_r)$. Then F is an isomorphism of Banach spaces.

<u>Proof</u>. The continuity of F is immediate from (B §2), and ker F = 0 follows from $M = \bigcup_i M_i$, so by the open mapping theorem* it will suffice to prove F surjective. Let $\varphi_1, \ldots \varphi_r$

* Douady points out that we actually construct a continuous inverse, so the open mapping theorem is not needed.

be a C^∞ partition of unity for M with $\mathrm{supp}(\varphi_i) \subseteq$ interior of M_1. Given $(s_1, \ldots, s_r) \in \widetilde{\mathcal{M}}(\xi)$ it follows from the remark following (B§1) that $\varphi_i s_i \in \mathcal{M}(\xi|M_i)$. If we define $\tilde{s}_i \in S(\xi)$ by $\tilde{s}_i | M_i = \varphi_i s_i$ and $\tilde{s}_i = 0$ outside M_i then it follows from Theorem 4.1 that $\tilde{s}_i \in \mathcal{M}(\xi)$, hence $\tilde{s} = \tilde{s}_1 + \ldots \tilde{s}_r \in \mathcal{M}(\xi)$. If $p \in M_i$ then either $\tilde{s}_j(p) = \varphi_j(p)\, s_i(p)$ (if $p \in M_j$) or else if $p \notin M_j$ then $\tilde{s}_j(p) = 0$ and since $\varphi_j(p) = 0$ $\tilde{s}_j(p) = \varphi_j(p) s_i(p)$ again. Thus $\tilde{s}(p) = s_i(p)$ or $\tilde{s}|M_i = s_i$ and $F(\tilde{s}) = (s_1, \ldots, s_r)$.

q.e.d

Let $S(D^n, \mathbb{R}^q)$ denote the vector space of all functions from the n-disc $D^n = \{x \in \mathbb{R}^n \,|\, ||x|| \leq 1\}$ into $\mathbb{R}^q$, regarded as the sections of the product bundle $D^n \times \mathbb{R}^q$ over D^n, and let $\mathcal{M}(D^n, \mathbb{R}^q)$ denote the Banach space assigned by $\mathcal{M}$ to this bundle, so $\mathcal{M}(D^n, \mathbb{R}^q)$ is a vector subspace of $S(D^n, \mathbb{R}^q)$ with a Banach space topology. The above theorem implies that knowledge of $\mathcal{M}(D^n, \mathbb{R}^q)$ completely determines $\mathcal{M}$ as a functor on VB(M) for any compact n-dimensional manifold M. In fact, given a vector bundle ξ over M let $\varphi_i : D^n \longrightarrow M$ $i = 1, \ldots, r$ be charts for M such that $M \subseteq \bigcup_{i=1} \varphi_i(\overset{\circ}{D^n})$, and let $\psi_i : \varphi_i^* \xi \approx D^n \times \mathbb{R}^q$ be trivializations. Define a linear function $F : S(\xi) \longrightarrow \bigoplus_{i=1}^r S(D^n, \mathbb{R}^q)$ by $F(s) = (s_1, \ldots, s_r)$ where $s_i(x) = \psi_i(s(\varphi_i(x)))$. Then $s \in \mathcal{M}(\xi)$ if and only if $F(s) \in \bigoplus_{i=1}^r \mathcal{M}(D^n, \mathbb{R}^q)$ and in fact it follows from Theorem 4.3 that F maps $\mathcal{M}(\xi)$ isomorphically onto $\{(s_1, \ldots, s_r) \in \bigoplus_{i=1}^r \mathcal{M}(D^n, \mathbb{R}^q) \,|\, \psi_i^{-1} s_i \varphi_i^{-1} = \psi_j^{-1} s_j \varphi_j^{-1}\}$. In particular note that if $|| \;||$ is an admissible norm for $\mathcal{M}(D^n, \mathbb{R}^q)$ then $\sum_{i=1}^r ||\psi_i s\, \varphi_i||$ is an admissible norm for $\mathcal{M}(\xi)$.

We can carry the reduction one step further. According to Theorem 4.2 if $f \in S(D^n, \mathbb{R}^q)$, say $f(x) = (f_1(x), \ldots, f_q(x))$ with $f_i \in S(D^n, \mathbb{R})$, then $f \in \mathcal{M}(D^n, \mathbb{R}^q)$ if and only if each $f_i \in \mathcal{M}(D_n, \mathbb{R})$, and in fact $f \longmapsto (f_1, \ldots, f_q)$ is a Banach space isomorphism of $\mathcal{M}(D^n, \mathbb{R}^q)$ with $\bigoplus_{i=1}^q \mathcal{M}(D^n, \mathbb{R})$. In particular if $|| \;||$ is an admissible norm for $\mathcal{M}(D^n, \mathbb{R})$ then $\sum_{i=1}^q ||f_i||$ (or $(\sum_{i=1}^q ||f_i||^2)^{1/2}$) is an admissible norm $\mathcal{M}(D^n, \mathbb{R}^q)$.

Thus to know $\mathcal{M}$ as a functor on VB(M) when M is a compact n-dimensional manifold, it suffices to know $\mathcal{M}(D^n, \mathbb{R})$. It is natural to ask, given a linear subspace $\mathcal{M}(D^n, \mathbb{R})$ of $S(D^n, \mathbb{R})$ with a Banach space topology, when can we find a function $\mathcal{M}$ satisfying (B§1) and (B§2) giving rise to it. If $\xi = D^n \times \mathbb{R}$ we can identify $\mathrm{Hom}(\xi, \xi)$ with $C^\infty(D^n, \mathbb{R})$, the C^∞ maps of D^n into $\mathbb{R}$.

Given $f \in Hom(\xi,\xi)$ and $s \in S(\xi)$ we have $(f_* s)(x) = f(x)s(x)$. Then (B §1) gives:

Axiom (B §1'). For each $f \in C^\infty(D^n,\mathbb{R})$ multiplication by f is a continuous linear map of $\mathcal{M}(D^n,\mathbb{R})$ into itself.

And (B §2) leads to:

Axiom (B §2'). If $\varphi: D^n \longrightarrow D^n$ is a diffeomorphism into then $s \longmapsto s \circ \varphi$ is a continuous linear map of $\mathcal{M}(D^n,\mathbb{R})$ into itself.

Conversely if $\mathcal{M}(D^n,\mathbb{R})$ is a linear subspace of $S(D^n,\mathbb{R})$ with a Banach space topology which satisfies (B §1') and (B §2') then using the constructions indicated above we can define first $\mathcal{M}(D^n,\mathbb{R}^q)$ and then $\mathcal{M}(\xi)$ for ξ any C^∞ vector bundle over a compact C^∞ n-dimensional manifold M. That $\mathcal{M}(\xi)$ is independent of the choice of charts $\varphi_i: D^n \longrightarrow M$ and of the trivializations $\psi_i: \varphi^*\xi \approx D^n \times \mathbb{R}^q$ follows easily from (B §2'), and so does (B §2), while (B §1) follows from (B §1').

In addition to the two functoriality axioms (B §1) and (B §2) there are two regularity axioms which are perhaps less natural but which again are satisfied in practice:

Axiom (B §3). If ξ is a C^∞ vector bundle over a compact C^∞ n-manifold M then $C^0(\xi) \cap \mathcal{M}(\xi)$ is dense in $\mathcal{M}(\xi)$.

Let $C_0^\infty(\xi)$ denote the space of C^∞ sections s of the vector bundle ξ over M such that support (s) is a compact set disjoint from ∂M.

Axiom (B§4). Let M be a compact C^∞ n-manifold with a strictly positive smooth measure and let ξ be a C^∞ Riemannian vector bundle over M. Then for each $\sigma \in C_0^\infty(\xi)$ the map $s \longmapsto \langle s,\sigma \rangle_\xi$ extends from $C^0(\xi) \cap \mathcal{M}(\xi)$ to a continuous linear functional ℓ_σ on $\mathcal{M}(\xi)$. Moreover $\{\ell_\sigma | \sigma \in C_0^\infty(\xi)\}$ is a total (i.e. point separating) linear subspace of $\mathcal{M}(\xi)^*$, the dual space of $\mathcal{M}(\xi)$.

Remark. Any other Riemannian structure on ξ is given by an element $T \in \mathrm{Hom}(\xi,\xi)$ such that T_x is a strictly positive operator on ξ_x for each $x \in M$. Any other strictly positive, smooth measure ν on M is related to μ by $d\nu = \rho\, d\mu$ where $\rho \in C^\infty(M,\mathbb{R})$ is strictly positive. If $\sigma \in C_0^\infty(\xi)$ and $\tilde{\ell}_\sigma$ denotes the linear functionals corresponding to the new choices of measure and Riemannian structure then clearly $\tilde{\ell}_\sigma = \ell_{\rho\sigma}\, \mathcal{M}(T)$. It is immediate that if (B §4) holds for one choice of measure and Riemannian structure then it holds for any other also.

4.4 Theorem. Let M be a compact C^∞ n-manifold and let ξ and η be C^∞ Riemannian vector bundles over M and $D \in \mathrm{Diff}_k(\xi,\eta)$. Then for $\sigma \in C_0^\infty(\xi)$ such that $Ds \in \mathcal{M}(\eta)$, $\ell_\sigma(Ds) = \ell_{D^*\sigma}(s)$.

Proof. Immediate from the definition of D^*.

We should of course answer the question, what are the properties of $\mathcal{M}(D^n,\mathbb{R})$ that will guarantee that (B §3) and (B §4) hold? The answer is completely evident.

Axiom (B §3') $C^0(D^n,\mathbb{R}) \cap \mathcal{M}(D^n,\mathbb{R})$ is dense in $\mathcal{M}(D^n,\mathbb{R})$

Axiom (B §4') For each $\sigma \in C_0^\infty(D^n,\mathbb{R})$ the map $s \longmapsto \int_{D^n} s(x)\sigma(x)dx$ extends from $C^0(D^n,\mathbb{R}) \cap \mathcal{M}(D^n,\mathbb{R})$ to a continuous linear functional ℓ_σ on $\mathcal{M}(D^n,\mathbb{R})$. Moreover $\{\ell_\sigma \mid \sigma \in C_0^\infty(D^n,\mathbb{R})\}$ is a total linear subspace of $\mathcal{M}(D^n,\mathbb{R})^*$

5. DERIVATIVE FUNCTORS

In this section we shall show how from a section functor $\mathcal{M}$ satisfying (B §1) - (B §4) we can construct a sequence of "derivative" functors $\mathcal{M}_k$ $k = 0,1,2 \ldots$ also satisfying these axioms, with $\mathcal{M}_0 = \mathcal{M}$.

Given a C^∞ vector bundle ξ over a compact C^∞ n-manifold M let

$$\mathcal{M}_{(k)}(\xi) = \{s \in C^k(\xi) \mid j_k(s) \in \mathcal{M}(J^k(\xi))\}$$

Then j_k is a continuous linear injection of $\mathcal{M}_{(k)}(\xi)$ into the Banach space $\mathcal{M}(J^k(\xi)$ and hence $\mathcal{M}_{(k)}(\xi)$ becomes a normable topological vector space if we topologize it by the requirement that j_k be a homeomorphism into. We define $\mathcal{M}_k(\xi)$ to be the completion of $\mathcal{M}_{(k)}(\xi)$, so that j_k extends to a continuous linear isomorphism of $\mathcal{M}_k(\xi)$ onto a closed linear subspace of $\mathcal{M}(J^k(\xi))$.

Note that we have a canonical identification of $J^0(\xi)$ with ξ in which j_0 becomes the identity map. It follows that $\mathcal{M}_{(0)}(\xi) = C^0(\xi) \cap \mathcal{M}(\xi)$ and hence by (B §3) it follows that $\mathcal{M}_0(\xi) = \mathcal{M}(\xi)$.

5.1 <u>Theorem</u>. If k and ℓ are integers with $k > \ell \geq 0$ then $\mathcal{M}_{(k)}(\xi) \subseteq \mathcal{M}_{(\ell)}(\xi)$ and the inclusion map $i_{(\ell k)}$ is continuous and therefore extends to a continuous linear map $i_{\ell k}: \mathcal{M}_k(\xi) \longrightarrow \mathcal{M}_\ell(\xi)$. These maps are all injective and satisfy $i_{m\ell} i_{\ell k} = i_{mk}$, so if we identify the underlying vector space of $\mathcal{M}_k(\xi)$ with its image under i_{0k} in $\mathcal{M}_0(\xi) = \mathcal{M}(\xi)$ then we have inclusions

$$\cdots \mathcal{M}_{k+1}(\xi) \subseteq \mathcal{M}_k(\xi) \subseteq \cdots \subseteq \mathcal{M}_0(\xi)$$

and all the inclusion maps are continuous.

<u>Proof</u>. Let $P_{\ell k} \in \mathrm{Hom}\,(J^k(\xi), J^\ell(\xi))$ denote the map characterized by $(P_{\ell k})_* (j_k(s)) = j_\ell(s)$ $\quad s \in C^\infty(\xi)$. If $s \in \mathcal{M}_k(\xi)$ then $j_\ell(s) = \mathcal{M}(P_{\ell k}) j_k(s) \in \mathcal{M}(J^\ell(\xi))$ so

$\mathcal{M}_{(k)}(\xi) \subseteq \mathcal{M}_{(\ell)}(\xi)$ and we have a commutative diagram

$$\begin{tikzcd}
\mathcal{M}_{(k)}(\xi) \arrow[r, "i_{(\ell k)}"] \arrow[d, "j_k"'] & \mathcal{M}_{(\ell)}(\xi) \arrow[d, "j_\ell"] \\
\mathcal{M}(J^k(\xi)) \arrow[r, "\mathcal{M}(P_{\ell k})"'] & \mathcal{M}(J^\ell(\xi))
\end{tikzcd}$$

The continuity of $i_{(\ell k)}$ is now immediate from the definition of the topology on $\mathcal{M}_{(k)}(\xi)$ and $\mathcal{M}_{(\ell)}(\xi)$ and the relation $i_{m\ell}i_{\ell k} = i_{mk}$ follows from the obvious relation $i_{(m\ell)}i_{(\ell k)} = i_{(mk)}$. Finally since $i_{ok} = i_{o\ell}i_{\ell k}$, to prove $i_{\ell k}$ injective it suffices to prove i_{ok} injective. Suppose $i_{ok}(s) = 0$, $s \in \mathcal{M}_k(\xi)$. Since $j_k: \mathcal{M}_k(\xi) \longrightarrow \mathcal{M}(J^k(\xi))$ is injective it will suffice to show that $j_k(s) = 0$. Let $\{s_n\}$ be a sequence in $\mathcal{M}_{(k)}(\xi)$ with $s_n \longrightarrow s$ in $\mathcal{M}_k(\xi)$, i.e. $j_k(s_n) \longrightarrow j_k(s) \in \mathcal{M}(J^k(\xi))$. Choose a strictly positive smooth measure on M and a Riemannian structure on ξ. By (B §4) it will suffice to prove that for each $\sigma \in C_o^\infty(\xi)$ we have $\ell_\sigma(j_k(s)) = 0$ or that $\ell_\sigma(j_k(s_n)) \longrightarrow 0$. Now by Theorem 4.4 we have $\ell_\sigma(j_k(s_n)) = \ell_{j_k^*\sigma}(s_n)$ and since $\ell_{j_k^*\sigma}$ is a continuous linear functional on $\mathcal{M}(\xi)$ it will suffice to show that $s_n \longrightarrow 0$ in $\mathcal{M}(\xi)$. In fact by the continuity of i_{ok}, $s_n = i_{ok}(s_n) \longrightarrow i_{ok}(s) = 0$.

q.e.d.

<u>Remark</u>. Henceforth we shall always regard the $\mathcal{M}_k(\xi)$ as subspaces of $\mathcal{M}(\xi)$ having topologies finer than the induced topology.

5.2 <u>Theorem</u>. Each $\mathcal{M}_k$ satisfies (B §1) - (B §4).

<u>Proof</u>. Given $f \in \mathrm{Hom}(\xi,\eta)$ we have $J^k(f) \in \mathrm{Hom}(J^k(\xi), J^k(\eta))$ satisfying $J^k(f)_*(j_k s) = j_k(f_* s)$, $s \in C^\infty(\xi)$. It follows that we have a commutative diagram

$$\begin{tikzcd}
\mathcal{M}_{(k)}(\xi) \arrow[r, "\mathcal{M}_{(f)}"] \arrow[d, "j_k"'] & \mathcal{M}_{(k)}(\eta) \arrow[d, "j_k"] \\
\mathcal{M}(J^k(\xi)) \arrow[r, "\mathcal{M}(J^k(f))"'] & \mathcal{M}(J^k(\eta))
\end{tikzcd}$$

from which by the definition of the topology on $\mathcal{M}_{(k)}$ it follows that $\mathcal{M}(f)$ is continuous and therefore extends to a continuous linear map $\mathcal{M}_k(f):\mathcal{M}_k(\xi) \longrightarrow \mathcal{M}_k(\eta)$ which proves (B §1). The proof of (B §2) is similar using the induced map $J^k(\varphi^*)$ in place of $J^k(f_*)$.

Since by definition $\mathcal{M}_k(\xi)$ is the completion of $\mathcal{M}_{(k)}(\xi) \subseteq C^k(\xi)$, (B §4) is trivial and we even have that $C^k(\xi) \cap \mathcal{M}_k(\xi)$ is dense in $\mathcal{M}_k(\xi)$.

Choose a strictly positive smooth measure on M and a Riemannian structure on $J^k(\xi)$. Since the sequence $0 \longrightarrow \ker P_{ok} \longrightarrow J^k(\xi) \longrightarrow \xi \longrightarrow 0$ is exact we can regard $J^k(\xi)$ as the orthogonal direct sum of $\ker P_{ok}$ and ξ, identified via P_{ok} with $(\ker P_{ok})$. Moreover since $P_{ok*}j_k(s) = s$, if $s \in C^k(\xi)$ we have $j_k(s) = s \oplus g_k(s)$ where $g_k(s) \in \ker P_{ok}$ and is as smooth as s. If $\gamma \in C_o^\infty(\xi)$ then for $s \in \mathcal{M}_{(k)}(\xi)$ we have

$$\ell_\gamma(s) = \int < s(x), \gamma(x) >_{\xi_x} d\mu(x) = \int < s(x) \oplus g_k(s), \gamma(x) \oplus 0 >_{J^k(\xi)_x} d\mu(x)$$

$$= \ell_{\gamma \oplus 0}(g_k s) \; .$$

Since by (B §4) $\ell_{\gamma\oplus 0}$ is in $\mathcal{M}(J^k(\xi))^*$ and since $j_k: \mathcal{M}_{(k)}(\xi) \longrightarrow \mathcal{M}(J^k(\xi))$ is continuous it follows that $\ell_\gamma \in \mathcal{M}_k(\xi)^*$. Since j_k maps $\mathcal{M}_k(\xi)$ isomorphically onto a closed subspace of $\mathcal{M}(J^k(\xi))$ and since $\{\ell_\sigma | \sigma \in C_o^\infty(J^k(\xi))\}$ is total in $\mathcal{M}(J^k(\xi))^*$ it follows that $\{\ell_\sigma | \circ j_k \sigma \in C_o^\infty(J^k(\xi))\}$ is total in $\mathcal{M}_k(\xi)^*$. But by Theorem 4.4 $\ell_\sigma \circ j_k = \ell_{j_k^* \sigma}$, hence

$$\{\ell_{j_k^*\sigma} \mid \sigma \in C_o^\infty(J^k(\xi))\} \subseteq \{\ell_\gamma \mid \gamma \in C_o^\infty(\xi)\}$$

is total in $\mathcal{M}_k(\xi)^*$. This proves (B §4) for $\mathcal{M}_k$. q.e.d.

5.3 <u>Theorem</u>. $(\mathcal{M}_k)_\ell = \mathcal{M}_{k+\ell}$

<u>Proof</u>. By the discussion following Theorem 4.3 it follows that it will suffice to prove that $(\mathcal{M}_k)_{(\ell)}(D^n,V) = \mathcal{M}_{(k+\ell)}(D^n,V)$. But $\mathcal{M}_{(k+\ell)}(D^n,V)$ consists of all $C^{k+\ell}$ maps $f\colon D^n \longrightarrow V$ such that $D^\alpha f \in \mathcal{M}(D^n,V)$ for all $|\alpha| \leq k+\ell$ and the topology is the least fine making all the maps $f \longmapsto D^\alpha f$ into $\mathcal{M}(D^n,V)$ continuous. On the other hand $(\mathcal{M}_k)_{(\ell)}(D^n,V)$ consists of all C^ℓ maps $f\colon D^n \longrightarrow V$ such that $D^\beta f \in \mathcal{M}_{(k)}(D^n,f)$ for all $|\beta| \leq \ell$, i.e. all $C^{k+\ell}$ maps $f\colon D^n \longrightarrow V$ such that $D^\alpha D^\beta f = D^{\alpha+\beta} f$ is in $\mathcal{M}(D^n,V)$ for $\alpha \leq k$, and the topology is the least fine making each of the maps $f \longmapsto D^{\alpha+\beta} f$ to $\mathcal{M}(D^n,V)$ continuous for $|\alpha| \leq k, |\beta| \leq \ell$. Clearly these are the same. q.e.d.

5.4 <u>Theorem</u>. If ξ and η are C vector bundles over a compact C^∞ manifold M and $D \in \mathrm{Diff}_r(\xi,\eta)$ then for $k \geq r$ D maps $\mathcal{M}_{(k)}(\xi)$ into $\mathcal{M}_{(k-r)}(\eta)$ and extends to a continuous linear map $D\colon \mathcal{M}_k(\xi) \longrightarrow \mathcal{M}_{k-r}(\eta)$.

<u>Proof</u>. First consider the case $k=r$. Recall that $D = T_* j_r$, $T \in \mathrm{Hom}(J^r(\xi),\eta)$ so we have a commutative diagram

$$\begin{tikzcd}
\mathcal{M}_{(r)}(\xi) \arrow[r, "D"] \arrow[d, "j_r"'] & \mathcal{M}(\eta) \\
\mathcal{M}(J^r(\xi)) \arrow[ur, "\mathcal{M}(T)"'] &
\end{tikzcd}$$

Since $\mathcal{M}(T)$ is continuous and j_r is by definition of the topology of $\mathcal{M}_{(r)}(\xi)$ an isomorphism into,

D is continuous. If $k > r$ then we have that D defines a continuous map $(\mathcal{M}_{k-r})_r \longrightarrow \mathcal{M}_{k-r}$, and Theorem 5.3 completes the proof.

5.5 Definition. We say $\mathcal{M}$ satisfies the Rellich condition if the inclusion $\mathcal{M}_1(\xi) \longrightarrow \mathcal{M}(\xi)$ is always completely continuous.

Remark. It is easily seen that it suffices for the inclusion $\mathcal{M}_1(D^n,\mathbb{R}) \longrightarrow \mathcal{M}(D^n,\mathbb{R})$ to be completely continuous (see Theorem 4.3 and the discussion which follows it).

5.5 Theorem. If $\mathcal{M}$ satisfies the Rellich condition then each of the inclusions

$\mathcal{M}_{k+\ell}(\xi) \longrightarrow \mathcal{M}_k(\xi)$, $\ell > 0$ is completely continuous.

Proof. Since the composition of a continuous map and a completely continuous map is completely continuous we can suppose $\ell = 1$. Then we have a commutative diagram (in which horizontal arrows are inclusions).

$$\begin{tikzcd}
\mathcal{M}_{k+1}(\xi) \arrow[r] \arrow[d, "j_k"'] & \mathcal{M}_k(\xi) \arrow[d, "j_k"] \\
\mathcal{M}_1(J^k(\xi)) \arrow[r] & \mathcal{M}(J^k(\xi))
\end{tikzcd}$$

Since $j_k \in \mathrm{Diff}_k(\xi, J^k(\xi))$, the left hand vertical arrow is continuous by 5.3 while the right hand vertical arrow is an isomorphism into by definition of $\mathcal{M}_k(\xi)$. Since the lower horizontal arrow is completely continuous by assumption, so is the upper one. q.e.d.

6. DUAL FUNCTORS

Two Banach spaces X and Y are said to be dually paired if there is given a continuous bilinear map $B: X \times Y \longrightarrow \mathbb{R}$ such that the maps $x \longmapsto B(x, \circ)$ and $y \longmapsto B(\circ, y)$ are respectively isomorphisms of X onto the dual of Y and of Y onto the dual of X.

Recall that given an n-dimensional C^∞ manifold M with a strictly positive smooth measure μ and a C^∞ Riemannian vector bundle ξ over M we write

$$< s_1, s_2 >_\xi = \int_M < s_1(x), s_2(x) >_{\xi_x} d\mu(x) \quad \text{for} \quad s_1, s_2 \in C^0(\xi).$$

6.1 Definition. Let $\mathcal{M}$ and $\mathcal{M}^*$ be two section functors satisfying (B §1) - (B §4). We say that $\mathcal{M}$ and $\mathcal{M}^*$ are dual section functors if given a Riemannian vector bundle ξ over a compact C^∞ n-manifold M with strictly positive smooth measure μ, there is a dual pairing $< , >_o$ of $\mathcal{M}(\xi)$ with $\mathcal{M}^*(\xi)$ such that $< s_1, s_2 >_o = < s_1, s_2 >_\xi$ for $s_1 \in \mathcal{M}(\xi) \cap C^0(\xi)$ and $s_2 \in \mathcal{M}^*(\xi) \cap C^0(\xi)$.

Remark. The discussion following (§4) shows that the choice of Riemannian structure and measure is irrelevant in 6.1.

As usual it suffices to verify that the condition of 6.1 is satisfied for the trivial line bundle over the n-disc, i.e. that there is a dual pairing of $\mathcal{M}(D^n, \mathbb{R})$ with $\mathcal{M}^*(D^n, \mathbb{R})$ which extends the bilinear map $(s_1, s_2) \longrightarrow \int_{D^n} s_1(x) s_2(x) dx$ of $C^0(D_n, \mathbb{R}) \times C^0(D^n, \mathbb{R}) \longrightarrow \mathbb{R}$.

6.2. Theorem. If $\mathcal{M}$ and $\mathcal{M}^*$ are dual section functors then for any C^∞ vector bundle ξ over a compact C^∞ n-dimensional manifold, $C_0^\infty(\xi)$ is dense in $\mathcal{M}(\xi)$ and in $\mathcal{M}^*(\xi)$.

Proof. Immediate from (B §4), since by Hahn-Banach a total subspace is dense.

6.3 Definition. If $\mathcal{M}$ and $\mathcal{M}^*$ are dual section functors then for any C^∞ vector bundle ξ over a compact C^∞ n-manifold M we define $\mathcal{M}_k^o(\xi)$ and $\mathcal{M}^*{}_k{}^o(\xi)$ to be respectively the closures of $C_o(\xi)$ in $\mathcal{M}_k(\xi)$ and in $\mathcal{M}_k^*(\xi)$. For $k > o$ we denote the dual space of $\mathcal{M}_k^{*\,o}(\xi)$ by $\mathcal{M}_{-k}(\xi)$. If M has a strictly positive smooth measure and if ξ is Riemannian then we identify $\mathcal{M}_o(\xi)$ with the dual of $\mathcal{M}_o^*(\xi)$ via the dual pairing $< , >_o$.

Remark. Note that by Theorem 6.2 $\mathcal{M}_0^{*\,0}(\xi) = \mathcal{M}_0^{*}(\xi)$ so that $\mathcal{M}_{-k}(\xi) =$ dual of $\mathcal{M}_k^{*\,0}(\xi)$ is still true when $k=0$.

6.4. Theorem. For $k \geq 0$ $\mathcal{M}_{k+1}^{0}(\xi)$ is a dense linear subspace of $\mathcal{M}_k^{0}(\xi)$ and the inclusion map is continuous. Thus restriction gives a continuous linear injection of $\mathcal{M}_{-k}(\xi)$ onto a dense linear subspace of $\mathcal{M}_{-k-1}(\xi)$. If $\mathcal{M}$ is a Rellich functor this injection is completely continuous.

Proof. The first statement is an immediate consequence of Theorem 5.1, and the definition of $\mathcal{M}_{k+1}^{0}$. The second follows because the restriction map of $\mathcal{M}_{-k}(\xi)$ into $\mathcal{M}_{-k-1}(\xi)$ is the adjoint of the inclusion $i\colon \mathcal{M}_{k+1}^{*\,0}(\xi) \longrightarrow \mathcal{M}_{k}^{*\,0}(\xi)$. The injectivity of i implies that i^* has dense range while the dense range of i implies the injectivity of i^*. The final remark follows because the adjoint of a completely continuous map is completely continuous. q.e.d.

In general we will identify each $\mathcal{M}_{-k}$ with its image in $\mathcal{M}_{-k-1}(\xi)$. Thus if ξ is a Riemannian vector bundle over a compact n-dimensional C^∞ manifold M and if M has a strictly positive smooth measure we have a doubly infinite chain of Banach spaces

$$\cdots \mathcal{M}_{k+1}(\xi) \subseteq \mathcal{M}_k(\xi) \subseteq \mathcal{M}_0(\xi) \subseteq \cdots \subseteq \mathcal{M}_{-k}(\xi) \subseteq \mathcal{M}_{-k-1}(\xi) \subseteq \cdots$$

Moreover each $\mathcal{M}_k(\xi)$ is dense in the next $\mathcal{M}_{k-1}(\xi)$ and the inclusion maps are continuous (and are completely continuous if $\mathcal{M}$ is a Rellich functor). Let $k \geq 0$ and let $s \in \mathcal{M}_k^{0}(\xi) \subseteq \mathcal{M}_0(\xi)$ and let $\ell \in \mathcal{M}_{-k}(\xi)$. If it should happen that $\ell \in \mathcal{M}_0(\xi) \subseteq \mathcal{M}_{-k}(\xi)$ then $\ell(s) = \langle s, \ell \rangle_0$. We may therefore regard the natural pairing of $\mathcal{M}_k^{0}(\xi)$ and $\mathcal{M}_{-k}(\xi)$ as an extension of $\langle\ ,\ \rangle_0$ and we shall write $\langle s, \ell \rangle_0$ for $\ell(s)$ whenever $\ell \in \mathcal{M}_{-k}(\xi)$ and $s \in \mathcal{M}_k^{0}(\xi)$.

6.5 Theorem. Let M be a compact C^∞ n-dimensional manifold with a strictly positive smooth measure and let ξ and η be C^∞ Riemannian vector bundles over M. If $D \in \mathrm{Diff}_r(\xi,\eta)$ then for every integer k, D extends to a continuous linear map $D_k\colon \mathcal{M}_k(\xi) \longrightarrow \mathcal{M}_{k-r}(\eta)$.

Proof. First suppose $r=1$. By Theorem 5.4 D extends to a continuous linear map $D_k: \mathcal{M}_k(\xi) \longrightarrow \mathcal{M}_{k-1}(\xi)$ for $k \geq 1$. Similarly $D^* \in \mathrm{Diff}_1(\eta,\xi)$ extends to a continuous linear map $D_k^*: \mathcal{M}_k^*(\eta) \longrightarrow \mathcal{M}_{k-1}^*(\xi)$ and since differential operators reduce supports, D_k^* restricts to a continuous linear map $D_k^{*\,o}: \mathcal{M}_k^{*\,o}(\eta) \longrightarrow \mathcal{M}_{k-1}^{*\,o}(\xi)$. Let $D_{-k+1}: \mathcal{M}_{-k+1}(\xi) \longrightarrow \mathcal{M}_{-k}(\eta)$ be adjoint to $D_k^{*\,o}$. If $\sigma \in C_o(\xi) \subseteq \mathcal{M}_{-k+1}(\xi)$ and $s \in C_o(\eta) \subseteq \mathcal{M}_k^{*\,o}(\eta)$ then

$< s, D_{-k+1}\sigma >_o = < D_k^{*\,o}s, \sigma >_o = < D^*s, \sigma > = < s, D\sigma >_o$.

Since $C_o(\eta)$ is a dense subspace of $\mathcal{M}_k^{*\,o}(\eta)$ it now follows that $D_{-k+1}\sigma = D\sigma$. Since k is any integer ≥ 1, $-k+1$ is any integer ≤ 0, so the theorem is proved for $r=1$. We now proceed by induction and assume that $r > 1$ and that the theorem holds when r is replaced by $r-1$. By §3 we can write $D = D_1D_2$ where $D_2 \in \mathrm{Diff}_{r-1}(\xi, J^{r-1}(\xi))$ and $D_1 \in \mathrm{Diff}_1(J^{r-1}(\xi),\eta)$. If $(D_2)_k: \mathcal{M}_k(\xi) \longrightarrow \mathcal{M}_{k-r+1}(J^{r-1}(\xi))$ is a continuous extension of D_2 and if $(D_1)_{k-r+1}: \mathcal{M}_{k-r+1}(J^{r-1}(\xi)) \longrightarrow \mathcal{M}_{k-1}(\eta)$ is a continuous extension of D_1 then putting $D_k = (D_1)_{k-r+1} \circ (D_2)_k$, $D_k: \mathcal{M}_k(\xi) \longrightarrow \mathcal{M}_{k-r}(\eta)$ is a continuous extension of D. q.e.d.

7. NON-COMPACT BASE SPACES

Let $\mathcal{M}$ satisfy axioms (B §1) - (B §4) of section 4 and let ξ be a C^∞ vector bundle over a not necessarily compact n-dimensional C^∞ manifold M. We define a vector subspace $\mathcal{M}(\xi)$ of $S(\xi)$, consisting of all sections s of ξ such that for each $p \in M$ there is a neighborhood N of p in M which is a compact C^∞ n-dimensional submanifold of M and $s|N \in \mathcal{M}(\xi|N)$. It is immediate from Theorem 4.1 that if N is any C^∞ n-dimensional compact submanifold of M then $s|N \in \mathcal{M}(\xi|N)$. We topologize $\mathcal{M}(\xi)$ by the requirement that the topology be the least fine locally convex topology such that for each such N the map $s \longmapsto s|N$ of $\mathcal{M}(\xi)$ into $\mathcal{M}(\xi|N)$ is continuous. If $\{N_\alpha\}_{\alpha\varepsilon A}$ is a family of compact n-dimensional submanifolds of M whose interiors cover $\mathcal{M}$ then the topology is the least fine locally convex topology such that each of the maps $s \longmapsto s|N_\alpha$ of $\mathcal{M}(\xi)$ into $\mathcal{M}(\xi|N_\alpha)$ is continuous. This follows easily from Theorem 4.3 since given N as above we can find $N_{\alpha_1},\ldots,N_{\alpha_m}$ whose union covers N. Indeed it follows from Theorem 4.3 that $s \longmapsto \{s|N_\alpha\}_{\alpha\varepsilon A}$ is a topological vector space isomorphism of $\mathcal{M}(\xi)$ with the subspace $\{ \{s_\alpha\} \in \prod_\alpha \mathcal{M}(\xi|N_\alpha) \mid s_\alpha|N_\beta = s_\beta|N_\alpha \}$ of $\prod_\alpha \mathcal{M}(\xi|N_\alpha)$. It follows that if M is σ-compact (or, what is the same, if M satisfies the second axiom of countability) then $\mathcal{M}(\xi)$ is a Frechet space (however it is Banach only when M is compact). If η is a second C^∞ vector bundle over M and $f \in \mathrm{Hom}(\xi,\eta)$ then given a compact n-dimensional submanifold N of M we have a commutative diagram

$$\begin{array}{ccc} \mathcal{M}(\xi) & \xrightarrow{\mathcal{M}(f)} & \mathcal{M}(\eta) \\ \downarrow & & \downarrow \\ \mathcal{M}(\xi|N) & \xrightarrow[\mathcal{M}(f|N)]{} & \mathcal{M}(\eta|N) \end{array}$$

from which it follows that (B §1) continues to hold if we omit the word "compact" and replace "Banach space" by "locally convex topological vector space". The properties (B §2) - (B §3) also generalize in the same way (and with equal ease) and hence so do their consequences.

We define $\mathcal{M}_\infty(\xi) = \bigcap_{n=1}^{\infty} \mathcal{M}_k(\xi)$ with the least fine locally convex topology making each of the inclusions $\mathcal{M}_\infty(\xi) \longrightarrow \mathcal{M}_k(\xi)$ continuous. If the base space M of ξ is compact or σ-compact then $\mathcal{M}_\infty(\xi)$ is Frechet (but it is not Banach even when M is compact).

In all interesting cases $C^\infty(\xi) \subseteq \mathcal{M}(\xi)$ and hence $C^\infty(\xi) \subseteq \mathcal{M}_k(\xi)$, so that $C^\infty(\xi) \subseteq \mathcal{M}_\infty(\xi)$. Moreover it also seems to be the case in all the usual examples that there is a positive integer ℓ such that $\mathcal{M}_\ell(\xi) \subseteq C^0(\xi)$ and that the inclusion is continuous. It then follows from Theorem 5.3 that $\mathcal{M}_{k+\ell}(\xi) \subseteq (C^0)_k(\xi) = C^k(\xi) = C^k(\xi)$ and that the inclusion is continuous and hence that in fact $\mathcal{M}_\infty(\xi) = C^\infty(\xi)$.

8. SOME EXAMPLES

Of course the most obvious example of a Banach space value section functor satisfying (B §1) - (B §4) is C^o with the compact open topology. If ξ is a C^∞ vector bundle over a compact n-dimensional manifold M and if $< , >_{\xi_x}$ is a Riemannian structure for ξ then $||s|| = \text{Sup}\ \{ < s(x), s(x) >_{\xi_x}^{1/2} \mid x \in M\}$ gives an admissible norm for $C^o(\xi)$, proving it is normable (completeness is well-known). The verifications of (B §1) - (B §4) are trivial. It is clear that $(C^o)_k = C^k$ with the usual "C^k-topology".

An only slightly less well-known example is C^α where $0 < \alpha < 1$. This is most easily described in terms of $C^\alpha(D^n,\mathbb{R})$ which consists of all real valued functions s on D^n which satisfy a Holder condition of order α (i.e. there exists $K > 0$ such that $|s(x) - s(y)| \leq K||x-y||^\alpha$) and with the norm $||s|| = \text{Sup}\ \{ |s(x) - s(y)|/||x-y||^\alpha | x \neq y\}$. Once again the axioms are easily verified. In this case $(C^\alpha)_k$ is usually denoted by $C^{k+\alpha}$: it consists of C^k sections whose partial derivatives of order k in local coordinates satisfy a Holder condition of order α. If we replace α by 1 in the above construction we get Lipschitz maps. Clearly C^1 would not be an appropriate symbol and we shall use the somewhat unusual symbol C^{1-}. For $(C^{1-})_k$ we write C^{k+1-}, these consist of C^k sections whose k^{th} partial derivatives satisfy a Lipschitz condition. Clearly if $1 > \alpha > \beta > 0$ we have

$$\ldots C^{k+1}(\xi) \subseteq C^{k+1-}(\xi) \subseteq C^{k+\alpha}(\xi) \subseteq C^{k+\beta}(\xi) \subseteq C^k(\xi) \subseteq \cdots$$

Moreover each of the inclusion maps is continuous and indeed by the Ascoli-Arzela Theorem it follows that the inclusion maps are even completely continuous. In particular all of the functors C^o, C^α, C^{1-} satisfy the Rellich condition. (Definition 5.5)

By the way, it might have seemed more natural to assume, instead of (B §3), that $C^\infty(\xi)$ was a dense subspace of $\mathcal{M}(\xi)$. Our reason for not doing so can now be explained. The closure of $C^\infty(\xi)$ in the space $C^{k+1-}(\xi)$ turns out to be $C^{k+1}(\xi)$ (this is most easily seen in the case of $C^{1-}(D^1,\mathbb{R})$, for if $f \in C^1(D^1,\mathbb{R})$ then clearly $\text{Sup}|f'(x)| = \text{Sup}|f(x) - f(y)|/|x-y|$ by the mean value theorem, hence the C^1 norm and the C^{1-} norm agree on $C^1(D^1,\mathbb{R})$, and since C^1 is complete in the C^1 norm it is a <u>closed</u> subspace of C^{1-}).

A somewhat different family of examples is provided by the functors L^p, $1 \leq p < \infty$. In terms of a local description, $L^p(D^n,\mathbb{R})$ consists of all measureable real valued functions s on D^n such that $||s|| = (\int |s(x)|^p dx)^{1/p} < \infty$. For a global description let ξ be a C^∞ vector bundle over a compact C^∞ n-manifold M. Choose a strictly positive smooth measure μ on M and a Riemannian structure $< , >_{\xi_x}$ for ξ and let $\mathcal{M}(\xi)$ denote all Borel measureable sections s of ξ such that $||s|| = (\int < s(x),s(x) >^{p/2} d\mu(x))^{1/p} < \infty$. An argument similar to that following (B §4) shows that if we change μ or the Riemannian structure, then we get an equivalent norm and hence the same Banach space $L^p(\xi)$. Note this is a case such as mentioned at the beginning of section 4, where the closure of the origin is all sections which vanish almost everywhere. Again the verification of (B §1) - (B §4) is trivial. If $p > 1$ and $\frac{1}{p} + \frac{1}{q} = 1$ then it is classical that L^p and L^q are dual functors (Definition 6.1), or rather it is classical that $(s_1,s_2) \longmapsto \int s_1(x)s_2(x)dx$ is a dual pairing of $L^p(D^n,\mathbb{R})$ with $L^q(D^n,\mathbb{R})$ which is all that is necessary. The spaces $L^p_k(\xi)$ are usually called Sobolev spaces. In particular L^2_k is often denoted by H^k. The L^2_k are particularly interesting because they are Hilbert spaces. Moreover L^2 is a self-dual functor and in fact it is essentially immediate from definition 6.1 that L^2 is the unique self-dual functor.

There are a great many more exotic Banach space section functors, however we shall not explicitly describe any of them but rather refer the interested reader to M. H. Taibleson s Research Announcement "Lipschitz classes of functions and distributions in E_n", Bull. Amer. Math. Soc., July 1963, pp. 487-493 and to the further references given there.

Finally a remark on notation. If ξ is a C^∞ vector bundle over a non-compact manifold M then it is customary to write $L^p_k(\xi)_{loc}$ for what in our notational convention would be $L^p_k(\xi)$. If ξ is Riemannian and μ is a strictly positive smooth measure on M then we can define a Banach space $L^p_k(\xi,\mu) = \{s \in (L^p_k(\xi))_{loc} | \infty > ||s||^p = \int ||s(x)||^p d\mu(x)\}$. Contrary to what happens when M is compact $L^p_k(\xi,\mu)$ does depend both on μ and on the Riemannian structure.

v = 1

9. SOME BASIC LEMMAS OF NON-LINEAR ANALYSIS

In this section M will denote a compact n-dimensional manifold, and ξ a C^∞ Riemannian vector bundle over M. We denote by R_M the trivial line bundle over M. If $1 \leq p < \infty$ we define $\bar{p}$ by $\frac{1}{p} + \frac{1}{\bar{p}} = 1$ so $L^{\bar{p}}(\xi)$ is the dual of $L^p(\xi)$. We recall that we define $L^p_k(\xi)^\circ$ for $k \geq 0$ to be the closure of $C^\infty_0(\xi)$ in $L^p_k(\xi)$ and $L^{\bar{p}}_{-k}(\xi)$ is defined to be the dual of $L^p_k(\xi)^\circ$. For reasons that will soon become evident it will be convenient to represent the spaces $L^p_k(\xi)$ as $L^p_{\frac{n}{p} - s}(\xi)$. We define $\bar{s} = n - s$, so that the dual of $L^p_{\frac{n}{p} - s}(\xi)$ is $L^{\bar{p}}_{\frac{n}{\bar{p}} - \bar{s}}(\xi)$. The spaces $L^p_k(\xi)$ are defined for non-integral values of k by the "complex method of interpolation" or equivalently by means of Bessel potentials (see "Lebesque spaces of differentiable functions and distributions" by A. P. Calderón, vol. 4 of the AMS Symposia in Pure Mathematics). The next few theorems state conditions for $L^p_k(\xi)$ to be included in $L^q_\ell(\xi)$ (or $C^{\ell+\alpha}(\xi)$) and are known collectively as the Sobolev embedding theorems. The proofs may be found in the paper of Calderón mentioned above.

<u>9.1 Theorem.</u> Let $1 \leq p, q < \infty$ and let k and ℓ be real numbers with $k - \frac{n}{p} \geq \ell - \frac{n}{q}$ and $k \geq \ell$. Then $L^p_k(\xi) \subseteq L^q_\ell(\xi)$ and the inclusion map is continuous. If $k - \frac{n}{p} > \ell - \frac{n}{q}$ and $k > \ell$ then the inclusion map is even completely continuous.

<u>Corollary.</u> If $1 \leq p < \infty$ then L^p satisfies the Rellich condition.

<u>9.2 Theorem.</u> Let $1 \leq p < \infty$ and let $k - \frac{n}{p} = \ell + \alpha$ where ℓ is a positive integer and $0 < \alpha < 1$. Then $L^p_k(\xi) \subseteq C^{\ell+\alpha}(\xi)$ and the inclusion map is continuous.

<u>Corollary.</u> If $k - \frac{n}{p} > \ell$ then $L^p_k(\xi) \subseteq C^\ell(\xi)$ and the inclusion map is completely continuous.

9.3 Theorem. Let N be a closed $n-j$ dimensional C^∞ submanifold of M and suppose $k - \frac{j}{p} > 0$. Then the restriction map of $C^\infty(\xi) \longrightarrow C^\infty(\xi|N)$ extends to a continuous map of $L^p_k(\xi)$ into $L^p_{k - \frac{j}{p}}(\xi|N)$.

Remark. We note that Theorem 9.1 can be rewritten in the form

$L^p_{\frac{n}{p}-s}(\xi) \subseteq L^q_{\frac{n}{q}-t}(\xi)$ if $\frac{n}{p} - s \geq \frac{n}{q} - t$ and $s \leq t$, and the inclusion is completely continuous if $\frac{n}{p} - s > \frac{n}{q} - t$ and $s < t$. In particular we have

$$L^p_{\frac{n}{p}-s}(\xi) \subseteq \begin{cases} L^\infty_o(\xi) & & s < 0 \\ L^q_o(\xi) & q < \infty & s = 0 \\ L^q_o(\xi) & \frac{1}{q} \geq \frac{s}{n} & 0 < s \leq \frac{n}{p} \end{cases}$$

and the inclusion map is continuous and is even completely continuous in case $s < 0$ or $0 < s < \frac{n}{p}$ and $\frac{1}{q} > \frac{s}{n}$.

The next few theorems seem to play a basic role in proving the differentiability of certain types of non-linear differential operators acting between Sobolev spaces. I would like to thank Mrs. Karen Uhlenbeck for suggesting several important improvements in these results.

$k-s$

9.4 Theorem. Let $1 \leq p_i < \infty$, $1 \leq q < \infty$ and let $\frac{n}{p_i} - s_i \geq 0$, $i = 1,\ldots,r$. Then multiplication is a continuous r-linear map of

$$\bigoplus_{i=1}^{r} L^{p_i}_{\frac{n}{p_i} - s_i}(M, \mathbb{R}) \longrightarrow L^q_o(M, \mathbb{R}) \text{ provided } \frac{1}{q} \geq \frac{1}{n} \sum_{s_i > 0} s_i$$

(where the inequality must be strict if some $s_i = 0$).

Proof. If $s_i < 0$ put $\frac{1}{q_i} = 0$, if $s_i > 0$ put $\frac{1}{q_i} = \frac{s_i}{n}$ and if some $s_i = 0$ (in which case $\frac{1}{q} > \frac{1}{n} \sum_{s_i>0} s_i$) choose the corresponding q_i so large that $\frac{1}{q} > \sum_i \frac{1}{q_i} = \sum_{s_i=0} \frac{1}{q_i} + \frac{1}{n} \sum_{s_i>0} s_i$.

Then by Holder's inequality multiplication is a continuous r-linear map of $\bigoplus_{i=1}^{r} L_o^{q_i}(M, \mathbb{R}) \longrightarrow L_o^q(M, \mathbb{R})$ and by the remark following Theorem 9.3 we have a continuous inclusion $L^{p_i}_{\frac{n}{p_i} - s_i}(M, \mathbb{R}) \subseteq L_o^{q_i}(M, \mathbb{R})$.

q.e.d.

9.5 Theorem. Let $1 \leq p_i < \infty$, $1 \leq q < \infty$, and $\frac{n}{p_i} - s_i \geq \max(0, \ell)$.

1) If each $s_i < -\ell < 0$ then multiplication is a continuous r-linear map of

$\bigoplus_{i=1}^{r} L^{p_i}_{\frac{n}{p_i} - s_i}(M, \mathbb{R})$ into $C^\ell(M, \mathbb{R})$.

2) If all $s_i \leq 0$ then multiplication is a continuous r-linear map of

$\bigoplus_{i=1}^{r} L^{p_i}_{\frac{n}{p_i} - s_i}(M, \mathbb{R})$ into $L^q_\ell(M, \mathbb{R})$ provided $\ell \leq \frac{n}{q} - \max(s_1, \ldots s_r)$ (where the inequality must be strict if $\frac{n}{q}$ is integral).

3) If some $s_i > 0$ then multiplication is a continuous r-linear map of

$\bigoplus_{i=1}^{r} L^{p_i}_{\frac{n}{p_i} - s_i}(M, \mathbb{R})$ into $L^q_\ell(M, \mathbb{R})$ provided $\ell \leq \frac{n}{q} - \sum_{s_i>0} s_i$ (where the inequality must be strict if some $s_i = 0$) and if moreover, in case $\ell < 0$, $\sum_{s_i>0} s_i < n$.

Proof. Case 1) is immediate from the fact that C^ℓ is a Banach algebra and that (Corollary of Theorem 9.2) for $k - \frac{n}{p} > \ell$, $L^q_k(M, \mathbb{R}) \subseteq C^\ell(M, \mathbb{R})$.

For the remaining two cases we can easily reduce consideration to a coordinate neighborhood in M, so we can assume $M = D^n$. Also by interpolation theory we need only consider integral ℓ. To begin with we assume $\ell \geq 0$ (which is no restriction in case 2) in which case by the definition of $L^q_\ell(D^n, \mathbb{R})$ it will suffice to show for each multi-index α with $|\alpha| \leq \ell$ that

$(f_1, \ldots, f_r) \longmapsto D^\alpha(f_1 \ldots f_r)$ is a continuous r-linear map of $\bigoplus_{i=1}^{r} L^{p_i}_{\frac{n}{p_i} - s_i}(D^n, \mathbb{R})$ into $L^q_0(D^n, \mathbb{R})$.

Now by Leibniz' formula $D^\alpha(f_1 \ldots f_r)$ is a linear combination with integer coefficeints of terms of of the form $(D^{\beta_1}f_1)\ldots(D^{\beta_r}f_r)$ with $\beta_1 + \ldots + \beta_r = \alpha$. Also $f_i \longmapsto D^{\beta_i}f_i$ is a continuous linear map of $L^{p_i}_{\frac{n}{p_i} - s_i}(D^n, \mathbb{R})$ into $L^{p_i}_{\frac{n}{p_i} - (s_i + |\beta_i|)}(D^n, \mathbb{R})$. Thus it will suffice to show that multiplication is a continuous r-linear map of $\bigoplus_{i=1}^{r} L^{p_i}_{\frac{n}{p_i} - (s_i + t_i)}(D^n, \mathbb{R})$ into $L^q_0(D^n, \mathbb{R})$ for all r-tuples $(t_1, \ldots, t_r)$ of non-negative integers with $t_1 + \ldots + t_r \leq \ell$.

By Theorem 9.4 this will be the case provided the maximum of the sums $\sum_{s_i + t_i > 0} w_i + t_i$, for all such $(t_1, \ldots, t_r)$, is $\leq \frac{n}{q}$. We can assume some $s_i \geq -\ell$, for otherwise Case 2 follows from Case 1. If all $s_i < 0$ the maximum of these sums is $\ell + \max(s_1, \ldots s_r)$. On the other hand if some $s_i > 0$ the maximum sum is $\ell + \sum_{s_i > 0} s_i$. Finally if $\ell < 0$ then recalling that $L^q_\ell(D^n, \mathbb{R})$ is the dual of $L^{\bar{q}}_{-\ell}(D^n, \mathbb{R})^o$ and that the pairing is given by $\langle g, f \rangle = \int g(x)f(x)dx$ when $g, f \in C^\infty(D^n, \mathbb{R})$, it will clearly suffice in this case to prove that multiplication is continuous from $\bigoplus_{i=1}^{r} L^{p_i}_{\frac{n}{p_i} - s_i}(D^n, \mathbb{R}) \oplus L^{\bar{q}}_{-\ell}(D^n, \mathbb{R})$ into $L^1_0(D^n, \mathbb{R})$. Since

$-\ell = \frac{n}{\bar{q}} - (\ell + \frac{n}{\bar{q}}) = \frac{n}{\bar{q}} - (\ell + n - \frac{n}{q})$ the condition for this by Theorem 9.4 is

$\frac{n}{1} \geq \ell + n - \frac{n}{q} + \sum_{s_i > 0} s_i$ or $\ell \leq \frac{n}{q} - \sum_{s_i > 0} s_i$ provided $(\ell + n - \frac{n}{q}) > 0$. If $\ell + n - \frac{n}{q} \leq 0$ then the condition from Theorem 9.4 becomes $n > \sum_{s_i > 0} s_i$ which is satisfied by hypothesis. q.e.d.

The following is just a restatement of the second two conclusions of Theorem 9.5 making the substitution $\ell = \frac{n}{q} - \sigma$.

<u>9.6 Theorem</u>. Let $1 \leq p_i < \infty$, $1 \leq q < \infty$, $\frac{n}{p_i} - s_i \geq \frac{n}{q} - \sigma$ and let $\sum_{s_i > 0} s_i < n$. Then multiplication is a continuous r-linear map of

$$\bigoplus_{i=1}^{r} L^{p_i}_{\frac{n}{p_i} - s_i}(M, \mathbb{R}) \text{ into } L^{q}_{\frac{n}{q} - \sigma}(M, \mathbb{R})$$

provided $\sigma \geq \sum_{s_i > 0} s_i$ (where the inequality must be strict if some $s_i = 0$). If all $s_i < 0$ then the weaker condition $\sigma \geq \max(s_1, \ldots, s_r)$ (where the inequality must be strict if $\max(s_1, \ldots, s_r)$ is an integer) suffices.

<u>9.7 Corollary</u>. Let $1 \leq p < \infty$ and $k > \frac{n}{p}$. Then $L^p_k(M, \mathbb{R})$ is a Banach algebra under pointwise multiplication and for $0 \leq j \leq k$, $L^p_j(M, \mathbb{R})$ is a topological $L^p_k(M, \mathbb{R})$ module.

<u>Proof</u>. In 9.6 take $r = 2$, $p = p_1 = p_2 = q$, $s_1 = \frac{n}{p} - k$, $s_2 = \frac{n}{p} - j$, $\sigma = \frac{n}{p} - j$. Then $\frac{n}{p_1} - s_1 = k$, $\frac{n}{p_2} - s_2 = j$ $\frac{n}{q} - \sigma = j$ so $\frac{n}{p_i} - s_i \geq \frac{n}{q} - \sigma \geq 0$ and $\sigma = \frac{n}{p} - j > (\frac{n}{p} - j) + (\frac{n}{p} - k) = s_1 + s_2$.

The conclusion is that multiplication is a continuous bilinear map of $L^p_k(M, \mathbb{R}) \oplus L^p_j(M, \mathbb{R})$ into $L^p_j(M, \mathbb{R})$. q.e.d.

<u>9.8 Lemma</u>. Let $1 \leq p < \infty$, $k > \frac{n}{p}$ and let $\beta_1, \ldots, \beta_r$ be n-multi-indices with $|\beta_i| > 0$ and $\sum_{i=1} |\beta_1| \leq k$. Then

$$(f_1, \ldots, f_r) \longmapsto D^{\beta_1} f_1 \ldots D^{\beta_r} f_r$$

is a continuous r-linear map of $\oplus^r L^p_k (D^n, \mathbb{R})$ into $L^p_o (D^n, \mathbb{R})$.

<u>Proof</u>. Since $D^{\beta_i}: L^p_k(D^n, \mathbb{R}) \longrightarrow L^p_{k-|\beta_i|}(D^n, \mathbb{R})$ is a continuous linear map we must show that multiplication is continuous from $\oplus^r L^p_{\frac{n}{p} - (|\beta_i| + (\frac{n}{p} - k))}(D^n, \mathbb{R})$ into $L^p_o(D^n, \mathbb{R})$. By Theorem 9.4 it suffices to show that $\frac{n}{p} > \sum_{|\beta_i| > k - \frac{n}{p}} (|\beta_i| + (\frac{n}{p} - k))$ or, writing t for the number of indices $i = 1, \ldots r$ such that $|\beta_i| > k - \frac{n}{p}$, we can rewrite this as $(\sum_{|\beta_i| > k - \frac{n}{p}} |\beta_i|) - \frac{n}{p} < t(k - \frac{n}{p})$. We consider three cases. First if $t = 0$ then the inequality becomes $-\frac{n}{p} < 0$. Second if $t > 1$ then since $\sum_{i=1}^{r} |\beta_i| < k$ the inequality is also clear. Finally if $t = 1$ then the inequality becomes $k > \sum_{|\beta_i| > k - \frac{n}{p}} |\beta_i|$. Since we can assume $r \geq 2$, and since all $|\beta_i| > 0$ we have $\sum_{|\beta_i| > k - \frac{n}{p}} |\beta_i| < \sum_{i=1}^{r} |\beta_i| \leq k$. q.e.d.

<u>9.9 Lemma</u>. If $f \in C^\infty(D^n \times \mathbb{R}^r, \mathbb{R})$ then for $k > \frac{n}{p}$ the map $(s_1, \ldots, s_r) \longmapsto f(\cdot, s_1(\cdot), \ldots, s_r(\cdot))$ of $\oplus^r C^o(D^n, \mathbb{R})$ into $C^o(D^n, \mathbb{R})$ restricts to a continuous map of $\oplus^r L^p_k(D^n, \mathbb{R})$ into $L^p_k(D^n, \mathbb{R})$.

<u>Proof</u>. By definition of the topology of $L^p_k(D^n, \mathbb{R})$ we must show that for $|\alpha| \leq k$ the map $(s_1, \ldots, s_r) \longmapsto D^\alpha f(\cdot, s_1(\cdot), \ldots, s_r(\cdot))$ of $\oplus^r C^\infty(D^n, \mathbb{R})$ to $C^\infty(D^n, \mathbb{R})$ extends to a continuous map of $\oplus^r L^p_k(D^n, \mathbb{R})$ into $L^p_o(D^n, \mathbb{R})$. By the chain rule and induction

$$D^\alpha f(\cdot, s_1(\cdot), \ldots, s_r(\cdot)) = \sum_{\beta_1 + \ldots + \beta_r \leq \alpha} \varphi_{\beta_1 \ldots \beta_r}(\cdot, s_1(\cdot), \ldots, s_r(\cdot)) D s_1^{\beta_1} \ldots D s_r^{\beta_r}$$ where

$\varphi_{\beta_1 \cdots \beta_r}$ is in $C(D^n \times \mathbb{R}^r, \mathbb{R})$. Now

$$(s_1, \ldots, s_r) \longmapsto \varphi_{\beta_1 \cdots \beta_r}(\cdot\, ,\ s_1(\cdot), \ldots, s_r(\cdot))$$

is clearly continuous from $\oplus^r C^o(D^n, \mathbb{R})$ into $C^o(D^n, \mathbb{R})$ and by the corollary of Theorem 9.2 it is also continuous from $\oplus^r L^p_k(D^n, \mathbb{R})$ into $C^o(D^n, \mathbb{R})$. Also multiplication is continuous from $C^o(D^n, \mathbb{R}) \times L^p(D^n, \mathbb{R})$ into $L^p(D^n, \mathbb{R})$ hence it suffices to show that $(s_1, \ldots, s_r) \longmapsto D^{\beta_1} s_1 \ldots D^{\beta_r} s_r$ is a continuous map of $\oplus^r L^p_k(D^n, \mathbb{R})$ into $L^p_o(D^n, \mathbb{R})$ if $|\beta_1| + \ldots + |\beta_r| \leq |\alpha| \leq k$. But this is immediate from Lemma 9.8.

<u>9.10 Theorem</u>. Let ξ and η be C^∞ vector bundles over M and let $f\colon \xi \longrightarrow \eta$ be a C^∞ fiber preserving (but not necessarily linear on fibers) map. Then the composition $s \longmapsto f \circ s$ is a continuous map of $L^p_k(\xi)$ into $L^p_k(\eta)$. $(k > \frac{n}{p}\ ?)$

Proof. As usual we can reduce to the case $M = D^n$ and we can assume $\xi = D^n \times \mathbb{R}^n$ and $\eta = D_n \times \mathbb{R}$, in which case the theorem is just a restatement of the preceding lemma. q.e.d.

The following lemma allows us to extend the results of Theorems 9.5 and 9.6 to the case when exactly one of the $\frac{n}{p_i} - s_i$ is negative. As usual $L^r(U_1, \ldots, U_r; V)$ denotes the continuous r-linear maps of $U_1 \oplus \cdots \oplus U_r$ into V.

<u>9.11 Lemma</u>. Let $V_1, \ldots, V_r$, W, Z be Banach spaces with W reflexive. Then there is a natural isomorphism of $L^{r+1}(V_1, \ldots, V_r, W^*; Z^*)$ with $L^{r+1}(V_1, \ldots, V_r, Z; W)$, $T \longmapsto \tilde{T}$, defined by $(\tilde{T}(v_1, \ldots, v_r, z)) = (T(v_1, \ldots, v_r, \ell))(z)$.

<u>Proof</u>. This is the composite of the following natural isomorphisms

$$L^{r+1}(V_1, \ldots, C_r, W^*; Z^*) \approx L^r(V_1, \ldots, V_r; L(W^*, Z^*)) \approx L^r(V_1, \ldots, V_r; L(W^*, L(Z, \mathbb{R})))$$

$\approx L^r(V_1,\ldots,V_r\ ;\ L^2(W^*,\ Z;\ \mathbb{R})) \approx L^r(V_1,\ldots,V_r;\ L(Z,L(W^*,\mathbb{R}))) \approx L^r(V_1,\ldots,V_r,\ L(Z,W))$

$\approx L^{r+1}(V_1,\ldots,V_r,\ Z;\ W).$ q.e.d.

<u>9.12 Lemma</u>. Let $1 \leq p_i < \infty$, $1 < q < \infty$, $k_i \geq 0$ $i = 1,\ldots,r$ $k_{r+1} < 0$, $\ell < 0$ and suppose that multiplication extends from $\oplus^{r+1} C^\infty(M, \mathbb{R}) \longrightarrow C^\infty(M, \mathbb{R})$ to a continuous $(r+1)$ - linear map of $\bigoplus_{i=1}^{r} L_{k_i}^{p_i}(M, \mathbb{R}) \oplus L_{-\ell}^{\bar{q}}(M, \mathbb{R})$ into $L_{-k_{r+1}}^{\bar{p}_{r+1}}(M, \mathbb{R})$. Then it also extends to a continuous $(r+1)$ - linear map of $\bigoplus_{i=1}^{r+1} L_{k_i}^{p_i}(M, \mathbb{R})$ into $L_\ell^q(M, \mathbb{R})$.

<u>Proof</u>. Recall that $L_{-\ell}^{\bar{q}}(M, \mathbb{R}) = (L_\ell^q(M, \mathbb{R})^o)^*$ and if $g \in C^\infty(M, \mathbb{R})$, then regarding g as an element of $L_{-\ell}^{\bar{q}}(M, \mathbb{R})$, for $f \in L_\ell^q(M, \mathbb{R})^o$ we have $g(f) = \int gf d\mu$ where μ is the Riemannian measure on M. Now since the product of elements of $C^\infty(M, \mathbb{R})$ has smaller support than any of the factors it is clear that multiplication maps $\bigoplus_{i=1}^{r} L_{k_i}^{p_i}(M, \mathbb{R}) \oplus L_{-\ell}^{\bar{q}}(M, \mathbb{R})^o$ into $L_{-k_{r+1}}^{\bar{p}_{r+1}}(M, \mathbb{R})^o$ and Lemma 9.12 now follows directly from Lemma 9.11 q.e.d.

<u>9.13 Theorem</u>. If $1 \leq p < \infty$, $k > \frac{n}{p}$ then for $0 \geq j \geq -k$, multiplication extends from $C^\infty(M, \mathbb{R}) \oplus C^\infty(M, \mathbb{R}) \longrightarrow C^\infty(M, \mathbb{R})$ to a continuous bilinear map of $L_k^p(M, \mathbb{R}) \oplus L_j^{\bar{p}}(M, \mathbb{R})$ into $L_j^{\bar{p}}(M, \mathbb{R})$ where $\frac{1}{\bar{p}} = 1 - \frac{1}{p}$. In particular (for $p = 2$), $L_j^2(M, \mathbb{R})$ is an $L_k^2(M, \mathbb{R})$ module for $|j| \leq k$ if $k > \frac{n}{2}$.

<u>Proof</u>. Corollary 9.7 and Lemma 9.12.

<u>9.14 Theorem</u>. Let $1 \leq p_i < \infty$, $1 < p_{r+1} < \infty$, $1 < q < \infty$ and suppose

$\frac{n}{q} - \sigma \leq \frac{n}{p_{r+1}} - s_{r+1} \leq 0$, $\frac{n}{p_i} - s_i \geq -(\frac{n}{p_{r+1}} - s_{r+1})$ for $1 \leq i \leq r$, and

$\sum_{s_i \geq 0} s_i < n$. Then multiplication extends from $\oplus^{r+1} C^\infty(M, \mathbb{R}) \longrightarrow C^\infty(M, \mathbb{R})$

to a continuous $(r+1)$ - linear map of $\bigoplus_{i=1}^{r+1} L^{p_i}_{\frac{n}{p_i} - s_i}(M, \mathbb{R})$ into $L^q_{\frac{n}{q} - \sigma}(M, \mathbb{R})$

provided $\sigma \geq \sum_{s_i > 0} s_i$ (where the inequality must be strict if some $s_i = 0$).

Proof. By Lemma 9.12 we must show that multiplication is continuous from $\bigoplus_{i=1}^{r} L^{p_i}_{\frac{n}{p_i} - s_i}(M, \mathbb{R}) \oplus L^{\bar{q}}_{\frac{n}{\bar{q}} - \bar{\sigma}}(M, \mathbb{R})$ into $L^{\bar{p}_{r+1}}_{\frac{n}{\bar{p}_{r+1}} - \bar{s}_{r+1}}(M, \mathbb{R})$ (where we recall, $\bar{\sigma} = n - \sigma$, $\bar{s}_{r+1} = n - s_{r+1}$). Since $\sum s_i < n$ we can choose $\sigma' \leq \sigma$ with $n > \sigma' \geq \sum_{s_i > 0} s_i$ and since there is a continuous inclusion of $L^q_{\frac{n}{q} - \sigma'}(M, \mathbb{R})$ into $L^q_{\frac{n}{q} - \sigma}(M, \mathbb{R})$ we can assume $\sigma = \sigma'$, or in other words we can assume $\sigma < n$. Then $\bar{\sigma} = n - \sigma > 0$, thus by Theorem 9.6 it suffices to verify that $\bar{s}_{r+1} > \bar{\sigma} + \sum_{\substack{s_i > 0 \\ i = 1 \ldots r}} s_i$ which is equivalent to $\sigma > \sum_{s_i > 0} s_i$, with equality permissible if no $s_i = 0$. q.e.d.

9.15 Lemma. Let $k > \frac{n}{2}$ and $k \geq s \geq 0$. Then multiplication extends from $C^\infty(M, \mathbb{R}) \oplus C^\infty(M, \mathbb{R}) \longrightarrow C^\infty(M, \mathbb{R})$ to a continuous bilinear map of $L^2_{k-s}(M, \mathbb{R}) \oplus L^2_{s-k}(M, \mathbb{R})$ into $L^2_{-k}(M, \mathbb{R})$.

Proof. By 9.12 it suffices to show that multiplication is continuous from $L^2_k(M, \mathbb{R}) \oplus L^2_{k-s}(M, \mathbb{R})$ into $L^2_{k-s}(M, \mathbb{R})$ and this is immediate from Corollary 9.7.

q.e.d.

<u>9.16 Theorem</u>. Let $k > \frac{n}{2}$, $s_i \geq 0$ with $s = \sum_{i=1}^{r} s_i \leq 2k$. Then multiplication extends from $\oplus^r C^\infty(M, \mathbb{R})$ to a continuous r-linear map of

$$\bigoplus_{i=1}^{r} L^2_{k-s_i}(M, \mathbb{R}) \text{ into } L^2_{k-w}(M, \mathbb{R}).$$

<u>Proof</u>. Since $s_i + s_j \leq \sum_{i=1}^{r} s_i \leq 2k$ at most of the s_i can be greater than k, hence at most one of the $k-s_i$ can be negative. Note that $k-s_i = \frac{n}{2} - (s_i - (k - \frac{n}{2}))$ and similarly $k-w = \frac{n}{2} - (w - (k - \frac{n}{2}))$, thus from Theorems 9.6 and 9.14 we see that what we must verify is that $s_i \leq w$ (which is trivial) and the two inequalities

$$\text{(A)} \qquad \sum_{s_i > k - \frac{n}{2}} (s_i - (k - \tfrac{n}{2})) < n$$

$$\text{(B)} \qquad w > \sum_{s_i > k - \frac{n}{2}} (s_i - (k - \tfrac{n}{2}))$$

If we define t to be the number of indices i with $s_i > k - \frac{n}{2}$ these inequalities become (remembering that $w = \Sigma s_i$ so that

$$w - \sum_{s_i > k - \frac{n}{2}} s_i = \sum_{s_i < k - \frac{n}{2}} s_i)$$

$$\text{(A')} \qquad \sum_{s_i > k - \frac{n}{2}} s_i < 2k + (t-2)(k - \tfrac{n}{2})$$

$$\text{(B')} \qquad \sum_{s_i < k - \frac{n}{2}} s_i > t(\tfrac{n}{2} - k)$$

Now (B') is trivially satisfied since $s_i \geq 0$ and the right hand side is negative. The inequality (A') is also clear if $t > 2$ since $\sum_{i=1}^{r} s_i \leq w \leq 2k$. Since we can ignore any $s_i = 0$

(by 9.13), in case $t = 2$ (A') is also clear unless $r = 2$ and $s_1 + s_2 = 2k$. But in this case the theorem reduces to Lemma 9.15. Finally in the special cases $t = 0$ and $t = 1$, when (A') in fact does not hold, the theorem follows easily from Theorem 9.13.

q.e.d.

10. THE CATEGORIES FB(M) AND FVB(M)

Given finite dimensional C^∞ manifolds E and M and a C^∞ map $p:E \longrightarrow M$ let us put $E_m = p^{-1}(m)$ for each $m \in M$. We shall say that E (or more precisely the map $p:E \longrightarrow M$) is a C^∞ fiber bundle over M if p is differentiably locally trivial, i.e., if, for each $m_o \in M$, E_{m_o} is a C^∞ submanifold of E and there is a neighborhood U of m_o and a C^∞ diffeomorphism $\varphi: p^{-1}(U) \approx U \times E_{m_o}$ making the following diagram commutative

$$\begin{tikzcd} p^{-1}(U) \arrow[rr, "\varphi"] \arrow[dr, "p"'] & & U \times E_{m_o} \arrow[dl, "\Pi"] \\ & U & \end{tikzcd}$$

where Π is the natural projection. Given a second C^∞ fiber bundle $p':E' \longrightarrow M$ over M and a C^∞ map $f:E \longrightarrow E'$ we call f a fiber bundle morphism if it is fiber preserving, i.e if $f(E_m) \subseteq E'_m$ for all $m \in M$. Of course for a C^∞ fiber bundle E over M, as for a vector bundle, we have the notion of the set $S(E)$ of all sections of E as well as that of the spaces $C^k(E)$ of all C^k sections of E, $k = 0,1,\ldots,\infty$, but these have no linear structure. Also if $f:E \longrightarrow E'$ is a fiber bundle morphism then we have an induced composition map $f_*;S(E) \longrightarrow S(E')$, defined by $s \longmapsto f \circ s$, which maps $C^k(E)$ into $C^k(E')$ for all k.

We define FB(M) to be the category of C^∞ fiber bundles over M and fiber bundle morphisms. Our goal in the next few sections is to show how with one extra assumption, (B §5) introduced in the next section, our functor $\mathcal{M}$ from VB(M) to Banach spaces and continuous linear maps "extends" to a functor from FB(M) to Banach manifolds and C^∞ maps. This extension is of course at the heart of the relation of non-linear analysis to the theory of infinite dimensional manifolds.

In carrying out this extension it is convenient to pass through an intermediate category, FVB(M), whose objects are C^∞ vector bundles over M (i.e. objects of VB(M)) but whose morphisms are all fiber bundle morphisms (i.e. morphisms of FB(M)) rather than only the vector bundle homomorphisms. Then for an object ξ of FVB(M) $\mathcal{M}(\xi)$ is a well-defined Banach space. The axiom (B §5) demands that $\mathcal{M}(\xi) \subseteq C^0(\xi)$ and that for a morphism $f:\xi \longrightarrow \eta$ of FVB(M), the map $f_*: C^0(\xi) \longrightarrow C^0(\eta)$ restrict to a continuous map $\mathcal{M}(f): \mathcal{M}(\xi) \longrightarrow \mathcal{M}(\eta)$. The somewhat surprising fact is that it follows automatically that $\mathcal{M}(f)$ is a fact a C^∞ map, hence $\mathcal{M}$ extends to a functor from the category FVB(M) to the category of Banach spaces and C^∞ maps. This is the first step in the extension.

The remainder of the extension is based on the notion of an "open vector bundle neighborhood" in a C^∞ fiber bundle E over M. This is C^∞ vector bundle ξ over M such that ξ is an open submanifold of E and the inclusion map of ξ in E is a fiber bundle morphism. If $s \in C^0(E)$ then a fundamental existence theorem states that we can always find such a ξ with $s \in C^0(\xi)$. Whether or not $s \in \mathcal{M}(\xi)$ turns out to be independent of the choice of such a ξ, and the set of all $s \in C^0(E)$ for which $s \in \mathcal{M}(\xi)$ is denoted by $\mathcal{M}(E)$. It follows from the first step in the extension that there is a unique C^∞ Banach manifold structure for $\mathcal{M}(E)$ such that for each open vector bundle neighborhood ξ of E, $\mathcal{M}(\xi)$ is an open submanifold of $\mathcal{M}(E)$, and it also follows that if $f:E \longrightarrow E'$ is a fiber bundle morphism then $s \longmapsto f \circ s$ defines a C^∞ map $\mathcal{M}(f):\mathcal{M}(E) \longrightarrow \mathcal{M}(E')$, and this completes the extension of $\mathcal{M}$ to a functor from FB(M) to the category of C^∞ Banach manifolds and C^∞ maps. We now proceed to fill in the details of this sketch.

11. THE AXIOM (§5)

In what follows n is some fixed positive integer. We now add to our axioms (B §1)- (B §4) introduced in section 4 the following axiom:

Axiom (B §5). If ξ is a vector bundle over a compact C^∞ n-dimensional manifold M then $\mathcal{M}(\xi) \subseteq C^0(\xi)$ and the inclusion map is continuous. Moreover if η is a second vector bundle over M and $f:\xi \longrightarrow \eta$ is a C^∞ fiber preserving map (i.e. a morphism of FVB(M)) then $f_*:C^0(\xi) \longrightarrow C^0(\eta)$ restricts to a continuous map $\mathcal{M}(f):\mathcal{M}(\xi) \longrightarrow \mathcal{M}(\eta)$.

We note that (B §5) is not independent of (B §1) - (B §4). Indeed (B §5) clearly implies (B §1), (B §3), and (B §4), so in effect the axioms we will henceforth be assuming are (B §2) and (B §5).

It follows from the corollary of Theorem 9.2 and Theorem 9.10 that for $k > n/p$ axiom (B §5) is satisfied with $\mathcal{M} = L^p_k$. It is trivial that (B §5) is satisfied with $\mathcal{M} = C^0$. A more or less obvious modification of the proof of Theorem 9.10 shows that (B §5) is also satisfied if $\mathcal{M} = C^k$, $C^{k+\alpha}$, or C^{k-} .

It is not hard to show from the localization and direct sum theorems of section 4 that in order to verify that (B §5) holds in general it suffices to consider the case $M = D^n$, $\xi = D^n \times \mathbb{R}^r$, and $\eta = D^n \times \mathbb{R}$, or in other words to prove an analogue of Lemma 9.9 with $\mathcal{M}(D^n, \mathbb{R})$ replacing $L^p_k(D^n, \mathbb{R})$.

The following is an interesting necessary condition for (B §5) to hold

<u>11.1 Theorem</u>. If (B §5) holds then, for each compact n-dimensional M, $\mathcal{M}(M, \mathbb{R})$ is a Banach algebra under pointwise multiplication.

<u>Proof</u>. Let $\mathbb{R}_M$ denote the trivial line bundle $M \times \mathbb{R}$ over M, so that $\mathcal{M}(M, \mathbb{R}) = \mathcal{M}(\mathbb{R}_M)$ and define a fiber bundle morphism $f: \mathbb{R}_M \longrightarrow \mathbb{R}_M$ by $f(x,y) = xy$. Then since the map $(s_1, s_2) \longmapsto s_1 s_2$ of $C^0(M, \mathbb{R}) \oplus C^0(M, \mathbb{R}) \longrightarrow C^0(M, \mathbb{R})$ is just f_*, it follows directly from (B §5) that $\mathcal{M}(M, \mathbb{R})$ is a Banach algebra.

q.e.d.

The really essential information contained in (B §5) is that $\mathcal{M}$ is a functor from FBV(M) to the category of Banach spaces and continuous maps. In the remainder of this section we will prove the fact, crucial to the rest of the development, that this automatically entails that $\mathcal{M}$ is really a functor from FVB(M) to the category of Banach spaces and C^∞ maps, i.e. that if $f: \xi \longrightarrow \eta$ is a fiber bundle morphism of vector bundles ξ and η over a compact n-dimensional manifold M, then $\mathcal{M}(f): \mathcal{M}(\xi) \longrightarrow \mathcal{M}(\eta)$, which is only guaranteed to be C^0 by (B §5), is in fact C^∞, and indeed we will explicitly compute $d^k\mathcal{M}(f)$ for all k.

Given vector bundles ξ and η over M let $L^r(\xi,\eta)$ denote the vector bundle over M whose fiber at x is the space $L^r(\xi_x,\eta_x)$ of r-linear maps of ξ_x into η_x, and let $L^r_s(\xi,\eta)$ denote the sub-bundle of symmetric r-linear maps. We note that there is a canonical identification of $L(\xi,L^r(\xi,\eta))$ with $L^{r+1}(\xi,\eta)$. Given a fiber bundle morphism $f: \xi \longrightarrow \eta$, for each $x \in M$ $f|\xi_x$ is a C^∞ map of ξ_x into η_x, hence for each positive integer r its r^{th} differential at $e \in \xi_x$ is an element $\delta^r f(e) = d^r(f|\xi_x)(e) \in L^r_s(\xi_x,\eta_x) = L^r_s(\xi,\eta)_x$. It is clear that $\delta^r f: \xi \longrightarrow L^r_s(\xi,\eta)$ is a fiber bundle morphism (for r=1 we write simply δf) called the vertical r^{th} differential of f (or the r^{th} differential of f along the fiber). Moreover if we regard $\delta^r f$ as a morphism into $L^r(\xi,\eta)$ and make the above identification of $L(\xi,L^r(\xi,\eta))$ with $L^{r+1}(\xi,\eta)$ then $\delta(\delta^r f) = \delta^{r+1} f$.

There is a canonical fiber bundle morphism $\gamma: L^r(\xi,\eta) \oplus (\oplus^r \xi) \longrightarrow \eta$ given by $\gamma(T, e_1,\ldots,e_r) = T(e_1,\ldots,e_r)$. By (B §5) and Theorem 4.4 we have a continuous mao

$\mathcal{M}(\gamma): \mathcal{M}(L^r(\xi,\eta)) \oplus (\oplus^r\mathcal{M}(\xi)) \longrightarrow \mathcal{M}(\eta)$ given by $\mathcal{M}(\gamma)\ (T,s_1,\ldots s_r)\ (x) = T(x)(s_1(x),\ldots,s_r(x))$. From this it follows that the map $\sim$ of the following definition is well-defined and continuous.

<u>11.2 Definition</u>. We define a continuous linear map $T \longrightarrow \tilde{T}$ of $\mathcal{M}(L^r(\xi,\eta))$ into $L^r(\mathcal{M}(\xi), \mathcal{M}(\eta))$ by $\tilde{T}(s_1,\ldots,s_r)(x) = T(x)\ (s_1(x),\ldots,s_r(x))$. If $F:\xi \longrightarrow L^r(\xi,\eta)$ is a fiber bundle morphism then we define a continuous map $\tilde{\mathcal{M}}(F): \mathcal{M}(\xi) \longrightarrow L^r(\mathcal{M}(\xi),\mathcal{M}(\eta))$ by $\tilde{\mathcal{M}}(F)(s) = \widetilde{\mathcal{M}(F)(s)}$. In particular if $f:\xi \longrightarrow \eta$ is a fiber bundle morphism then for each positive integer r we define a continuous map $\tilde{\mathcal{M}}(\delta^r f):\mathcal{M}(\xi) \longrightarrow L^r_s(\mathcal{M}(\xi),\mathcal{M}(\eta))$ which is given explicitly by

$$\tilde{\mathcal{m}}(\delta^r f)(s_o)(s_1,\ldots,s_r)(x) = \delta^r f(s_o(x))(s_1(x),\ldots,s_r(x))$$

11.3 Theorem. If ξ and η are vector bundles over a compact C^∞ manifold M and if $f:\xi \longrightarrow \eta$ is a fiber bundle morphism then $\mathcal{m}(f)\colon \mathcal{m}(\xi) \longrightarrow \mathcal{m}(\eta)$ is a C^∞ map and for each positive integer r

$$d^r\,\mathcal{m}(f) = \tilde{\mathcal{m}}(\delta^r f).$$

Proof. It suffices to consider the case $r=1$, for then we can proceed inductively as follows:

$$d^{r+1}\,\mathcal{m}(f) = d(d^r\,\mathcal{m}(f)) = d(\tilde{\mathcal{m}}(\delta^r f)).$$

But since $\sim\ \mathcal{m}(L^r(\xi,\eta)) \longrightarrow L^r(\mathcal{m}(\xi),\ \mathcal{m}(\eta))$ is a continuous linear map it commutes with d and we have:

$$d^{r+1}\,\mathcal{m}(f) = (d\,\mathcal{m}(\delta^r f))^{\sim} = (\tilde{\mathcal{m}}\,(\delta\,\delta^r f))^{\sim}$$

But making the identifications of $L^{r+1}(\xi,\eta)$ with $L(\xi,L^r(\xi,\eta))$ and of $L^{r+1}(\mathcal{m}(\xi),\ \mathcal{m}(\eta))$ with $L\,(\mathcal{m}(\xi),\ L^r(\mathcal{m}(\xi),\ \mathcal{m}(\eta)))$ entails the identification of $\tilde{\mathcal{m}}(\delta\,\delta^r f))^{\sim}$ with $\tilde{\mathcal{m}}(\delta^{r+1} f)$, which completes the inductive step. The case $r = 1$ is handled as follows. We claim first that it suffices to prove that for s_o and σ in $\mathcal{m}(\xi)$:

$$(*) \qquad \mathcal{m}(f)(s_o + \sigma) = \mathcal{m}(f)(s_o) - \tilde{\mathcal{m}}(\delta f)(s_o)(\sigma)$$

$$= \int_o^1 \left(\tilde{\mathcal{m}}(\delta f)(s_o + t\sigma) - \tilde{\mathcal{m}}(\delta f)(s_o)\right) dt\ (\sigma)$$

To see this choose norms for $\mathcal{m}(\xi)$ and $\mathcal{m}(\eta)$ and, given $\varepsilon > 0$, choose $\delta > 0$, by the continuity of $\mathcal{m}(\delta f)$, so that if $||\sigma|| < \delta$ then

$$||\tilde{\mathfrak{M}}(\delta f)(s_0 + t\sigma) - \tilde{\mathfrak{M}}(\delta f)(s_0)|| < \varepsilon \quad 0 \leq t \leq 1.$$

Then for $||\sigma|| < \delta$ it follows from (*) that $||\mathfrak{M}(f)(s_0 + \sigma) - \mathfrak{M}(f)(s_0) - \tilde{\mathfrak{M}}(\delta f)(s_0)(\sigma)|| \leq ||\sigma||\, \varepsilon$ and so by definition $\mathfrak{M}(f)$ is differentiable at s_0 and $d\,\mathfrak{M}(f)(s_0) = \tilde{\mathfrak{M}}(\delta f)(s_0)$.

To prove (*) note that since both sides of the equality are sections of ξ, in fact elements of $\mathfrak{M}(\xi)$, it must be shown that both sides have the same value for each $x \in M$. If we put $s_0(x) = p$ and $\sigma(x) = v$ then evaluating the left hand side gives $f(p+v) - f(p) - \delta f(p)$. On the other hand by the first part of (B§5) evaluation at x is a continuous linear map of $\mathfrak{M}(\xi)$ into ξ_x and hence passes under the integral sign on the right of (*), which becomes

$$\int_0^1 (\delta f(p+tv) - \delta f(p))(v)dt \quad .$$

Finally, since by definition $\delta f(p+tv) = d(f|\xi_x)(p+tv)$, the equality of $f(p+v) - f(p) - \delta f(p)$ and $\int_0^1 (\delta f(p+tv) - \delta f(p))(v)dt$ is just one of the versions of the Mean Value Theorem.

q.e.d.

<u>11.4</u> <u>Corollary</u>. If $s \in C^\infty(\xi)$ then $d\,\mathfrak{M}(f)(s) = \mathfrak{M}(\delta_s f)$ where $\delta_s f : \xi \longrightarrow \eta$ is a vector bundle homomorphism, called the vertical differential of f along s, defined by $\delta_s f = \delta f \circ s$.

Proof. For $\sigma \in \mathfrak{M}(\xi)$ and $x \in M$ $\mathfrak{M}(\delta_s f)(\sigma)(x) = \delta_s f(x)(\sigma(x)) = \delta f(s(x))(\sigma(x)) =$ $\widetilde{\mathfrak{M}(\delta f)(s)}(\sigma)(x) = \tilde{\mathfrak{M}}(\delta f)(s)(\sigma) = d\,\mathfrak{M}(f)(s)(\sigma)$.

q.e.d.

12. VECTOR BUNDLE NEIGHBORHOODS

If $\pi: E \longrightarrow M$ is a C^∞ fiber bundle over M then by local triviality it follows that π is a submersion and hence that $\ker(d\pi)$ is a vector sub-bundle of $T(E)$, whose fiber at $e \in E_x$ is clearly $T(E_x)_e$. This vector bundle over E is denoted by $TF(E)$ and called the tangent bundle along the fiber of E. Because of the conventional picture of a fiber bundle it is also often referred to as the vertical tangent bundle (or bundle of vertical vectors) of E. If $s \in C^\infty(E)$ then $s^*TF(E)$ is a vector bundle over M denoted by $T_s(E)$. Note that if we identify M with its embedded image under s then $T_s(E) = TF(E)|M$. If ξ is a vector bundle over M then for each $e \in \xi_x$ we have a canonical identification of $TF(\xi)_e = T(\xi_x)_e$ with ξ_x hence, if $s \in C^\infty(\xi)$, a canonical identification of $T_s(\xi)_x = TF(\xi)_{s(x)}$ with ξ_x, and hence

12.1 Theorem. If ξ is a vector bundle over M then for each $s \in C^\infty(\xi)$ there is a canonical isomorphism of $T_s(\xi)$ with ξ.

If E_1 and E_2 are C^∞ fiber bundles over M and $f: E_1 \longrightarrow E_2$ is a fiber bundle morphism then since f is fiber preserving it follows that the vector bundle homomorphism $df: T(E_1) \longrightarrow f^*T(E_2)$ maps $TF(E_1)$ into $f^*TF(E_2)$ and we denote by $\delta f: TF(E_1) \longrightarrow f^*TF(E_2)$ the corresponding restriction. We call δf the "differential of f along the fiber" (or the "vertical differential of f"). Given $s \in C^\infty(E_1)$ we define $\delta_s f = \delta f \circ s$. Then $\delta_s f: T_s(E_1) \longrightarrow T_{f\circ s}(E_2)$ is a vector bundle homomorphism called the vertical differential of f along s.

12.2 Definition. If E_1 and E_2 are C^∞ fiber bundles over M we say that E_1 is a sub-bundle of E_2 if $E_1 \subseteq E_2$ and if the inclusion map $E_1 \longrightarrow E_2$ is a fiber bundle morphism. If in addition E_1 is open in E_2 we call E_1 an open sub-bundle of E_2.

Remark. If E_1 is an open sub-bundle of E_2 then clearly $TF(E_1) = TF(E_2)|E_1$ hence if $s \in C^\infty(E_1)$ then

$$T_s(E_1) = s^*TF(E_1) = s^*TF(E_2) = T_s(E_2).$$

12.3 Definition. If E is a C^∞ fiber bundle over M and $s \in C^0(E)$ then a vector bundle neighborhood (abbreviated VBN) of s in E is a vector bundle ξ over M such that ξ is an open sub-bundle of E and $s \in C^0(\xi)$.

12.4 Lemma. If E is a C^∞ fiber bundle, $s \in C^\infty(E)$, and ξ is a VBN of s in E then $\xi \approx T_s(E)$.

Proof. Theorem 12.1 and the remark following Definition 12.2.

12.5 Uniqueness Theorem for VBN. If E is a C^∞ fiber bundle, $\sigma \in C^0(E)$, and if ξ_1 and ξ_2 are both VBN of σ then $\xi_1 \approx \xi_2$.

Proof. By standard approximation theory we can find $s \in C^\infty(\xi_1) \cap C^\infty(\xi_2)$ and by Lemma 12.4 we have $\xi_i \approx T_s(E)$. q.e.d.

The remainder of this section will be devoted to the proof of an existence theorem for VBN. The proof is closely analagous to the construction of tubular neighborhoods of submanifolds which is based on the notion of a spray (see Lang's Introduction to Differentiable Manifolds, Chapter IV). We begin with the notion of a "bundle spray".

First recall that if W is a C^∞ manifold then a "second order differential equation in W" is defined to be a vector field X on $T(W)$ such that for each $v \in T(W)$ we have $dp(X(v)) = v$ where $p\colon T(W) \longrightarrow W$ is the natural projection (cf. Lang, loc. cit. page 68). Given $\alpha \in \mathbb{R}$ let $\tilde{\alpha}\colon T(W) \longrightarrow T(W)$ denote the map $v \longmapsto \alpha v$. If X is a second order differential equation in W then for each $v \in T(W)$ $d\tilde{\alpha}\,(\alpha X(v))$ is a tangent vector to $T(W)$ at αv. If for all $v \in T(W)$ we have

$$X(\alpha v) = d\tilde{\alpha}\,(\alpha X(v))$$

then X is called a spray over W (Lang, loc. cit. p. 69).

Now one intuitive way of thinking about a C^∞ fiber bundle $\pi\colon E \longrightarrow M$ is as a family of C^∞ manifolds $\{E_m\}_{m\varepsilon M}$, "smoothly parameterized by M". The corresponding parameterized family of tangent bundles $\{T(E_m)\}_{m\varepsilon M}$ is represented by $TF(E) = \bigcup_{m\varepsilon M} T(E_m)$ (which "explains" its importance). Similarly the parameterized family of second tangent bundles $\{T(T(E_m))\}_{m\varepsilon M}$ is represented by the sub-bundle of $T(TF(E))$ given by

$$dp^{-1}(TF(E)) = \{x \;\varepsilon\; T(TF(E)) \mid dp(x) \;\varepsilon\; TF(E)\}$$

where $p\colon TF(E) \longrightarrow E$ is the natural projection; indeed since $T(E_m) = p^{-1}(E_m)$ and p is a submersion $T(T(E_m)) = dp^{-1}(T(E_m))$.

The notion of a "bundle spray" over a C^∞ fiber bundle E is supposed to capture the intuitive notion of a smoothly parameterized family of sprays (one over each fiber). In view of the above the proper definition is clear.

<u>12.6</u> <u>Definition</u>. Let $\pi\colon E \longrightarrow M$ be a C^∞ fiber bundle and $p\colon TF(E) \longrightarrow E$ its tangent bundle along the fiber. A C^∞ vector field X on $TF(E)$ is called a <u>bundle second order differential equation in</u> E if

1) $dp(X(v)) = v$

for all $v \;\varepsilon\; TF(E)$. If in addition for all $v \;\varepsilon\; TF(E)$ and all $\alpha \;\varepsilon\; \mathbb{R}$ we have

2) $X(\alpha v) = d\tilde{\alpha}(\alpha X(v))$

(where $\tilde{\alpha}\colon TF(E) \longrightarrow TF(E)$ is the map $v \longmapsto \alpha v$) then we call X a <u>bundle spray over</u> E.

<u>Remark</u>. If X is a bundle second order differential equation for E then for each $m \;\varepsilon\; M$ it follows that $X|T(E_m)$ is a second order differential equation in E_m. The point is that while X being a vector field on $TF(E)$ only implies for $v \;\varepsilon\; T(E_m)$ that $X(v) \;\varepsilon\; T(TF(E))_v$, which is bigger than $T(T(E_m))_v$, condition 1) gives that $dp(X(v)) \;\varepsilon\; T(E_m)$ which by the remark preceding the definition gives $X(v) \;\varepsilon\; T(T(E_m))$. In particular it follows that if $\sigma\colon (a,b) \longrightarrow TF(E)$ is a solution curve of a second order differential equation in E and $\sigma(t_o) \;\varepsilon\; T(E_m)$ then $\sigma((a,b)) \subseteq T(E_m)$ and so $p\sigma((a,b)) \subseteq E_m$.

Secondly we note that both conditions 1) and 2) are convex. Also since if E has paracompact fiber there is a spray over each fiber (Lang, loc. cit. p. 70) and hence a bundle spray over each product neighborhood, it follows from a partition of unity argument that

<u>12.7 Theorem</u>. If $\pi: E \longrightarrow M$ is a C^∞ fiber bundle with paracompact base and paracompact fiber then there exists a bundle spray over E.

Now let X be a bundle spray over E and define an open subset D of $TF(E)$ and a map $Exp_X: D \longrightarrow E$ as follows: for each $v \in TF(E)$ let σ_v be the maximum solution curve of X such that $\sigma_v(0) = v$; let $D = \{v \in TF(E) | \sigma_v(1)$ is defined$\}$ and let $Exp_X(v) = \sigma_v(1)$ for $v \in D$. Exp is called the exponential map associated with X.

<u>12.8 Lemma</u>. Let $E = M \times F$ be a trivial C^∞ fiber bundle whose fiber F is a finite dimensional normed vector space and let Exp be the exponential map of a bundle spray over E. Given any $e_0 \in E$ there is a neighborhood $\mathcal{O}(e_0)$ of e_0 in E and an $r > 0$ such that for each $e \in \mathcal{O}(e_0)$ Exp maps the ball of radius r about zero in F (F being identified with $TF(E)_e$) C^∞ isomorphically onto a neighborhood of e in F (identified with the fiber containing e) which includes the ball of radius $r/2$ about e.

<u>Proof</u>. For each $e \in E$ identify $TF(E)_e$ with F in the canonical way. Then $Exp|TF(E)_e$ becomes a map f^e of a neighborhood of 0 in F, into F. As in Lang, loc. cit. page 72 we have $f^e(0) = e$ and the differential of f^e at zero is the identity map of F. Now we define $g^e(x) = x+e - f^e(x)$, then $g^e(0) = 0$ and the differential of g^e at zero is zero. By continuity there is a neighborhood $\mathcal{O}(e_0)$ of e_0 in E, and an $r > 0$ such that if $e \in \mathcal{O}(e_0)$ and $||x|| < 2r$ then the norm of dg^e at x is less than $1/2$. The lemma now follows from the proof of the implicit function theorem (Lang, loc. cit. p. 12). q.e.d.

<u>12.9 Lemma</u>. Let E be a paracompact C^∞ fiber bundle over a C^∞ manifold M and let Exp be the exponential map of a bundle spray over E. Choose a Riemannian structure for $TF(E)$ and let ρ_x denote the corresponding Riemannian metric in the fiber E_x. Then there are strictly positive C^∞

functions λ and μ on E such that if $e \in E_x$ then Exp maps the open disc of radius $\lambda(e)$ in $T(E_x)_e$ C^∞ isomorphically onto a neighborhood U of e in E_x which contains all $e' \in E_x$ with $\rho_x(e,e') < \mu(e)$.

<u>Proof</u>. Given $e_o \in E$ it follows from Lemma 12.8 that there is a neighborhood $\mathcal{O}(e_o)$ of e_o in E and positive numbers $\varepsilon(e_o)$ and $\delta(e_o)$ such that if $e \in \mathcal{O}(e_o) \cap E_x$ then Exp maps the disc of radius $\delta(e_o)$ about the origin in $T(E_x)_e$ C^∞ isomorphically onto a neighborhood of e in E_x which contains all $e' \in E_x$ with $\rho_x(e,e') < \varepsilon(e_o)$. (Note that since this is a local statement we can assume that E is trivial and the fiber a vector space). Let $\{V_\beta\}_{\beta\in B}$ be a locally finite cover of E by relatively compact open sets which refines $\{\mathcal{O}(e)\}_{e\in E}$, and choose $e(\beta)$ so that $V_\beta \subset \mathcal{O}(e(\beta))$. Let $\{\varphi_\beta\}_{\beta\in B}$ be a C^∞ partition of unity with support $\varphi_\beta \subseteq V_\beta$. Put $\lambda = \sum_\beta \delta(e(\beta))\varphi_\beta$ and $\mu = \sum_\beta \gamma(\beta)\varphi(\beta)$ where $\gamma(\beta) = \min\{\varepsilon(e(\beta'))\,|\,V_\beta \cap V_{\beta'} \neq \emptyset\}$.

Now given $e \in E_x$, Exp maps the disc of radius $\delta(e(\beta))$ about the origin in $T(E_x)_e$ C^∞ isomorphically onto a neighborhood of e in E_x which contains all $e' \in E_x$ with $\rho(e,e') < \varepsilon(e(\beta))$, provided $e \in V_\beta$. Since
$\min\{\delta(\varepsilon(\beta))\,|\,e \in V_\beta\} \leq \max\{\delta(e(\beta))\,|\,e \in V_\beta\}$ and $\mu(e) \leq \min\{\varepsilon(e(\beta))\,|\,e \in V_\beta\}$
the lemma follows. q.e.d.

<u>12.10 Existence Theorem for VBN</u>. Let E be a C^∞ fiber bundle with paracompact fiber over a paracompact C^∞ manifold M and let $g \in C^0(E)$. Given a neighborhood $\mathcal{O}$ of $g(M)$ in E there is a VBN ξ of g in E with $\xi \subseteq \mathcal{O}$. Moreover if $g \in C^\infty(E)$ we can choose ξ so that g is the zero section of ξ.

<u>Proof</u>. Choose a Riemannian structure for $TF(E)$ and a bundle spray over E and let Exp, λ, μ, and ρ_x be as in Lemma 12.9. By standard approximation theory we can find $f \in C^\infty(E)$ which is so close to g that on the one hand $\mu(f(x)) > \frac{1}{2}\mu(g(x))$ while on the other hand $\rho_x(f(x),g(x)) < \frac{1}{2}\mu(g(x))$. If g is C^∞ we take $f = g$. Then clearly $\rho_x(f(x),g(x)) < \mu(f(x))$. Now let $\Lambda = \{v \in T_f(E)\,|\,\|v\| < \lambda(f(p(v))\}$ where $p\colon T_f(E) \longrightarrow M$ is the bundle projection. Let $\varphi\colon [0,\infty) \longrightarrow [0,1)$ be a C^∞ diffeomorphism such that $\varphi(t) = t$ for t near zero, and

define a diffeomorphism $\tilde{\varphi} : T_f(E) \approx \Lambda$ by $\tilde{\varphi}(v) = \lambda(f(p(v)) \frac{\varphi(||v||)}{||v||} v$. Then $\psi = \mathrm{Exp} \circ \varphi$ is a C^∞ fiber preserving isomorphism of $T_f(E)$ onto an open subset ξ of E which we make into an open vector sub-bundle of E by demanding that ψ shall be a vector bundle isomorphism (note that f is the zero section of ξ). If $x \varepsilon M$ then since $\rho_x(f(x), g(x)) < \mu(f(x))$ there is a $v \varepsilon T(E_x)_{f(x)} = T_f(E)_x$ with $||v|| < \lambda(f(x))$ (so $v \varepsilon \Lambda$) such that $\mathrm{Exp}(v) = g(x)$ and hence $g(x) = \psi(\tilde{\varphi}^{-1}(v)) \varepsilon \xi$. Thus $g \varepsilon C^0(\xi)$ and ξ is a VBN of g in E. Finally, we note that we can replace μ by any strictly positive smaller C^∞ function so that in particular we can assume that if $e \varepsilon E_x$ and $\rho_x(g(x),e) < 2\mu(g(x))$ then $e \varepsilon \mathcal{O}$. It then follows that $\xi \subseteq \mathcal{O}$. q.e.d.

<u>12.11 Definition</u>. Let E_1 be a C^∞ fiber bundle over M and let E_2 be a closed C^∞ sub-bundle of E_1. By a <u>bundle tubular neighborhood</u> of E_2 in E_1 we mean a C^∞ vector bundle $r:\mathcal{O} \longrightarrow E_2$ over E_2 such that $\mathcal{O}$ is an open subset of E_1, and in fact an open sub-bundle of E_1, and r is a C^∞ fiber bundle morphism over M.

<u>12.12</u>. <u>Existence Theorem for Bundle Tubular Neighborhoods</u>.
Let E_1 be a paracompact C^∞ fiber bundle over M and let E_2 be a closed C^∞ sub-bundle of E_1. Then E_2 has a bundle tubular neighborhood in E_1.

<u>Proof.</u> Choose a Riemannian structure for $TF(E_1)$ and let Exp be the exponential map of a bundle spray over E_1. Let ν denote the normal bundle to $TF(E_2)$ in $TF(E_1)|E_2$. Then there is a C^∞ strictly positive function λ on E_2 such that Exp maps the disc of radius $\lambda(e)$ in ν_e C^∞ isomorphically onto a neighborhood of e in E_1. As in the proof of the existence of tubular neighborhoods in Lang, loc. cit., p. 74, it then follows that there is a neighborhood U of the zero section of ν such that Exp maps U diffeomorphically onto a neighborhood of E_2 in E_1. Replacing λ by a possibly smaller strictly positive C^∞ function we can suppose that $\Lambda = \{v \varepsilon \nu_e | \; ||v|| < \lambda(e)\}$ is included in U. Then defining $\tilde{\varphi}: \nu \longrightarrow U$ as in the proof of Theorem 12.10, $\mathrm{Exp} \circ \tilde{\varphi}$ defines a diffeomorphism of ν onto a neighborhood $\mathcal{O}$ of E_2 in E_1, and carrying the vector bundle structure of ν over to $\mathcal{O}$ via this diffeomorphism it is clear that $\mathcal{O}$ is a bundle tubular neighborhood of E_2 in E_1. q.e.d.

<u>Remark</u>. We can think of a cross section of a C^∞ fiber bundle E (or rather, its image) as being a sub-bundle whose fiber is a point. In this case the notion of bundle tubular neighborhood and VBN coincide.

13. THE DIFFERENTIABLE STRUCTURE FOR $\mathcal{M}(E)$

In this section we again assume that M is a compact n-dimensional manifold and that $\mathcal{M}$ satisfies condition (B§2) of §4 and (B§5) of §11 (we recall that (B§5) implies conditions (B§1), (B§3), and (B§4) of §4).

13.1. Definition. If E is a C^∞ fiber bundle over M then we define $\mathcal{M}(E)$ to be the set of all sections s of E such that $s \in \mathcal{M}(\xi)$ for some open vector sub-bundle ξ of E, i.e. $\mathcal{M}(E) = \bigcup_{\xi} \mathcal{M}(\xi)$ where the union is over all open vector sub-bundles ξ of E.

Remark. From (B§5) we have $C^\infty(E) \subseteq \mathcal{M}(E) \subseteq C^0(E)$.

13.2 Lemma. Let E be a C^∞ fiber bundle over M, $\sigma \in C^0(E)$ and let ξ_1 and ξ_2 be VBN of σ in E. Then there is a C^∞ fiber bundle morphism $f: \xi_1 \longrightarrow \xi_2$ which is the identity in a neighborhood of $\sigma(M)$.

Proof. Choose a Riemannian structure for ξ_1 and an $r > 0$ such that for each $x \in M$ the disc of radius $3r$ about $\sigma(x)$ in $(\xi_1)_x$ is included in $(\xi_2)_x$. Let $\gamma \in C^\infty(\xi_1)$ such that $||\gamma(x) - \sigma(x)|| < r$ for all $x \in M$ and let $\varphi: [0,\infty) \longrightarrow [0,2r)$ be a C^∞ map such that $\varphi(t) = t$ for $0 \leq t \leq \frac{3}{2}r$. Define $f: \xi_1 \longrightarrow \xi_2$ by the condition that on $(\xi_1)_x$

$$f(v) = \gamma(x) + \frac{\varphi(||v - \gamma(x)||)}{||v - \gamma(x)||} (v - \gamma(x)) .$$

Then $||f(v) - \gamma(x)|| = \varphi(||v - \gamma(x)||) < 2r$ so $||f(v) - \sigma(x)|| < 2r + ||\sigma(x) - \gamma(x)|| < 3r$; so $f(v) \in (\xi_2)_x$. Also if $||v - \sigma(x)|| < \frac{1}{2} r$ then $||v - \gamma(x)|| < \frac{1}{2} r + ||\gamma(x) - \sigma(x)|| < \frac{3}{2} r$, so $\varphi(||v-\gamma(x)||) = ||v-\gamma(x)||$ and $f(v) = v$.

q.e.d.

13.3 Theorem. If E is a C^∞ fiber bundle over M and $\sigma \in \mathcal{M}(E)$ then for each VBN of σ in E, say ξ, we have $\sigma \in \mathcal{M}(\xi)$.

<u>Proof</u>. Let ξ_1 be a VBN of σ in E such that $\sigma \in \mathcal{M}(\xi_1)$. Taking $\xi = \xi_2$ in Lemma 13.1 we have a C^∞ fiber bundle morphism $f\colon \xi_1 \longrightarrow \xi$ which is the identity in a neighborhood of $\sigma(M)$, and in particular $f\circ\sigma = \sigma$. But by (B§5) $f\circ\sigma = \mathcal{M}(f)(\sigma) \in \mathcal{M}(\xi)$. q.e.d.

<u>13.4 Theorem</u>. Let E_1 and E_2 be C^∞ fiber bundles over M and let $f\colon E_1 \longrightarrow E_2$ be a C^∞ fiber bundle morphism. Then $f_*\colon C^0(E_1) \longrightarrow C^0(E_2)$ restricts to a function $\mathcal{M}(f)\colon \mathcal{M}(E_1) \longrightarrow \mathcal{M}(E_2)$. Moreover if ξ_1 and ξ_2 are open vector sub-bundles of E_1 and E_2 respectively and $\mathcal{O} = \{\sigma \in \mathcal{M}(\xi_1) | \mathcal{M}(f)(\sigma) \in \mathcal{M}(\xi_2)\}$ then $\mathcal{O}$ is open in $\mathcal{M}(\xi_1)$ and $\mathcal{M}(f)$ maps $\mathcal{O}$ C^∞ into $\mathcal{M}(\xi_2)$.

<u>Proof</u>. Let $\sigma \in \mathcal{M}(E_1)$ and let $\gamma = f_*(\sigma) \in C^0(E_2)$. Let ξ_2 be a VBN of γ in E_2 and let ξ_1 be a VBN of σ in E_1 such that $\xi_1 \subseteq f^{-1}(\xi_2)$ (see Theorem 12.10). Then $g = f|\xi_1\colon\xi_1 \longrightarrow \xi_2$ is a C^∞ fiber bundle morphism. By 13.3 $\sigma \in \mathcal{M}(\xi_1)$ so by (B§5) $g_*(\sigma) \in \mathcal{M}(\xi_2)$ and so by definition $g_*(\sigma) \in \mathcal{M}(E_2)$. But clearly $f_*(\sigma) = g_*(\sigma)$ so $f_*(\sigma) \in \mathcal{M}(E_2)$.

If now ξ_1, ξ_2 and $\mathcal{O}$ are as in the statement of the theorem, then since $\xi_1 \cap f^{-1}(\xi_2)$ is open in ξ_1 it follows from the fact that the topology of $\mathcal{M}(\xi_1)$ is stronger than the compact-open topology (by (B§5)) that $\mathcal{O}$ is open in $\mathcal{M}(\xi_1)$. Let $\sigma \in \mathcal{O}$ and by Theorem 12.10 choose η a VBN of σ In E_1 with $\eta \subseteq \xi_1 \cap f^{-1}(\xi_2)$. By Lemma 13.2 there is a C^∞ fiber bundle morphism $g\colon\xi_1 \longrightarrow \eta$ which is the identity in a neighborhood U of $\sigma(M)$. Then $f\circ g\colon\xi_1 \longrightarrow \xi_2$ is a C^∞ fiber bundle morphism so by Theorem 11.3 $\mathcal{M}(f\circ g)\colon \mathcal{M}(\xi_1) \longrightarrow \mathcal{M}(\xi_2)$ is a C^∞ map. But $\tilde{U} = \{\gamma \in \mathcal{M}(\xi_1) | \gamma(M) \subseteq U\}$ is a neighborhood of σ in $\mathcal{M}(\xi_1)$ and clearly $\mathcal{M}(f)$ and $\mathcal{M}(f\circ g)$ agree in $\tilde{U}$. Hence $\mathcal{M}(f)$ is C^∞ near σ in $\mathcal{O}$, and since σ was arbitrary, $\mathcal{M}(f)|\mathcal{O}$ is a C^∞ map of $\mathcal{O}$ into $\mathcal{M}(\xi_2)$. q.e.d.

We now come to one of our main theorems which in fact is little more than a rewording of Theorem 13.4.

<u>13.5 Theorem</u>. Let $\mathcal{M}$ satisfy (B§2) and (B§5) and let M be a compact n-dimensional manifold. Then for each C^∞ fiber bundle E over M there is a unique C^∞

differentiable structure for $\mathcal{M}(E)$ such that for each open vector sub-bundle ξ of E $\mathcal{M}(\xi)$ is an open submanifold of $\mathcal{M}(E)$. If $f:E_1 \longrightarrow E_2$ is a C^∞ fiber bundle morphism of C^∞ fiber bundles over M then $\mathcal{M}(f):\mathcal{M}(E_1) \longrightarrow \mathcal{M}(E_2)$ is a C^∞ map with respect to the above differentiable structures for $\mathcal{M}(E_1)$ and $\mathcal{M}(E_2)$. In other words $\mathcal{M}$ extends to a functor from the category FB(M) to the category of C^∞ Banach manifolds and C^∞ maps.

Proof. In Theorem 13.4 if we take $E_1 = E_2 = E$ and let f be the identity map, then the second conclusion of the theorem states that the charts $\mathcal{M}(\xi_1)$ and $\mathcal{M}(\xi_2)$ for $\mathcal{M}(E)$ corresponding to two open vector sub-bundles are C^∞ related. This proves the existence of the required C^∞ structure for $\mathcal{M}(E)$ and uniqueness is of course trivial. The remainder of the theorem now becomes just a restatement of Theorem 13.4. q.e.d.

Henceforth, when E is a C^∞ fiber bundle over M, $\mathcal{M}(E)$ will denote the C^∞ Banach manifold whose differentiable structure is given by Theorem 13.5. We note that by (B§5) the inclusion $\mathcal{M}(E) \longrightarrow C^0(E)$ is continuous.

13.6 Theorem. Let E_1 be a C^∞ fiber bundle over M. If $s \in C^\infty(E_1)$ then $T(\mathcal{M}(E_1))_s$, tangent space to $\mathcal{M}(E_1)$ at s, can be identified canonically with $\mathcal{M}(T_s(E_1))$. Moreover if $f: E_1 \longrightarrow E_2$ is a C^∞ fiber bundle morphism then the differential of $\mathcal{M}(f):\mathcal{M}(E_1) \longrightarrow \mathcal{M}(E_2)$ at s, when regarded via the above canonical identification as a linear map of $\mathcal{M}(T_s(E_1))$ into $\mathcal{M}(T_{f\circ s}(E_2))$ is given by

$$d(\mathcal{M}(f))_s = \mathcal{M}(\delta_s f)$$

(where $\delta_s f = \delta f \circ s: T_s(E_1) \longrightarrow T_{f\circ s}(E_2)$ is defined following Theorem 12.1).

Proof. By Theorem 12.10 we can find a VBN ξ of s such that s is the zero section of ξ. For any such ξ we have a canonical identification of ξ_x with $T(\xi_x)_{s(x)}$, and since

ξ_x is open in $(E_1)_x$ we have $T(\xi_x)_{s(x)} = T((E_1)_x)_{s(x)} = TF(E_1)_{s(x)}$. (We have essentially repeated the proof of Theorem 12.5). Thus ξ is canonically identified with $T_s(E_1)$ and since by definition of the differentiable structure on $\mathcal{M}(E_1)$, $\mathcal{M}(\xi)$ is a chart about s, we have a canonical identification of $T(\mathcal{M}(E_1))_s$ with $\mathcal{M}(\xi)$. The remainder of the theorem now follows from Corollary 11.4. q.e.d.

Recall that if E_1 and E_2 are C^∞ fiber bundles over M then their fiber product $E_1 \times_M E_2$ is their product in the category $FB(M)$. Concretely it is $p_1^* E_2$ or equivalently $p_2^* E_1$ where $p_i\colon E_i \longrightarrow M$ are the bundle projections. There is a natural map of $C^0(E_1) \times C^0(E_2)$ into $C^0(E_1 \times_M E_2)$ given by $(\sigma_1,\sigma_2) \longrightarrow \sigma_1 \times \sigma_2$ where $(\sigma_1 \times \sigma_2)(x) = (\sigma_1(x),\sigma_2(x))$. If E_1 and E_2 are vector bundles then of course $E_1 \times_M E_2 = E_1 \oplus E_2$. The next result then follows directly from Theorem 4.2 and the definition of the differentiable structure on $\mathcal{M}(E)$.

<u>13.7 Theorem</u>. $\mathcal{M}$ is a multiplicative functor from $FB(M)$ to C^∞ Banach manifolds. That is if $p_i\colon E_i \longrightarrow M$ $(i = 1,2)$ are C^∞ fiber bundles over M then $(\varphi,\psi) \longmapsto \varphi \times \psi$ is a C^∞ isomorphism of $\mathcal{M}(E_1) \times \mathcal{M}(E_2)$ with $\mathcal{M}(E_1 \times_M E_2)$, and if $\pi_i\colon E_1 \times_M E_2 \longrightarrow E_i$ are the natural projections (C^∞ fiber bundle morphisms over M) then we have commutativity in the diagrams:

$$\begin{tikzcd}
\mathcal{M}(E_1) \times \mathcal{M}(E_2) \arrow[rr] \arrow[dr] & & \mathcal{M}(E_1 \times_M E_2) \arrow[dl, "\mathcal{M}(\pi_i)"] \\
& \mathcal{M}(E_i) &
\end{tikzcd}$$

<u>13.8 Lemma</u>. Let X be a finite dimensional C^∞ manifold, $E = M \times X$ the product bundle over M and for each $x \in X$ let $\bar{x}$ denote the constant section, $m \longmapsto x$ of E. Then $x \longmapsto \bar{x}$ is a C^∞ embedding of X in $\mathcal{M}(E)$.

Proof. If V is a neighborhood of $x_o \varepsilon X$ with a vector space structure, then $\xi = M \times V$ is a VBN of $\bar{x}_o$ in E, so it suffices to prove that $x \longmapsto \bar{x}$ is C^∞ embedding of V in $\mathcal{M}(\xi)$. But this map is clearly linear, and since V is finite dimensional it is a continuous linear map and hence C^∞.

q.e.d.

13.9 Theorem. Let E_1 and E_2 be C^∞ fiber bundles over M and let X be a C^∞ manifold. Let $f: E_1 \times X \longrightarrow E_2$ be a C^∞ map such that for each $x \varepsilon X$ the map $f_x: E_1 \longrightarrow E_2$ given by $f_x(e) = f(e,x)$ is a fiber bundle morphism. Then $(x,\sigma) \longmapsto \mathcal{M}(f_x)(\sigma)$ is a C^∞ map of $X \times \mathcal{M}(E_1)$ into $\mathcal{M}(E_2)$.

Proof. Put $E_3 = M \times X$. Then $E_1 \times_M E_3 = E_1 \times X$. Clearly $f: E_1 \times_M E_3 \longrightarrow E_2$ is a C^∞ fiber bundle morphism so $\mathcal{M}(f): \mathcal{M}(E_1 \times_M E_3) \longrightarrow \mathcal{M}(E_2)$ is C^∞. By 13.8 $x \longmapsto \bar{x}$ is a C^∞ map of X into $\mathcal{M}(E_3)$ so by 13.7 $(x,\sigma) \longmapsto \sigma \times \bar{x}$ is a C^∞ map of $X \times \mathcal{M}(E_1)$ into $\mathcal{M}(E_1 \times_M E_3)$ and so $(x,\sigma) \longmapsto \mathcal{M}(f)(\sigma \times \bar{x})$ is a C^∞ map of $X \times \mathcal{M}(E_1)$ into $\mathcal{M}(E_2)$. But $\mathcal{M}(f)(\sigma \times \bar{x})(m) = f(\sigma(m),\bar{x}(m)) = f(\sigma(m),x) = f_x(\sigma(m)) = \mathcal{M}(f_x)\sigma(m)$, i.e. $\mathcal{M}(f)(\sigma \times \bar{x}) = \mathcal{M}(f_x)(\sigma)$.

q.e.d.

13.10 Definition. Let E_1 and E_2 be C^∞ fiber bundles over M. We say that E_2 is a C^∞ bundle deformation retract of E_1 if there is a C^∞ deformation retraction $f: E_1 \times I \longrightarrow E_1$ of E_1 onto E_2 such that for each $t \varepsilon I$ the t-th stage $f_t: E_1 \longrightarrow E_1$ of the deformation is a bundle map. We note that this is the case if E_2 is a closed sub-bundle of a C^∞ bundle E over M and E_1 is a bundle tubular neighborhood of E_2 in E.

13.11 Theorem. If E_1 and E_2 are C^∞ fiber bundles over M and E_2 is a C^∞ bundle deformation retract of E_1 then $\mathcal{M}(E_2)$ is a C^∞ deformation retract of $\mathcal{M}(E_1)$ and in particular the includion $\mathcal{M}(E_2) \longrightarrow \mathcal{M}(E_1)$ is a homotopy equivalence. Hence if E_2 is a closed sub-bundle of a C^∞ fiber bundle E over M and E_1 is a bundle tubular neighborhood of E_2 in E then the inclusion $\mathcal{M}(E_2) \longrightarrow \mathcal{M}(E_1)$ is a homotopy equivalence.

Proof. If $f_t\colon E_1 \longrightarrow E_1$, $t \in I$ is a C^∞ bundle deformation retraction of E_1 onto E_2 then, by 13.9, $\mathcal{M}(f_t)\colon \mathcal{M}(E_1) \longrightarrow \mathcal{M}(E_1)$ is a C^∞ deformation retraction of $\mathcal{M}(E_1)$ on $\mathcal{M}(E_2)$. q.e.d.

In many applications we have homotopy type information about $C^0(E)$ and we would like to derive similar information about $\mathcal{M}(E)$, or vice versa. The next result we are after takes care of all such situations since it shows that the inclusion of $\mathcal{M}(E)$ in $C^0(E)$ is always a homotopy equivalence. First we state as a lemma Theorem 15 of "Homotopy theory of infinite dimensional manifolds" (Topology, Vol. 5 pp. 1-16 (1966)).

<u>13.12. Lemma</u>. Let V_1 and V_2 be metrizeable locally convex topological vector spaces and let $f\colon V_1 \longrightarrow V_2$ be a continuous linear map of V_1 onto a dense linear subspace of V_2. Given $\mathcal{O}$ open in V_2 let $\tilde{\mathcal{O}} = f^{-1}(\mathcal{O})$ and let $\tilde{f} = f|\tilde{\mathcal{O}}$ Then $\tilde{f}\colon \tilde{\mathcal{O}} \longrightarrow \mathcal{O}$ is a homotopy equivalence.

<u>13.13. Lemma</u>. If E is a C^∞ fiber bundle over M with paracompact fibers then we can imbed E as a closed C^∞ sub-bundle of a C^∞ vector bundle ξ over M.

<u>Proof</u>. By Whitney's embedding theorem we can find a C^∞ embedding $g\colon E \longrightarrow \mathbb{R}^n$ of the total space of E into a Euclidean space such that g is a proper map. Let $\xi = M \times \mathbb{R}^n$ and define a fiber bundle isomorphism $f\colon E \longrightarrow \xi$ by $f(e) = (\pi(e), g(e))$ where $\pi\colon E \longrightarrow M$ is the bundle projection. Clearly f is proper so $f(E)$ is closed in ξ.

q.e.d.

<u>13.14. Theorem on the Invariance of Homotopy Type</u>. If E is a C^∞ fiber bundle over M then the inclusion map $\mathcal{M}(E) \longrightarrow C^0(E)$ is a homotopy equivalence.

<u>Proof</u>. By 13.13 we can suppose that E is a closed sub-bundle of a C^∞ vector bundle ξ over M, and by Theorem 12.12 we can find a bundle tubular neighborhood U of E in ξ. Since

U is an open sub-bundle of ξ, $\mathfrak{M}(U)$ is open in the Banach space $\mathfrak{M}(\xi)$. The inclusion map $i: \mathfrak{M}(\xi) \longrightarrow C^0(\xi)$ is a continuous linear map whose image contains $C^\infty(\xi)$, and hence is dense in $C^0(\xi)$. Clearly $i^{-1}(C^0(U)) = \mathfrak{M}(U)$ so by 13.12 $\tilde{i}: \mathfrak{M}(U) \longrightarrow C^0(U)$ is a homotopy equivalence. We have a commutative diagram

$$\begin{array}{ccc} \mathfrak{M}(U) & \longrightarrow & C^0(U) \\ {\scriptstyle \mathfrak{M}(j)} \uparrow & & \uparrow {\scriptstyle C^0(j)} \\ \mathfrak{M}(E) & \longrightarrow & C^0(E) \end{array}$$

where $j: E \longrightarrow U$ is inclusion and the bottom arrow is likewise an inclusion. By 13.11 the vertical arrows are homotopy equivalences and since the top arrow is too, so also is the bottom arrow. q.e.d.

Remark. Essentially the same argument shows that the inclusion $C^\infty(E) \longrightarrow C^0(E)$ is also a homotopy equivalence.

14. SOME SPECIAL TYPES OF MORPHISMS

Our standing assumptions remain those mentioned at the beginning of §13.

If $f:E_1 \longrightarrow E_2$ is a C^∞ fiber bundle morphism over M then certain properties of f are reflected in corresponding properties of $\mathcal{M}(f): \mathcal{M}(E_1) \longrightarrow \mathcal{M}(E_2)$. We shall consider some specific instances of this in the present section.

If ξ and η are C^∞ vector bundles over M then recall there is a natural isomorphism between C^∞ vector bundle morphisms $f:\xi \longrightarrow \eta$ and C^∞ sections f of $L(\xi,\eta)$ which we regard as an identification. Similarly for each non-negative integer k, C^k vector bundle morphisms $f:\xi \longrightarrow \eta$ are identified with $f \in C^k(L(\xi,\eta))$. Since $\mathcal{M}(L(\xi,\eta)) \subseteq C^0(L(\xi,\eta))$ it is natural to call a C^0 vector bundle morphism $f:\xi \longrightarrow \eta$ of class $\mathcal{M}$ if $f \in \mathcal{M}(L(\xi,\eta))$ and we shall do so.

Recall that a C^k vector bundle morphism $f:\xi \longrightarrow \eta$ is called <u>strict</u> if $f(x)$ has the same rank at all $x \in M$ and that in this case $\ker f$ and $\mathrm{im}(f)$ are C^k vector sub-bundles of ξ and η respectively. We shall now see that there is a natural notion of a sub-bundle of class $\mathcal{M}$ of a C^∞ vector bundle over M and that if $f:\xi \longrightarrow \eta$ is a strict vector bundle homomorphism of class $\mathcal{M}$ then $\ker(f)$ and $\mathrm{im}(f)$ are sub-bundles of class $\mathcal{M}$ of ξ and respectively.

First recall that if $f:\xi \longrightarrow \eta$ is a vector bundle homomorphism of class $\mathcal{M}$ then we have a continuous linear map $\tilde{f}:\mathcal{M}(\xi) \longrightarrow \mathcal{M}(\eta)$ defined in 11.2 by $\tilde{f}(s)(x) = f(x)s(x)$. Secondly we note that if $s_1,\ldots,s_m \in \mathcal{M}(\xi)$ are linearly independent over an open set U and we construct $\sigma_1,\ldots,\sigma_m$ by the Gram-Schmidt process, then σ_i is a linear combination of $s_1,\ldots,s_i$ with coefficients which are in $\mathcal{M}(U,\mathbb{R})$ by (B§5), and so the $\sigma_i \in \mathcal{M}(\xi|U)$. We shall use then facts in the following.

<u>14.1 Theorem</u>. Let ξ be a C^∞ vector bundle over M and let η be a C^0 sub-bundle of ξ. Then the following five conditions are equivalent and if any one and hence all of them hold we shall call η a sub-bundle of class $\mathcal{M}$ of ξ.

1) For each $x \in M$ there is a neighborhood U and $s_1,\ldots,s_m \in \mathcal{M}(\xi)$ such that $s_1(y),\ldots,s_m(y)$ is a basis for η_y at all $y \in U$.

2) If $P:\xi \longrightarrow \xi$ is the orthogonal projection of ξ on η with respect to any C^∞ Riemannian metric for ξ then P is of class $\mathcal{M}$.

3) There is a vector bundle morphism $P:\xi \longrightarrow \xi$ of class $\mathcal{M}$ such that $P(x)$ is a projection of ξ_x on η_x for all $x \in M$.

4) There is a C^∞ vector bundle ζ and a strict vector bundle morphism $f:\zeta \longrightarrow \xi$ of class $\mathcal{M}$ such that $\eta = \mathrm{im}(f)$.

5) There is a C^∞ vector bundle ζ and a strict vector bundle morphism $f:\xi \longrightarrow \zeta$ of class $\mathcal{M}$ such that $\eta = \ker f$.

Proof. 1) => 2). Given a C^∞ Riemannian metric for ξ, let $P:\xi \longrightarrow \xi$ be the orthogonal projection on η. By (B§2) it will suffice to show that $P|U \in \mathcal{M}(L(\xi|U, \xi|U))$ where U is as in 1). By the Gram-Schmidt process we can assume that $s_1,\ldots,s_m$ are othonormal in U. Then for $y \in U$ $P_y e = \sum_{i=1}^{m} < e, s_i(y) > s_i(y)$ from which it follows easily from the remarks preceding 11.2 that P is of class $\mathcal{M}$ in U.

The implications 2) => 3) and 3) => 4) are trivial. We next prove 4) => 1). Given $x \in M$ choose $\sigma_1,\ldots,\sigma_m \in C^\infty(\zeta)$ such that $f(\sigma_1(x)),\ldots,f(\sigma_m(x))$ is a basis for η_x. Then $s_1 = f(\sigma_1),\ldots,s_m = f(\sigma_m)$ are in $\mathcal{M}(\xi)$ and by continuity $s_1(y),\ldots,s_m(y)$ are a basis for η_y for y near x.

At this point we know that 1) - 4) are equivalent and from 2) that if η satisfies any of these conditions then so does its orthogonal complement with respect to any C^∞ Riemannian metric. We complete the proof by showing that 5) is equivalent to the other conditions.

Given a strict vector bundle morphism $f:\xi \longrightarrow \zeta$ of class $\mathcal{M}$ choose C^∞ Riemannian metrics for ξ and ζ. Since $\gamma \longrightarrow \gamma^*$ is a C^∞ vector bundle isomorphism of $L(\xi,\zeta)$ with $L(\zeta,\xi)$ it follows that $f^*: \zeta \longrightarrow \xi$ is of class $\mathcal{M}$ and clearly is strict, so $\mathrm{im}(f^*)$ satisfies 1) - 4) and hence so does $\mathrm{im}(f^*)^\perp = \ker f$. Hence 5) implies 1) - 4). Finally 3) => 5 since $1-P:\xi \longrightarrow \xi$ is strict and of class $\mathcal{M}$ and $\eta = \ker(1-P)$.

q.e.d.

14.2 Corollary. If η is a sub-bundle of ξ of class $\mathcal{M}$ then so is the orthogonal complement of η with respect to any C^∞ Riemannian metric for ξ.

Before stating the next theorem we recall the elementary fact that if η is a C^0 sub-bundle of a C^∞ vector bundle then η has a C^∞ complementary sub-bundle.

14.3 Theorem. If ξ and η are C^∞ vector bundles over M and $f:\xi \longrightarrow \eta$ is a strict vector bundle morphism of class $\mathcal{M}$ then there is a strict vector bundle morphism $g:\eta \longrightarrow \xi$ of class $\mathcal{M}$ such that $f\circ g(e) = e$ for $e \in \mathrm{im}(f)$.

Proof.

Case 1. $f(x):\xi_x \longrightarrow \eta_x$ is bijective for all $x \in M$. Then $f \in \mathcal{M}(\mathrm{Iso}(\xi,\eta))$, where $\mathrm{Iso}(\xi,\eta)$ is the open C^∞ fiber bundle in $L(\xi,\eta)$ whose fiber at x is all linear isomorphisms of ξ_x onto η_x. Since $e \longmapsto e^{-1}$ is a C^∞ fiber bundle isomorphism of $\mathrm{Iso}(\xi,\eta)$ with $\mathrm{Iso}(\eta,\xi)$ it follows that $f^{-1}: \eta \longrightarrow \xi$ is a vector bundle morphism of class $\mathcal{M}$.

Case 2. $f(x):\xi_x \longrightarrow \eta_x$ is surjective for all $x \in M$.. Let ζ be a C^∞ sub-bundle of ξ complementary to $\ker(f)$ and let $j: \zeta \longrightarrow \xi$ be inclusion. Then $f\circ j:\zeta \longrightarrow \eta$ is of class $\mathcal{M}$ and bijective so by Case 1, $g = (f\circ j)^{-1} \in \mathcal{M}(L(\eta,\zeta)) \subseteq \mathcal{M}(L(\eta,\xi))$.

Case 3. General case. Let ζ be a C^∞ complement to $\mathrm{im}(f)$ in η. Then $f \oplus \mathrm{id}: \xi \oplus \zeta \longrightarrow \eta$ is of class $\mathcal{M}$ and is surjective on each fiber, so by case 2 there exists $h:\eta \longrightarrow \xi \oplus \zeta$ a vector bundle morphism of class $\mathcal{M}$ such that if $e \in \mathrm{im}(f)$ and $e \in \zeta$ then $(f \oplus \mathrm{id}) \circ h\,(e + e') = e + e'$, and hence $f \circ h(e) = e$. Thus we can take $g = P \circ h$ where P is the projection of $\xi \oplus \xi$ on ξ. q.e.d.

14.4 Definition. If η is a vector sub-bundle of class $\mathcal{M}$ of a C^∞ vector bundle ξ then we define $\mathcal{M}(\eta) = C^0(\eta) \cap \mathcal{M}(\xi)$.

<u>14.5 Theorem</u>. If ξ is a C^∞ vector bundle over M and if η is a vector sub-bundle of ξ of class $\mathcal{M}$ then $\mathcal{M}(\eta)$ is a closed complemented subspace of $\mathcal{M}(\xi)$ and in fact we can choose a vector sub-bundle ζ of ξ of class $\mathcal{M}$, complementary to η, such that $\mathcal{M}(\xi) = \mathcal{M}(\eta) \oplus \mathcal{M}(\zeta)$.

<u>Proof</u>. Let $P:\xi \longrightarrow \xi$ be a vector bundle morphism of class $\mathcal{M}$ such that $P(x)$ is a projection of ξ_x on η_x for all $x \in M$. Then $P:\mathcal{M}(\xi) \longrightarrow \mathcal{M}(\xi)$ is a continuous projection whose image is clearly $\mathcal{M}(\eta)$ and whose kernel is $\mathcal{M}(\zeta)$ where $\zeta = \ker P$. q.e.d.

<u>14.6 Theorem</u>. Let $f:\xi \longrightarrow \eta$ be a strict vector bundle morphism of class $\mathcal{M}$ and let $\tilde{f}:\mathcal{M}(\xi) \longrightarrow \mathcal{M}(\eta)$ be as defined in 11.2, i.e. $\tilde{f}(s)(x) = f(x)s(x)$. Then $\ker(\tilde{f}) = \mathcal{M}(\ker f)$ and $\mathrm{im}(\tilde{f}) = \mathcal{M}(\mathrm{im}\, f)$, hence in particular $\ker(\tilde{f})$ and $\mathrm{im}(\tilde{f})$ are closed complemented subspaces of $\mathcal{M}(\xi)$ and $\mathcal{M}(\eta)$ respectively.

<u>Proof</u>. That $\mathrm{im}(\tilde{f})$ is really all of $\mathcal{M}(\mathrm{im}\, f)$ follows from 14.3. Everything else is either obvious or follows directly from 14.5.

<u>14.7 Lemma</u>. Let $f:\xi \longrightarrow \eta$ be a C^∞ vector bundle morphism and let $s \in \mathcal{M}(\xi)$. Then $\delta_s f: x \longmapsto \delta f_{s(x)}$ is a vector bundle morphism $\xi \longrightarrow \eta$ of class $\mathcal{M}$ and $\widetilde{\delta_s f} = d\mathcal{M}(f)_s$.

<u>Proof</u>. Since $e \longmapsto \delta f_e$ is a C^∞ fiber bundle morphism of ξ into $L(\xi,\eta)$ it follows from (B§5) that $\sigma \longmapsto \delta_\sigma f$ is a continuous map of $\mathcal{M}(\xi)$ into $\mathcal{M}(L(\xi,\eta))$ and hence from 11.2 that $\sigma \longmapsto \delta_\sigma f$ is a continuous map of $\mathcal{M}(\xi)$ into $L(\mathcal{M}(\xi),\mathcal{M}(\eta))$. If $s \in C^\infty(\xi)$ then by definition $\widetilde{\delta_s f} = \mathcal{M}(\delta_s f)$ so in this case $\widetilde{\delta_s f} = d\mathcal{M}(f)_s$ by 1.4. If $s \in \mathcal{M}(\xi)$ then the argument of 11.4 gives the same result. q.e.d.

The following is one of the main results of this section.

<u>14.8 Splitting Theorem</u>. Let $f:E_1 \longrightarrow E_2$ be a C^∞ fiber bundle morphism over M and let $s \in \mathcal{M}(E_1)$ be such that $\delta f:TF(E_1) \longrightarrow f^* TF(E_2)$

has the same rank at $s(x)$ for all $x \varepsilon M$. Then the differential of $\mathcal{M}(f):\mathcal{M}(E_1) \longrightarrow \mathcal{M}(E_2)$ at s has a kernel which is a closed complemented subspace of $\mathcal{M}(E_1)$ and an image which is a closed complemented subspace of $\mathcal{M}(E_2)$. Moreover $d\mathcal{M}(f)_s$ is injective (surjective) if and only if $\delta f_{s(x)} : TF(E_1)_{s(x)} \longrightarrow TF(E_2)_{f(s(x))}$ is injective (surjective) for all $x \varepsilon M$.

Proof. Immediate from 14.6 and 14.7.

14.9 Definition. Let $f:E_1 \longrightarrow E_2$ be a C^∞ morphism of C^∞ fiber bundles over M. We say that f is a bundle immersion (bundle submersion) over M if for all $x \varepsilon M$ $f|(E_1)_x \longrightarrow (E_2)_x$ is an immersion (submersion).

Caution. In the context of infinite dimensional manifolds and in particular in the following theorem and corollary, we use the terms submanifold, submersion, and immersion in the sense they are used in Lang's Introduction to Differentiable Manifolds. In the terminology of some authors these are called split submanifolds, split submersion, and split immersions respectively, i.e. we assume that the tangent space to a submanifold, the kernel of the differential of a submersion, and the image of the differential of an immersion have at each point closed complementary subspaces.

14.10 Theorem. If $f:E_1 \longrightarrow E_2$ is a C^∞ bundle immersion (submersion) over M then $\mathcal{M}(f):\mathcal{M}(E_1) \longrightarrow \mathcal{M}(E_2)$ is an immersion (submersion).

Proof. Immediate from 14.8.

14.11 Corollary. If E_1 and E_2 are C^∞ fiber bundles over M and E_1 is a closed sub-bundle of E_2 then $\mathcal{M}(E_1)$ is a closed submanifold of $\mathcal{M}(E_2)$.

The following is well-known.

14.12 Theorem Consider a commutative diagram

$$\begin{tikzcd} E_1 \arrow[rr, "f"] \arrow[dr, "p_1"'] & & E_2 \arrow[dl, "p_2"] \\ & M & \end{tikzcd}$$

where $p_2: E_2 \longrightarrow M$ and $f: E_1 \longrightarrow E_2$ are C^∞ fibrations. Then $p_1: E_1 \longrightarrow M$ is also a C^∞ fibration and f is a C^∞ bundle morphism over M.

Proof. First suppose that M is contractible, so that we may assume $E_2 = F \times M$ with p_2 the projection. Let $q: F \longrightarrow F \times M$ be the map $x \longmapsto (x, m_o)$, for some fixed $m_o \in M$, and let $E_1^o = q^* E_1$. Since M is contractible q is a homotopy inverse of the projection $F \times M \longrightarrow F$ so, as a bundle over $F \times M$, $E_1 \approx E_1^o \times (F \times M)$, and it is immediate that $p_1 = p_2 \circ f: E_1 \longrightarrow M$ is a trivial bundle with fiber $E_1^o \times F$. In general each $x \in M$ has a neighborhood U which is contractible and it follows that $p_1: E_1 \longrightarrow M$ is trivial over U. The final remark is clear. q.e.d.

14.13 Definition Let $f: E_1 \longrightarrow E_2$ be a C^∞ fiber bundle morphism over M. We say that f is a fibered morphism of bundles over M if as a C^∞ map of E_1 into E_2 f is locally trivial (i.e. $f: E_1 \longrightarrow E_2$ is a C^∞ fiber bundle over E_2).

Remark. The point of 14.12 is of course that in 14.13 it suffices to assume that $p_2: E_2 \longrightarrow M$ and $f: E_1 \longrightarrow E_2$ are C^∞ fiber bundles and then automatically $p_1 = p_2 \circ f: E_1 \longrightarrow M$ is a C^∞ fiber bundle and f is a fibered morphism of bundles over M.

14.14 Lemma. Let $p: B \longrightarrow M$ and $q: \bar{E} \longrightarrow M$ be C^∞ fiber bundles over M and let $\alpha: p^* \bar{E} \longrightarrow B$ be the induced C^∞ fiber bundle over B. Then $p \circ \alpha: p^*\bar{E} \longrightarrow M$ is identical to the fiber product $B \times_M \bar{E} \longrightarrow M$, and moreover α is a C^∞ bundle morphism over M.

Proof. By definition, as topological spaces both $p^*\bar{E}$ and $B \times_M \bar{E}$ are equal to $\{(b,e) \in B \times \bar{E} \mid p(b) = q(e)\}$ and also by definition $\alpha(b,e) = b$, so $p \circ \alpha(b,e) = p(b)$ which is how the natural projection $B \times_M E \longrightarrow M$ is defined. Since for $x \in M$ $(B \times_M \bar{E})_x = B_x \times \bar{E}_x$ and α restricts to the projection on B_x the final remark is clear. q.e.d.

14.15 Lemma. Let $p:\xi \longrightarrow M$ be a C^∞ vector bundle over M and $\pi:E \longrightarrow \xi$ a C^∞ fiber bundle over ξ. If $\zeta:M \longrightarrow \xi$ is any C^∞ section of ξ then E is equivalent to the induced bundle $\alpha:p^* \bar{E} \longrightarrow \xi$, where $q:\bar{E} \longrightarrow M$ is the C^∞ fiber bundle $\zeta^* E$ induced by ζ.

Proof. This is of course well known and follows from the fact that since $\mathrm{im}(\zeta)$ is a deformation retract of ξ, ζ is a homotopy inverse for p and hence $E = (\zeta p)^* E = p^* \zeta^* E = p^*\bar{E}$.
q.e.d.

The following interesting result was discovered and pointed out to me by S. Greenfield and H. Levine independently. Their proofs were somewhat different from that which follows.

14.16 Theorem. Let $f:E_1 \longrightarrow E_2$ be a fibered morphism of C^∞ fiber bundles over M. Then $\mathcal{M}(f):\mathcal{M}(E_1) \longrightarrow \mathcal{M}(E_2)$ is a C^∞ fibration whose fiber at $\sigma \in \mathcal{M}(E_2)$ is $\mathcal{M}(\zeta^* E_1)$, where $\zeta:M \longrightarrow E_2$ is a C^∞ section which is contained in some VBN ξ of σ in E_2.

Proof. Let $E = E_1|\xi$. Clearly $\mathcal{M}(f)^{-1}(\mathcal{M}(\xi)) = \mathcal{M}(E)$ and since $\mathcal{M}(\xi)$ is a neighborhood of σ in $\mathcal{M}(E_2)$ it will suffice to find a diffeormophism of $\mathcal{M}(E)$ onto $\mathcal{M}(\zeta^* E_1) \times \mathcal{M}(\xi)$ commuting with $\mathcal{M}(f|E):\mathcal{M}(E) \longrightarrow \mathcal{M}(\xi)$ and the projection of $\mathcal{M}(\zeta^* E_1) \times \mathcal{M}(\xi)$ on $\mathcal{M}(\xi)$. Now by 14.15 and 14.14 there is a bundle isomorphism $\varphi : E \longrightarrow \zeta^* E_1 \times_M \xi$ over M which commutes with the bundle morphisms $f|E : E \longrightarrow \xi$ and $\zeta^* E_1 \times_M \xi \longrightarrow \xi$ over M. By the functoriality of $\mathfrak{M}$ and its multiplicativity (Theorem 13.7) it follows that $\mathfrak{M}(\varphi)$ is the desired diffeomorphism.

q.e.d.

14.17 Theorem. If E is a C^∞ fiber bundle over M with paracompact fibers then $\mathfrak{M}(E)$ is a paracompact C^∞ manifold.

Proof. By Lemma 13.8 E is a closed sub-bundle of a C^∞ vector bundle ξ over M, hence by 14.11 $\mathfrak{M}(E)$ is a closed C^∞ submanifold of the Banach space $\mathfrak{M}(\xi)$.

q.e.d.

15. NON-LINEAR DIFFERENTIAL OPERATORS

Let $\pi: E \longrightarrow M$ be a C^∞ fiber bundle over M. Given $e_o \in E$ with $\pi(e_o) = x_o$ and local sections s_1, s_2 of E defined near x_o with $s_1(x_o) = s_2(x_o) = e_o$, by choosing a chart at x_o in M, a local trivialization of E near x_o, and a chart near e_o in the fiber E_{x_o} we can define the k^{th} order Taylor expansions of s_1 and s_2 at x_o. While these Taylor expansions will depend on the various choices, if they are the same for one set of choices they will be the same for any other and we say in this case that s_1 and s_2 have the same k-jet at x_o. This defines an equivalence relation on the set of local sections s of E defined near x_o with $s(x_o) = e_o$. The set of equivalence classes is denoted by $J^k(E)_{e_o}$ and the equivalence class of s is denoted by $j_k(s)_{x_o}$. Let $J_o^k(E) = \bigcup_{e \in E} J^k(E)_e$ and let $\pi_o^k: J_o^k(E) \longrightarrow E$ be the function which maps $J_o^k(E)_e$ to e. It is easily seen that $\pi_o^k: J_o^k(E) \longrightarrow E$ has in a natural way the structure of a C^∞ fiber bundle over E whose fiber at e_o is $\bigoplus_{m=1}^{k} L_s^m(T(M)_{x_o}, T(E_{x_o})_{e_o})$, or equivalently all polynomial maps of degree less than or equal k from $T(M)_{x_o}$ into $T(E_{x_o})_{e_o}$. We define a C^∞ fiber bundle $\pi^k: J^k(E) \longrightarrow M$ whose total space is just $J_o^k(E)$ and whose projection is $\pi^k = \pi \circ \pi_o^k$, i.e. $\pi^k(j_k(s)_{x_o}) = x_o$. That this is indeed a C^∞ fiber bundle follows from 14.12. More generally if $0 \leq \ell < k$ we define a C^∞ fiber bundle $\pi_\ell^k: J_\ell^k(E) \longrightarrow J^\ell(E)$ whose total space is $J^k(E)$ and whose projection is given by $\pi_\ell^k(j_k(s)_{x_o}) = j_\ell(s)_{x_o}$. (There is a canonical identification of $J^o(E)$ with E so the notation is consistent).

We note that we have a natural map, the k-jet extension map, $j_k: C^\infty(E) \longrightarrow C^\infty(J^k(E))$, defined of course by $j_k(s)(x) = j_k(s)_x$. If ξ is a VBN of $s \in C^\infty(E)$ then (see §2) $J^k(\xi)$ is a vector bundle, and in fact it is clearly a VBN of $j_k(s) \in C^\infty(J^k(E))$. Moreover $j_k: C^\infty(E) \longrightarrow C^\infty(J^k(E))$ restricts to a <u>linear</u> map of $C^\infty(\xi)$ to $C^\infty(J^k(\xi))$.

Now let $f: E_1 \longrightarrow E_2$ be a C^∞ fiber bundle morphism over M. Then there is an induced C^∞ fiber bundle morphism over M $J^k(f): J^k(E_1) \longrightarrow J^k(E_2)$, defined by

$J^k(f)j_k(s)_x = j_k(f \circ s)_x$. Of course one must check that $J^k(f)$ is well defined (i.e. that $j_k(f \circ s)_x$ depends only on $j_k(s)_x$) and is C^∞, but this is elementary (and will follow from considerations in the next section when we compute $J^k(f)$ explicitly in "local coordinates"). Thus J^k is a functor from $FB(M)$ to itself.

Note that we can also consider $J^k(f)$ as going from $J^k_\ell(E_1)$ to $J^k_\ell(E_2)$ for $\ell < k$, in which case we denote it by $J^k_\ell(f)$. Clearly $J^k_\ell(f)$ is a fiber preserving map with induced map $J^\ell(f)$, i.e. we have a commutative diagram

$$\begin{tikzcd} J^k_\ell(E_1) \arrow[r, "J^k_\ell(f)"] \arrow[d, "\pi^k_\ell"'] & J^k_\ell(E_2) \arrow[d, "\pi^k_\ell"] \\ J^\ell(E_1) \arrow[r, "J^\ell(f)"'] & J^\ell(E_2) \end{tikzcd}$$

<u>15.1 Theorem</u>. If $\mathcal{M}$ satisfies (B§5) then so does each of the derivative functors $\mathcal{M}_k$ $k = 0,1,2 \ldots$ defined in Section 5.

<u>Proof</u>. Similar to that of 5.2.

<u>15.2 Theorem</u>. If $\mathcal{M}$ satisfies (B§2) and (B§5) and E is a C^∞ fiber bundle over M then $j_k : C^\infty(E) \longrightarrow C^\infty(J^k(E))$ extends to a C^∞ map $j_k : \mathcal{M}_{k+r}(E) \longrightarrow \mathcal{M}_r(J^k(E))$.

<u>Proof</u>. Given $s \in \mathcal{M}_{k+r}(E)$ let ξ be a VBN of s in E. Then $\mathcal{M}_{k+r}(\xi)$ is a neighborhood of s in $\mathcal{M}_{k+r}(E)$ and by 5.4 j_k (being an element of $\mathrm{Diff}_k(\xi, J^k(\xi))$ extends uniquely to a C^∞ (in fact linear) map of $\mathcal{M}_{k+r}(\xi)$ into $\mathcal{M}_r(J^k(\xi))$. q.e.d.

Recall that if ξ and η are vector bundles over M then a linear differential operator of order k from ξ to η (or from $C^\infty(\xi)$ to $C^\infty(\eta)$) is a linear map $D : C^\infty(\xi) \longrightarrow C^\infty(\eta)$ that can be factored as a composition

$$C^\infty(\xi) \xrightarrow{j_k} C^\infty(J^k(\xi)) \xrightarrow{f_*} C^\infty(\eta)$$

where $f:J^k(\xi) \longrightarrow \eta$ is a C^∞ vector bundle morphism over M. It is clear how to define a (non-linear) differential operator of order k by analogy.

15.3 Definition. Let E_1 and E_2 be C^∞ fiber bundles over M. A function $D:C^\infty(E_1) \longrightarrow C^\infty(E_2)$ will be called a (non-linear) differential operator of order k from E_1 to E_2 (or from $C^\infty(E_1)$ to $C^\infty(E_2)$) if it can be factored as

$$C^\infty(E_1) \xrightarrow{j_k} C^\infty(J^k(E_1)) \xrightarrow{F_*} C^\infty(E_2)$$

where $F:J^k(E_1) \longrightarrow E_2$ is a C^∞ fiber bundle morphism over M. We denote the set of all differential operators of order k from E_1 to E_2 by $Df_k(E_1,E_2)$.

15.4 Theorem. If $\ell \leq k$ then $Df_\ell(E_1,E_2) \subseteq Df_k(E_1,E_2)$.

Proof. Let $D \in Df_\ell(E_1,E_2)$, say $D = F_* j_\ell$ where $F:J^\ell(E_1) \longrightarrow E_2$ is a fiber bundle morphism. Then $f \circ \pi^k_\ell : J^k(E_1) \longrightarrow E_2$ is a fiber bundle morphism over M and clearly $D = (F \circ \pi^k_\ell)_* j_k$. q.e.d.

15.5 Lemma. Let $j_k:C^\infty(E) \longrightarrow C^\infty(J^k(E))$ and $\bar{j}_\ell:C^\infty(J^k(E)) \longrightarrow C^\infty(J^\ell(J^k(E)))$ be jet extension maps. Then $\bar{j}_\ell \circ j_k$ is a differential operator of order $k+\ell$ from E to $J^\ell(J^k(E))$.

Proof. If E is a vector bundle this is a special case of the fact that the composite of linear differential operators is a linear differential operator, so if ξ is a VBN in E then there is a C^∞ fiber bundle morphism (in fact a vector bundle morphism) $f:J^{k+\ell}(\xi) \longrightarrow J^\ell(J^k(\xi))$ such that $\bar{j}_\ell \circ j_k = f_* \circ j_{k+\ell}$. Since f is of course unique, the various f for different ξ are consistent and define a C^∞ fiber bundle morphism $f:J^{k+\ell}(E) \longrightarrow J^\ell(J^k(E))$ with the same property. q.e.d.

15.6 Theorem. If $D_1 \in Df_k(E_1,E_2)$ and $D_2 \in Df_\ell(E_2,E_3)$ then $D_2D_1 \in Df_{k+\ell}(E_1,E_3)$.

Proof. Let $D_1 = f_{1*}j_k$ and $D_2 = f_{2*}j_\ell$ where $f_1 : J^k(E_1) \longrightarrow E_2$ and $f_2 : J^\ell(E_2) \longrightarrow E_3$ are C^∞ fiber bundle morphisms. Then $D_2D_1 = f_{2*}(j_\ell f_{1*})j_k = f_{2*}(J^\ell(f_1)_*\bar{j}_\ell)j_k$ where $\bar{j}_\ell : C^\infty(J^k(E)) \longrightarrow C^\infty(J^\ell(J^k(E)))$ is the jet extension map and $J^\ell(f_1) : J^\ell(J^k(E)) \longrightarrow J^\ell(E_2)$ is the C^∞ fiber bundle morphism induced by f_1. By 15.5 $\bar{j}_\ell \circ j_k = g_* j_{k+\ell}$ where $g : J^{k+\ell}(E_1) \longrightarrow J^\ell(J^k(E_1))$ is a C^∞ fiber bundle morphism, so finally we have $D_2D_1 = h_* j_{k+\ell}$ where $h = f_2 \circ J^\ell(f_1) \circ g$ is a C^∞ fiber bundle morphism from $J^{k+\ell}(E_1)$ to E_3. q.e.d.

15.7 Theorem. Assume $\mathcal{M}$ satisfies (B§2) and (B§5) and let $D \in \mathrm{Df}_k(E_1,E_2)$ where E_1 and E_2 are C^∞ fiber bundles over a compact n-dimensional C^∞ manifold M. Then $D : C^\infty(E_1) \longrightarrow C^\infty(E_2)$ extends to a C^∞ map of $\mathcal{M}_{k+r}(E_1)$ to $\mathcal{M}_r(E_2)$ for $r = 0,1,2,\ldots$.

Proof. $D = f_* \circ j_k$ where $f : J^k(E_1) \longrightarrow E_2$ is a C^∞ fiber bundle morphism. By 15.2 j_k extends to a C^∞ map $j_k^{(r)} : \mathcal{M}_{k+r}(E_1) \longrightarrow \mathcal{M}_r(J^k(E_1))$ and by 15.1 and 13.4, f_* extends to a C^∞ map $\mathcal{M}_r(f) : \mathcal{M}_r(J^k(E_1)) \longrightarrow \mathcal{M}_r(E_2)$, so $\mathcal{M}_r(f) \circ j_k^{(r)}$ gives the desired extension. q.e.d.

Both to better justify the name non-linear differential operator and also to prepare for later computations we shall now see what a differential operator "really looks like" in local coordinates.

Let $\pi : E \longrightarrow M$ be a C^∞ fiber bundle. Let e_o E and let $x_o = \pi(e_o)$. Choose coordinates $x_1,\ldots,x_n$ in a neighborhood U of x_o in M, coordinates $y_1,\ldots,y_m$ in a neighborhood V of e_o in E_{x_o} and using a local trivialization identify a neighborhood of e_o in E with U x V so that π restricted to U x V is projection on the first component. Then we have natural coordinates y_i^α $(i = 1,\ldots,;\ 0 \le |\alpha| \le k)$ for the fibers of $J_o^k(E)$ over U x V defined as follows. A section σ of E over U with $\sigma(x) \in V$ for $x \in U$ is given by a map $x \longmapsto (x,s(x))$ of U into U x V. Putting $s_i(x) = y_i(s(x))$, $j_k(\sigma)$ is given in coordinate form by a map $x \longmapsto (x,s(x),\ y_i^\alpha(j_k(\sigma)(x)))$ where $y_i^\alpha(j_k(\sigma)(x)) = D^\alpha s_i(x)$, and as usual $D^\alpha = \partial^{|\alpha|}/\partial x_1^{\alpha_1}\ldots\partial x_n^{\alpha_n}$.

Now let $\pi':\zeta \longrightarrow M$ be a C^∞ vector bundle over M and let $D \varepsilon Df_k(E,\zeta)$, say $D = F_* j_k$ where $F:J^k(E) \longrightarrow \zeta$ is a C^∞ fiber bundle morphism. Let $v_1,\ldots,v_r$ be a basis of C^∞ sections of ζ over U. If we restrict F to the part of $J^k_o(E)$ over $U \times V$ it is given by certain functions $F_j(j = 1,\ldots,r)$ of the coordinates $(x_1,\ldots,x_n, y_1,\ldots,y_m, y_i^\alpha)$ by the rule

$$(x,y,y_i^\alpha) \longmapsto \sum_{j=1}^{r} F(x,y,y_i^\alpha)v_j(x)$$

Then the explicit expression for $D\sigma$ is

$$D\sigma(x) = \sum_{j=1}^{r} F_j(x,s_1(x),\ldots,s_m(x), D^\alpha s_i(x))v_j(x).$$

<u>15.8 Definition</u>. We shall call the ordered r-tuple of functions $F_j(x,y,y_i)$ defined above the parametric expressions for the operator D near e_o relative to the various choices made (i.e. the coordinates $x_1,\ldots,x_n$ in U, the coordinates $y_1,\ldots,y_m$ in V, the local trivialization of E over U, and the basis of sections $v_1,\ldots,v_r$ for ζ over U).

16. POLYNOMIAL DIFFERENTIAL OPERATORS

If E is a C^∞ fiber bundle over M and ζ is a C^∞ vector bundle over M, then by utilizing the linear structure in ζ it is possible to single out certain vector subspaces of the vector space $Df_k(E,\zeta)$ of k^{th} order differential operators from E to ζ which are "polynomial" functions of certain derivatives. These classes play an important role in many part of non-linear analysis, particularly in the calculus of variations, and in this section we shall define them and derive some of their most basic properties.

16.1 Definition. Let E be a C^∞ fiber bundle over M, ζ a C^∞ vector bundle over M and $D \in Df_k(E,\zeta)$. Let w and ℓ be integers with $w > 0$ and $0 \leq \ell < k$. We shall say that D is polynomial of weight $\leq w$ with respect to derivatives of order $> \ell$ (symbolically $D \in Df_k^{w;\ell}(E,\zeta)$) if for each parametric representation of D

$$D\sigma(x) = \sum_{j=1}^{r} F_j(x, s_1(x), \ldots, s_m(x), D^\alpha s_i(x))\; v_j(x)$$

(see Definition 15.8) each of the functions $F_j(x,y,y_i^\alpha)$ $j = 1,\ldots,r$ can be written as a sum of functions of the form

$$\Phi(x,y,y_i^\gamma)y_{\ell_1}^{\beta_1} \cdots y_{\ell_q}^{\beta_q}$$

where all $|\gamma| \leq \ell$, all $|\beta_i| > \ell$ and $|\beta_1| + \ldots + |\beta_q| \leq w$.

We abbreviate $Df_k^{w;o}(E,\zeta)$ to $Df_k^w(E,\zeta)$, and elements of this space we refer to as polynomial differential operators of order k and weight $\leq w$.

Elements of $Df_k^{k;k-1}(E,\zeta)$ (i.e. such that the F_j of an parametric representation are linear in derivatives of order k) are called quasi-linear differential operators of order k from E to ζ.

Remark. It is immediate from the definition that if $D \in Df_k^{w;\ell}(E,\zeta)$ then $D \in Df_w(E,\zeta)$, hence if D is really of order k (i.e. involves k^{th} order derivatives in an essential way) then $w \geq k$.

It is also immediate from the definition that $Df_k^{w;\ell}(E,\zeta)$ is a vector subspace of $Df_k(E,\zeta)$ and $Df_k^{w;\ell}(E,\zeta) \subseteq Df_k^{w;m}(E,\zeta)$ if $0 \leq \ell \leq m$.

Evidently $Df_o(E,\zeta) \subseteq Df_k^{o;\ell}(E,\zeta)$ for all k and $\ell < k$, however it is not a priori evident that $Df_k^{w;\ell}(E,\zeta)$ ever has positive dimension if $w > 0$. The next theorem however shows how to construct lots of operators in $Df^{w;\ell}(E,\zeta)$. For example if E is a vector bundle it shows that $\mathrm{Diff}_k(E,\zeta)$, the space of linear k-th order differential operators from E to ζ is $\subseteq Df_k^k(E,\zeta)$.

16.2 Theorem. Let E be a C^∞ fiber bundle over M, ζ a C^∞ vector bundle over M and $D \in Df_k(E,\zeta)$. In order that $D \in Df_k^{w;\ell}(E,\zeta)$ it is sufficient that for each $e_o \in E$ there exist at least one parametric expression for D near e_o satisfying the conditions of Definition 16.1.

Before commencing on the proof of Theorem 16.2 we note that it suffices to assume that E is a vector bundle. This follows from the following lemma which is an elementary consequence of Definition 16.1.

Lemma 16.3. Let D be as in Definition 16.1 and suppose that for each coordinate neighborhood U in M and each VBN ξ of $E|U$ the obvious "restriction" of D to a map of $C^\infty(\xi)$ into $C^\infty(\zeta|U)$ is an element of $Df_k^{w;\ell}(\xi,\zeta|U)$. Then $D \in Df_k^{w;\ell}(E,\zeta)$.

The crucial step in proving Theorem 16.2 is getting an explicit coordinate description of the induced fiber bundle morphism $J^k(f): J^k(\xi) \longrightarrow J^k(\eta)$ of a C^∞ fiber bundle morphism $f:\xi \longrightarrow \eta$ of vector bundles. Since this is a local question we can suppose $M = \mathbb{R}^n$ and we use $x = (x_1,\ldots,x_n)$ to denote both a point in $\mathbb{R}^n$ and the natural coordinates in $\mathbb{R}^n$. Also as usual we write $D^\alpha = \partial^{|\alpha|}/\partial x_1^{\alpha_1}\cdots\partial x_n^{\alpha_n}$. We can also assume $\xi = M \times \mathbb{R}^m$ and $\eta = M \times \mathbb{R}^p$ and we use $y = (y_1,\ldots,y_m)$ and $z = (z_1,\ldots,z_p)$ for the natural coordinates in $\mathbb{R}^m$ and $\mathbb{R}^p$ respectively.

We recall that we get coordinates $y_i^\alpha(i=1,\ldots,m;\ 0 < |\alpha| \leq k)$ for the fibers of $J_o^k(\xi)$ as follows: an element σ of $C^\infty(\xi)$ is given by a map $x \longmapsto (x,s(x)) = (x,s_1(x),\ldots,s_m(x))$ of M into $M \times \mathbb{R}^m$, where $s_i(x) = y_i(s(x))$. Then $j_k(\sigma)$ is given by a map

$$x \longmapsto (x, s(x),\ y_i^{\alpha}(j_k(\sigma)(x)))$$

where $y_i^{\alpha}(j_k(\sigma)(x)) = D^{\alpha}s_i(x)$. Clearly those y_i^{α} with $|\alpha| > \ell$ are coordinates for the fiber of the bundle $J_{\ell}^k(\xi)$ over $J^{\ell}(\xi)$.

In the same way we get coordinates $z_j^{\alpha}(j=1,\ldots p;\ 0 < |\alpha| \leq k)$ for the fiber of the bundle $J_o^k(\eta)$, where now for a section $\sigma: x \longmapsto (x, s(x))$ of η, $z_j^{\alpha}(J_k(\sigma)(x)) = D^{\alpha}z_j(s)(x)$.

Now a C^{∞} fiber bundle morphism $f: \xi \longrightarrow \eta$ is given by a map

$$(x,y) \longmapsto (x,\varphi(x,y)) = (x,\varphi_1(x,y),\ldots,\varphi_p(x,y))$$

where $\varphi: \mathbb{R}^n \times \mathbb{R}^m \longrightarrow \mathbb{R}^p$ is a C^{∞} map and $\varphi_j: \mathbb{R}^n \times \mathbb{R}^m \longrightarrow \mathbb{R}$ is $z_j \circ \varphi$. If $s \in C^{\infty}(\xi)$ as above then $f \circ \sigma(x) = (x,\varphi(x,s(x)))$, hence

$$z_j^{\alpha}(j_k(f \circ \sigma)) = D^{\alpha}z_j(f \circ s) = D^{\alpha}\varphi_j(x,s_1(x),\ldots,s_m(x)).$$

Hence using the chain rule and induction we get $z_j(j_k(f \circ \sigma)) =$

$$\sum_{0 \leq q \leq |\alpha|} \quad \sum_{\substack{\beta_1+\cdots+\beta_q \leq |\alpha| \\ |\beta_i|>0}} \quad \sum_{\ell_1,\ldots,\ell_q=1} \quad {}^{\alpha}_{j}\Phi^{\ell_1\cdots\ell_q}_{\beta_1\cdots\beta_q}(x,s(x))\, Ds^{\beta_1}_{\ell_1}\cdots Ds^{\beta_q}_{\ell_q}$$

(where the term in the sum corresponding to $q = 0$ is to be interpreted to be of the form ${}^{\alpha}_{j}\Phi(x,s(x))$. The ${}^{\alpha}_{j}\Phi^{\ell_1\cdots\ell_q}_{\beta_1\cdots\beta_q}$ are certain C^{∞} real valued functions on $\mathbb{R}^n \times \mathbb{R}^m$ and in fact are sums of partial derivatives of φ_j of order less than $|\alpha|$. Recalling That $J^k(f)j_k(\sigma) = j_k(f \circ \sigma)$ we have the following result.

16.4 Theorem. Let $M = \mathbb{R}^n$, $\xi = M \times \mathbb{R}^n$, $\eta = M \times \mathbb{R}^p$ and let y_i and z_j be the natural coordinates described above for the fibers of $J^k(\xi)$ and $J^k(\eta)$. Let $f:\xi \longrightarrow \eta$ be a C^∞ fiber bundle morphism given by $(x,y) \longmapsto (x,\varphi_1(x,y),\ldots,\varphi_p(x,y))$. Then the induced morphism $J^k(f): J^k(\xi) \longrightarrow J^k(\eta)$ is given by the formulas: $z_j^\alpha \circ J^k(f) =$

$$\sum_{0\le q\le|\alpha|} \sum_{\substack{\beta_1+\cdots+\beta_q\le\alpha \\ |\beta_i|>0}} \sum_{\ell_1,\ldots,\ell_q=1}^{m} \left({}^{\alpha}_{j}\Phi^{\beta_1\cdots\beta_q}_{\ell_1\cdots\ell_q} \circ \pi^k_\circ \right) y^{\beta_1}_{\ell_1}\cdots y^{\beta_q}_{\ell_q}$$

where the ${}^{\alpha}_{j}\Phi^{\beta_1\cdots\beta_q}_{\ell_1\cdots\ell_q}$ are C^∞ real valued functions on ξ and $\pi^k_\circ : J^k(\xi) \longrightarrow \xi$ is the natural projection.

16.5 Corollary. The induced morphism $J^k_\ell(f): J^k_\ell(\xi) \longrightarrow J^k_\ell(\eta)$ is given by the following formulas (with $|\alpha| > \ell$): $z_j^\alpha \circ J^k_\ell(f) =$

$$\sum_{0\le q\le|\alpha|} \sum_{\substack{\beta_1+\cdots+\beta_q\le\alpha \\ |\beta_i|>\ell}} \sum_{\ell_1\cdots\ell_q=1}^{m} \left({}^{\alpha}_{j}\Psi^{\beta_1\cdots\beta_q}_{\ell_1\cdots\ell_q} \circ \pi^k_\ell \right) y^{\beta_1}_{\ell_1}\cdots y^{\beta_q}_{\ell_q} .$$

where the ${}^{\alpha}_{j}\Psi^{\beta_1\cdots\beta_q}_{\ell_1\cdots\ell_q}$ are C^∞ functions on $J^\ell(\xi)$ and $\pi^k_\ell: J^k_\ell(\xi) \longrightarrow J^\ell(\xi)$ is the natural projection.

16.6 Definition. Let M be a manifold diffeomorphic to $\mathbb{R}^n$. Let ξ be a C^∞ fiber bundle over M with fiber $\mathbb{R}^m$ and let ζ be a C^∞ vector bundle over M. Given integers

ℓ, k, w with $w > 0$ and $0 \leq \ell < k$ we define a vector subspace $P_k^{w;\ell}(\xi,\zeta)$ of the vector space of C^∞ fiber bundle morphisms of $J^k(\xi)$ into ζ as follows. Make the following choices: coordinates $x = (x_1,\ldots,x_n)$ on M, a trivilization $\xi = M \times \mathbb{R}^m$ of ξ and coordinates $y = (y_1,\ldots,y_m)$ in $\mathbb{R}^m$, and let $v_1,\ldots,v_r$ be a basis of C^∞ sections of ζ. Let y_i^α be the natural coordinates in the fiber of $J_o^k(\xi)$ as above. Then a morphism $g:J^k(\xi) \longrightarrow \zeta$ is given by

$$g(x,y,y_i^\alpha) = \sum_{j=1}^{r} G_j(x,y,y_i^\alpha)v_j(x)$$

and we say that $g \in P_k^{w;\ell}(\xi,\zeta)$ if and only if we can write each G_j as a sum of terms of the form

$$(\Phi \circ \pi_\ell^k)\ y_{\ell_1}^{\beta_1}\cdots y_{\ell_q}^{\beta_q}$$

where $\pi_\ell^k: J^k(\xi) \longrightarrow J^\ell(\zeta)$ is the natural projection, Φ is a C^∞ real valued function on $J^\ell(\xi)$, all the $|\beta_i| > \ell$ and $|\beta_1|+\ldots+|\beta_q| \leq w$. We define $Df_k^{w;\ell}(\xi,\zeta)$ to be the set of $D \in Df_k(\xi,\zeta)$ such that $D = g_* j_k$ where $g \in P_k^{w;\ell}(\xi,\zeta)$.

<u>Remark</u>. Note that to say a function on $J^k(\xi)$ is of the form $\Phi \circ \pi_\ell^k$ where Φ is a function on J^ℓ is just to say that it can be written as a function of the coordinates (x,y,y_i^α) with $|\alpha| \leq \ell$. Thus it is clear that Definition 16.6 is consistent with Definition 16.1. In view of Lemma 16.3 the proof Theorem 16.2 is equivalent to showing that $P_k^{w;\ell}(\xi,\zeta)$ is really well defined, i.e. is independent of the various choices entering into the definition, and this we now proceed to show.

First suppose we replace $v_1,\ldots,v_r$ by a second basis of sections $\bar{v}_1,\ldots,\bar{v}_r$ of ζ. Then $v_k(x) = \Sigma a_{kj}(x)\,\bar{v}_j$ so

$$g(x,y,y_i^\alpha) = \sum_{j=1}^{r} \left(\sum_{k=1}^{r} a_{kj}(x)\ G_k(x,y,y_i^\alpha)\right)\bar{v}_j$$

and it is clear that if each G_k has the form stated in 16.6 then so does each $\Sigma\, a_{kj}\, G_k$.

Next suppose we replace the coordinates $x_1,\dots,x_n$ by new coordinates $\bar{x}_1,\dots,\bar{x}_n$ and write $\bar{D}^\alpha = \partial^{|\alpha|}/\partial\bar{x}_1^{\alpha_1}\cdots\partial\bar{x}_1^{\alpha_n}$ Then by elementary calculus $D^\alpha = \sum_{\beta\le\alpha} \varphi^\alpha_\beta\, \bar{D}^\beta$ where the φ^α_β are C^∞ functions on M. Hence the new coordinates $\bar{y}^\alpha_i$ are related to the coordinates y^α_i by $y^\alpha_i = \sum_{\substack{\beta\le\alpha \\ |\beta|>0}} (\varphi^\alpha_\beta \circ \pi^k)\, \bar{y}^\beta_i$ and it follows easily if g has the proper form when expressed in terms of the coordinates (x,y,y^α_i) then it also does when expressed in terms of $(\bar{x},y,\bar{y}_i^{\,\alpha})$.

Finally consider the effect of changing the trivilization $\xi\approx M\times\mathbb{R}^m$ and the coordinates in $\mathbb{R}^m$. The new coordinates $(x_1,\dots,x_n,\ \bar{y}_1,\dots,\bar{y}_m)$ in ξ are related to the old coordinates $(x_1,\dots,x_n,\ y_1,\dots,y_m)$ by $v_i = \bar{y}_i\circ f$ where $f:\xi \longrightarrow \xi$ is a C^∞ fiber bundle automorphism, and clearly the induced coordinates $\bar{y}^{\,\alpha}_{\,i}$ in $J^k_\ell(\xi)$ are related to the coordinates y^α_i by $y^\alpha_i = \bar{y}_i^{\,\alpha}\circ J^k_\ell(f)$ $(|\alpha| > \ell)$. Hence for $|\alpha| > \ell$ we have by Corollary 16.5 that y^α_i is a sum of terms of the $(\Psi\circ\pi^k_\ell)\, \bar{y}^{\,\beta_1}_{\ell_1}\cdots\bar{y}^{\,\beta_q}_{\ell_q}$ where each $|\beta_i| > \ell$ and $|\beta_1|+\dots+|\beta_q| \le |\alpha|$. It then follows immediately by substitution that if g has the required form to be in $P^{w;\ell}_k(\xi,\zeta)$ when expressed in terms of (x,y,y^α_i), it also does when expressed in terms of $(x,\bar{y},\bar{y}^\alpha_i)$. This completes the proof that $P^{w;\ell}_k(\xi,\zeta)$ in Definition 16.6 is really well-defined and hence also completes the proof of Theorem 16.2. q.e.d.

<u>16.7 Theorem</u>. Let $\xi,\eta_0,\dots,\eta_q,\zeta$ be C^∞ vector bundles over M and let T be a C^∞ section of the vector bundle $L^{q+1}(\eta_0,\dots,{}_q;\cdot\zeta)$ of $q+1$ - linear maps of $\eta_0\oplus\dots\oplus\eta_q$ into ζ. Let $D_0 \in Df_\ell(\xi,\eta_0)$ and let $D_i \in \mathrm{Diff}_{s_i}(\xi,\eta_i)$ $i = 1,\dots,q$, where $0\le\ell< s_i$ $i = 1,\dots,q$, and define $D: C^\infty(\xi) \longrightarrow C^\infty(\zeta)$ by $D\sigma = T\,(D_0\sigma,\ D_1\sigma,\dots,D_q\sigma)$. Then $D \in Df^{w;\ell}_k(\xi,\zeta)$, where $k = \max\,(s_1,\dots,s_q)$ and $w = s_1+\dots+s_q$.

Proof. Obvious

16.8 Corollary. Let ξ and η be C^∞ vector bundles over M with η Riemannian and let $(\ ,\ )_x$ denote the inner product in η_x. Let $D_i \in \mathrm{Diff}_{s_i}(\xi,\eta)$, $i = 1.2$ and define $D: C^\infty(\xi) \longrightarrow C^\infty(\mathbb{R}_M)$ (where $\mathbb{R}_M$ denotes the trivial line bundle over M) by $D\sigma(x) = (D_1\sigma(x), D_2\sigma(x))_x$. Then $D \in Df_k^w(\xi, \mathbb{R}_M)$ where $k = \max(s_1, s_2)$ and $w = s_1 + s_2$.

We recall that by Lemma 13.8 every C^∞ fiber bundle E over M with paracompact fiber can be embedded as a closed sub-bundle of a C^∞ vector bundle ξ over M. From this fact together with the preceding theorem and part 2) of the following theorem we see how to construct many polynomial differential operators from E to ζ.

16.9 Theorem. Let E be a C^∞ fiber bundle over M and let η and ζ be C^∞ vector bundles over M.

1) If $D \in Df_k^{w;\ell}(E,\eta)$ and $f: \eta \longrightarrow \zeta$ is a C^∞ vector bundle homomorphism then $f_* \circ D \in Df_k^{w;\ell}(E,\zeta)$.

2) If $f: E \longrightarrow \eta$ is a C^∞ fiber bundle morphism and $D \in Df_k^{w;\ell}(\eta,\zeta)$ then $D \circ f_* \in Df_k^{w;\ell}(E,\zeta)$. In particular if E is a closed sub-bundle of η then (taking f above to be the inclusion) it follows that the restriction of D to $C^\infty(E)$ is in $Df_k^{w;\ell}(E,\zeta)$.

3) If D_1 is a quasi-linear differential operator of order k_1 from E to η and D_2 is a quasi-linear operator of order k_2 from η to ζ then D_2D_1 is a quasi-linear operator of order $k_1 + k_2$ from E to ζ.

Proof. 1) is obvious, 2) follows easily from 16.5 and 3) from an easy direct calculation.

Remark. Actually all three of the above conclusions are special cases of a considerably more general theorem which says that if $D_1 \in Df_{k_1}^{w_1;\ell_1}(E,\eta)$ and $D_2 \in Df_{k_2}^{w_2,\ell_2}(\eta,\zeta)$ then $D_2D_1 \in Df_{k_1+k_2}^{w;k_1+\ell_2}(E,\zeta)$ where however w is in general a little hard to describe.

The importance of the classes $Df_r^{w;\ell}(E,\eta)$ is in part due to theorems of the following sort.

16.10 Theorem. Let $1 \leq p < \infty$, $1 \leq q < \infty$, $0 \leq \ell < k - \frac{n}{p} < r \leq k$ and $r \leq m$. Let $D \in Df_r^{w;\ell}(E,\eta)$ where E is a C^∞ fiber bundle over a compact C^∞ n-dimensional manifold M and η is a C^∞ vector bundle over M. If $m > k$ assume $w \leq \dfrac{pr}{1-\frac{p(k-r)}{n}}$. Then D extends to a C^∞ map of $L_k^p(E)$ into $L_{k-m}^q(\eta)$ provided

$$\frac{n}{q} \geq (k-m) + \frac{w}{r}\left(\frac{n}{p} - (k-r)\right)$$

Proof. It will suffice to prove that D extends to a C^∞ map of $L_k^p(\xi)$ into $L_{k-m}^q(\eta)$ for each open vector sub-bundle ξ of E, so there is no loss of generality in assuming that $E = \xi$ is a vector bundle. Also we can as usual restrict to a coordinate neighborhood on M, so we can assume that $M = D^n$, $\xi = D^n \times \mathbb{R}^j$ and $\eta = D^n \times \mathbb{R}^t$. Then by definition of $Df_r^{w;\ell}(\xi,\eta)$ if $f = (f_1,\ldots,f_j)$ then $Df = ((Df)_1,\ldots,(Df)_t) \in C^\infty(\eta)$ where each $(Df)_i$ is a sum of terms of the form

$$(Df)_i = \Delta(f)(D^{\alpha_1}f_{s_1})\ldots(D^{\alpha_u}f_{s_u})$$

where $\Delta \in Df_\ell(\xi,\mathbb{R}_M)$, $\ell < |\alpha_i| \leq r$ and $\sum_{i=1}^{u} |\alpha_i| \leq w$. Now Δ extends to a C^∞ map of $L_k^p(\xi)$ into $L_{k-\ell}^p(M,\mathbb{R})$ by Theorem 15.7 (since $k-\ell > \frac{n}{p}$ and hence $\mathcal{M} = L_{k-\ell}^p$ satisfies B§5) while $f \longmapsto D^{\alpha_i} f_i$ is an element of $\mathrm{Diff}_{|\alpha_i|}(\xi, \mathbb{R}_M)$ and hence extends to a continuous linear map of $L^p_k(\xi)$ into $L^p_{k-|\alpha_i|}(M,\mathbb{R})$. Thus it will suffice to prove that multiplication is a continuous multilinear map of

$$L^p_{k-\ell}(M,\mathbb{R}) \oplus \bigoplus_{i=1}^{u} L^p_{k-|\alpha_i|}(M,\mathbb{R})$$

into $L^q_{k-m}(M,\mathbb{R})$. Since $k - |\alpha_i| = \frac{n}{p} - (|\alpha_i| - (k - \frac{n}{p}))$ the condition for this is, by Theorem 9.5, that

$$k-m \leq \frac{n}{q} - \sum_{|\alpha_i| > k - \frac{n}{p}} (|\alpha_i| - (k - \tfrac{n}{p}))$$

where in case some $|\alpha_i| = k - \frac{n}{p}$ we must have strict inequality. Now since $|\alpha_i| \leq r$, $|\alpha_i| \left(1 - \frac{k-\frac{n}{p}}{r}\right) \geq |\alpha_i| - (k - \frac{n}{p})$, and since $\Sigma\, |\alpha_i| \leq w$, $w\left(1 - \frac{k-\frac{n}{p}}{r}\right) \geq \sum_{|\alpha_i| > k - \frac{n}{p}} (|\alpha_i| - (k - \frac{n}{p}))$ and the inequality is strict unless all $|\alpha_i| = r$ (in which case, since $r > k - \frac{n}{p}$, no $|\alpha_i|$ equals $k - \frac{n}{p}$). Thus it always suffices that $\frac{n}{q} \geq (k-m) + w\left(1 - \frac{k - \frac{n}{p}}{r}\right)$. In case $m > k$ then an additional requirement of Theorem 9.5 is that $\sum_{|\alpha_i| > k - \frac{n}{p}} (|\alpha_i| - (k - \frac{n}{p})) < n$, and by the above it suffices that

$w\left(1 - \frac{k - \frac{n}{p}}{r}\right) \leq n$, or $w \leq \dfrac{pr}{1 - \frac{p(k-r)}{n}}$ q.e.d.

16.11 Corollary. If $1 \leq p < \infty$, $k > \frac{n}{p} + \ell$ and $D \in Df_k^{w;\ell}(E,\eta)$ where $w \leq pk$ then D extends to a C^∞ map of $L^p_k(E)$ into $L^1_0(\eta)$.

<u>16.12 Corollary</u>. If $1 < p < \infty$, $k < \frac{n}{p} + \ell$, and $D \in Df_k^{pk-r;\ell}(E,\eta)$, $0 \leq r \leq k$, then D extends to a C^∞ map of $L_k^p(E)$ into $L_{r-k}^{\bar{p}}(\eta)$ where $\frac{1}{\bar{p}} = 1 - \frac{1}{p}$.

<u>Proof</u>. First we must show that $pk-r \leq \frac{pr}{1-\frac{p(k-r)}{n}}$. But $\frac{p}{n} > \frac{1}{k}$ so $\frac{pr}{1-\frac{p(k-r)}{n}} \geq \frac{pr}{1-\frac{k-r}{k}} = pk \geq pk-r$. Secondly we must show that

$$n\left(1-\frac{1}{p}\right) \geq (r-k) + \frac{pk-r}{r}\left(\frac{n}{p} - (k-r)\right)$$

or that $n \geq \frac{pk}{r}\left(\frac{n}{p} - (k-r)\right)$, or that $\frac{k}{r}\left(1-\frac{p(k-r)}{n}\right) \leq 1$. But again since $\frac{p}{n} > \frac{1}{k}$, $1 - \frac{p(k-r)}{n} \leq 1 - \frac{k-r}{k} = \frac{r}{k}$.

q.e.d.

<u>16.13 Theorem</u>. Let $k > \frac{n}{2} + \ell$, $0 \leq \ell \leq w \leq 2k$, and let $D \in Df_r^{w;\ell}(E,\eta)$, where E is a C^∞ fiber bundle over a compact C^∞ n-dimensional manifold M and η is a C^∞ vector bundle over M. Then D extends to a C^∞ map of $L_k^2(E)$ into $L_{k-w}^2(\eta)$.

<u>Proof</u>. Similar to that of Theorem 16.10 but using Theorem 9.16 in place of Theorem 9.5.

There is another special class of differential operators from fiber bundles to vector bundles, whose importance derives from the fact that they occur as the left hand side of the Euler-Lagrange equations of the Calculus of Variations. They are often referred to as "differential operators in divergence form", which explains the symbol we choose for them.

<u>16.4 Definition</u>. Let E be a C^∞ fiber bundle over M and η a C^∞ vector bundle over M. We define a vector subspace $Div_{2k}(E,\eta)$ of $Df_{2k}(E,\eta)$ as follows:

$D \in \mathrm{Div}_{2k}(E,\eta)$ if and only if D can be written as a sum of operators of the form $L\,\Delta$ where for some vector bundle ξ (depending on the summand) $\Delta \in \mathrm{Df}_k(E,\xi)$ and $L \in \mathrm{Diff}_k(\xi,\eta)$. Similarly we define a vector subspace $\mathrm{Div}_{2k}^{w;\ell}(\xi,\eta)$ of $\mathrm{Df}_{2k}^{w;\ell}(E,\eta)$ as follows

$D \in \mathrm{Div}_{2k}^{w;\ell}(\xi,\eta)$ if and only if D can be written as a sum of terms of the form $L\Delta$ where $L \in \mathrm{Diff}_r(\xi,\eta)$ $0 \le r \le k$ and $\Delta \in \mathrm{Df}_k^{w;\ell}(E,\xi)$ if $r \le \ell$ and $\Delta \in \mathrm{Df}_k^{w-r;\ell}(E,\xi)$ if $r > \ell$.

Remark. If $F_j(x,y,y_i^{\alpha})$ is a parametric representation of $D \in \mathrm{Df}_{2k}(E,\eta)$ (see Definition 15.8) then it follows easily from Definition 16.1 that $D \in \mathrm{Div}_{2k}(E,\eta)$ if and only if each $F_j(x,s_i(x), D^{\alpha}s_i(x))$ can be written as a sum of terms of the form

$$A_{\alpha}(x)\, D^{\alpha}\, \Psi_{\alpha}(x,s_i(x), D^{\beta}s_i(x))$$

where A_{α} is a C^{∞} real valued function, $|\alpha| \le k$ and the $|\beta|$ are $\le k$. And $D \in \mathrm{Div}_{2k}^{w;\ell}(E,\eta)$ if moreover the functions $\Psi_{\alpha}(x,s_i(x), D^{\beta}s_i(x))$ can be written as a sum of terms of the form

$$\Phi(x,s_i(x),D^{\gamma}s_i(x))\, D^{\beta_1}s_{\ell_1}(x)\cdots D^{\beta_q}s_{\ell_q}(x)$$

where $|\gamma| \le \ell$, the $|\beta_i| > \ell$, and $\sum_{i=1}^{q} |\beta_i| \le w$, and $\sum_{i=1}^{q} |\beta_i| \le w - |\alpha|$ if $|\alpha| > \ell$.

16.15 Theorem. Let $1 < p < \infty$, $k > \frac{n}{p} + \ell$ and $D \in \mathrm{Div}_{2k}^{pk;\ell}(E,\eta)$ where E is a C^{∞} fiber bundle over a compact C^{∞} manifold M and η is a C^{∞} vector bundle over M. Then D extends to a C^{∞} map of $L_k^p(E)$ into $L_{-k}^{\bar{p}}(\eta)$ where $\frac{1}{\bar{p}} = 1 - \frac{1}{p}$.

Proof. We can assume $D = L\,\Delta$, where $L \in \mathrm{Diff}_r(E,\eta)$ for some $r \le k$ and $\Delta \in \mathrm{Df}_k^{w;\ell}(E,\xi)$, where $w = pk-r$ if $r > \ell$ and $w = pk$ if $r \le \ell$. Since L extends to a C^{∞} (and in fact continuous linear) map of $L_{r-k}^{\bar{p}}(\xi)$ into $L_{-k}^{\bar{p}}(\eta)$ it will suffice to prove that Δ extends to a C^{∞} map of $L_k^p(E)$ into $L_{r-k}^{\bar{p}}(\xi)$. If $r > \ell$, so that

$w = pk-r$, this is precisely the content of Corollary 16.12. On the other hand if $r \leq \ell$, so that $w = pk$, then by Theorem 16.10 it suffices to verify that

$$n\left(1 - \frac{1}{p}\right) > (r-k) + \frac{pk}{k}\left(\frac{n}{p} - (k-k)\right)$$

or $k \geq \frac{n}{p} + r$, which is clear since by hypothesis $k > \frac{n}{p} + \ell$ and $r \leq \ell$. q.e.d.

17. LINEARIZATION AND THE SYMBOL OF A DIFFERENTIAL OPERATOR

Let E_1 and E_2 be C^∞ fiber bundles over a compact n-dimensional C^∞ manifold M and let $D: C^\infty(E_1) \longrightarrow C^\infty(E_2)$ be an element of $Df_r(E_1, E_2)$, say $D = F_* j_r$, where $F: J^r(E_1) \longrightarrow E_2$ is a C^∞ fiber bundle morphism. Let $\mathfrak{M}$ satisfy (B§2) and (B§5) so that by Theorem 15.7 D extends to a C^∞ map of $\mathfrak{M}_r(E_1)$ into $\mathfrak{M}(E_2)$, which we continue to denote by D. If $s \in C^\infty(E_1)$ then dD_s is a linear map of $T(\mathfrak{M}_r(E_1))_s$ into $T(\mathfrak{M}(E_2))_{Ds}$. Now we have (Theorem 13.6) canonical identifications

$$T(\mathfrak{M}_r(E_1))_s = \mathfrak{M}_r(T_s(E_1)) \quad \text{and} \quad T(\mathfrak{M}(E_2))_{Ds} = \mathfrak{M}(T_{Ds}(E_2))$$

(recall that $T_s(E_1)$ is the vector bundle $s^* TF(E_1)$ over M, where $TF(E_1)$ is the vector bundle over E_1 whose fiber at e is the tangent space to the fiber of E_1 containing e). We shall see that there exists $\Lambda(D)_s \in \mathrm{Diff}_r(T_s(E_1), T_{Ds}(E_2))$, called the "linearization of D at s" such that $\Lambda(D)_s : C^\infty(T_s(E_1)) \longrightarrow C^\infty(T_{Ds}(E_2))$ extends to dD_s. Moreover this $\Lambda(D)_s$ does not depend on $\mathfrak{M}$ but only on D. Now the linear differential operator $\Lambda(D)_s$ has a symbol $\sigma_r(\Lambda(D)_s)$; this is a function on $T^*(M)$, the cotangent bundle of M, and for $(v,x) \in T^*(M)$ (i.e. $v \in T^*(M)_x$), $\sigma(\Lambda(D)_s)\,(v,x)$ is a linear map of the fiber of $T_s(E_1)_x$ (namely $T((E_1)_x)_{s(x)}$) into the fiber of $T_{Ds}(E_2)_x$ (namely $T((E_2)_x)_{Ds(x)}$). Moreover $\sigma_r(\Lambda(D)_s)\,(v,x)$ depends only on $j_r(s)(x)$ and v and we therefore will denote it by $\sigma_r(D)(j_r(s)(x), (v,x))$. Note that $\sigma_r(D)$ is a function defined on the total space of a certain C^∞ fiber bundle over M, namely the fiber product $J^r(E_1) \times_M T^*(M)$, which we denote by $T_r^*(E_1)$. Note also we have two natural maps $\pi: T_r^*(E_1) \longrightarrow E_1$ and $\bar{F}: T_r^*(E_1) \longrightarrow E_2$ defined by $\pi(j_r(s)(x), (v,x)) = s(x)$ and $\bar{F}(j_r(s)(x), (v,x)) = F\, j_r(s)(x) = Ds(x)$. These give rise to two C^∞ vector bundles over $T_r^*(E_1)$, namely $\pi^*(TF(E_1))$ and $\bar{F}^*(TF(E_2))$ whose fibers at $(j_r(s)(x), (v,x))$ are respectively $T_s(E_1)_x$ and $T_{Ds}(E_2)_x$. Thus $\sigma(D)$ is an element of $\mathrm{Hom}\,(\pi^*(TF(E_1)), \bar{F}^*(TF(E_2)))$. This vector bundle homomorphism over $T_r^*(E_1)$ is called the (r^{th} order) symbol of the differential operator D. We now consider the above more formally and in detail. We maintain the notation already introduced.

17.1 Theorem. If E is a C^∞ fiber bundle over M and $s \in C^\infty(E)$ then there is a natural isomorphism $J^k(T_s(E)) \approx T_{j_k(s)}(J^k(E))$. Here "natural" means that if $f: E \longrightarrow \bar{E}$ is a C^∞ fiber bundle morphism and $\bar{s} = f \circ s$ then there is commutativity in the diagram

$$\begin{tikzcd} J^k(T_s(E)) \arrow[r, phantom, "\approx"] \arrow[d, "J_k(\delta_s f)"'] & T_{j_k(s)}(J^k(E)) \arrow[d, "\delta_{j_k(s)} J^k(f)"] \\ J^k(T_{\bar{s}}(\bar{E})) \arrow[r, phantom, "\approx"] & T_{j_k(\bar{s})}(J^k(\bar{E})) \end{tikzcd}$$

This natural isomorphism is characterized by the fact that if ξ is a C^∞ vector bundle over M and $s \in C^\infty(\xi)$ then the isomorphism $J^k(T_s(\xi)) \approx T_{j_k(s)}(J^k(\xi))$ is the composite of the isomorphisms $J^k(\lambda): J^k(T_s(\xi)) \approx J^k(\xi)$ where $\lambda: T_s(\xi) \approx \xi$ is the natural isomorphism given by Theorem 12.1, and the inverse of the isomorphism $\mu: T_{j_k(s)}(J^k(\xi)) \approx J^k(\xi)$ also given by Theorem 12.1.

Proof. Because of the existence of vector bundle neighborhoods it suffices to prove the theorem for the category FVB, which means that we must simply show the commutativity of the diagram when E and $\bar{E}$ are C^∞ vector bundles and f is a C^∞ fiber bundle morphism, and the isomorphisms are given of course by the final statement of the theorem. Now if $\sigma \in C^\infty(E) = C^\infty(T_s(E))$ then $J^k(f)(j_k(s) + t\, j_k(\sigma)) = J^k(f)(j_k(s + t\,\sigma)) = j_k(f \circ (s + t\,\sigma))$ so that

$$(\delta_{j_k(s)} J^k(f))(j_k(\sigma)) = \frac{d}{dt}\Big|_{t=0} j_k(f \circ (s + t\,\sigma))$$

while $\delta_s f(\sigma) = \frac{d}{dt}\Big|_{t=o} f(s + t\sigma)$ so that

$$(J^k(\delta_s f))(f_k(\sigma)) = j_k(\delta_s f(\sigma)) = j_k\left(\frac{d}{dt}\Big|_{t=o} f(s + t\sigma)\right)$$

and since $j_k : C^\infty(\xi) \longrightarrow C^\infty(J^k(\xi))$ is linear, its permutability with $\frac{d}{dt}\Big|_{t=o}$ is clear.

q.e.d.

Next recall (see remarks following Theorem 12.1) that the C^∞ bundle morphism $F : J^r(E_1) \longrightarrow E_2$ has a "vertical differential" $\delta F: TF(J^r(E_1)) \longrightarrow F^*TF(E_2)$ and given $s \in C^\infty(E_1)$ this gives rise to a C^∞ vector bundle homomorphism $\delta_{j_r(s)}F = \delta F \circ j_r(s)$ of $T_{j_r(s)}(J^r(E_1))$ into $T_{F \circ j_r(s)}(E_2) = T_{Ds}(E_2)$. Combining the fact with Lemma 17.1 we have

<u>17.2 Theorem</u>. Given $s \in C^\infty(E_1)$, $\delta_{j_r(s)} F$ is a C^∞ vector bundle homomorphism of $J^r(T_s(E_1))$ into $T_{Ds}(E_2)$ and hence defines an element $\Lambda(D)_s \in \mathrm{Diff}_r(T_s(E_1), T_{Ds}(E_2))$ called the linearization of D at s. If ξ is a VBN of s in E_1 (so that we may identify $T_s(E_1)$ with ξ) then, for $\sigma \in C^\infty(\xi)$, $\Lambda(D)_s\sigma\ (x)$ depends only on $j_r(s)(x)$ and $j_r(\sigma)(x)$ and indeed is equal to $\delta F_{j_r(s)(x)}(j_r(\sigma)(x))$. Moreover we have $\Lambda(D)_s(\sigma)(x) = \frac{d}{dt}\Big|_{t=o}(D(s + t\sigma)(x))$.

<u>Proof</u>. Everything, except perhaps the final remark is easy. Now $D(s + t\sigma)(x) = F(j_r(s + t\sigma)(x)) = F(j_r(s)(x) + tj_r(\sigma)(x))$. Hence

$$\frac{d}{dt}\Big|_{t=o} D(s + t\sigma)(x) = dF_{j_r(s)(x)}(j_r(\sigma)(x)) = \Lambda(D)_s(\sigma)(x).$$

q.e.d.

<u>17.3 Theorem</u>. If $\mathcal{M}$ satisfies (B§2) and (B§5) so that by Theorem 15.7 D extends to a C^∞ map of $\mathcal{M}_r(E_1)$ into $\mathcal{M}(E_2)$ (still denoted by D) then for a $s \in C^\infty(E_1)$, $dD_s : T(\mathcal{M}_r(E_1))_s \longrightarrow T(\mathcal{M}(E_2))_{Ds}$ is an extension of $\Lambda(D)_s$. In fact more generally if $s \in \mathcal{M}_r(E_1)$ and if ξ_1 and ξ_2 are VBN of s and Ds in E_1 and E_2 respectively then $dD_s : \mathcal{M}_r(\xi_1) \longrightarrow \mathcal{M}(\xi_2)$ is given by $\sigma \longmapsto \delta_{j_r(s)} F(j_r(\sigma))$.

<u>Proof</u>. We can suppose ξ_1 chosen so small that $F(J^r(\xi_1)) \subseteq \xi_2$ so that D restricts to a C^∞ map of $\mathcal{M}_r(\xi_1)$ into $\mathcal{M}(\xi_2)$ which is the composite of the continuous linear map $j_r : \mathcal{M}_r(\xi_1) \longrightarrow \mathcal{M}(J^r(\xi_1))$ and the C^∞ map $\mathcal{M}(F|J^r(\xi_1)) : \mathcal{M}(J^r(\xi_1)) \longrightarrow \mathcal{M}(\xi_2)$ and the Theorem then follows from Theorem 13.6, the chain rule, and the fact that a linear map is its own differential. q.e.d.

<u>17.4 Corollary</u>. Let $s \in C^\infty(E_1)$, ξ_1 a VBN of s in E_1 and ξ_2 a vector bundle neighborhood of Ds in E_2. Then for $\sigma \in C^\infty(\xi_1)$

$$\frac{D(s + t\sigma) - Ds}{t}$$

converges in the C^∞ topology to $\Lambda(D)_s(\sigma)$ as $t \longrightarrow 0$.

<u>Proof</u>. Given $k \geq 0$ it will suffice to prove that the convergence is C^k. Choose $\mathcal{M} = C^k$ in 17.3. Then we have $dD_s : C^{k+r}(\xi_1) \longrightarrow C^k(\xi_2)$ is an extension of $\Lambda(D)_s$. Hence $\Lambda(D)_s(\sigma) = dD_s(\sigma) = \lim_{t \to 0} \frac{1}{t}(D(s + t\sigma) - Ds)$ in $C^k(\xi_2)$. q.e.d.

<u>17.5 Corollary</u>. Suppose E_2 is a C^∞ vector bundle over M and $D \in Df_r(E_1, E_2)$ extends to a C^∞ map of $\mathcal{M}(E_1)$ into $\overline{\mathcal{M}}(E_2)$ where

$\bar{\mathcal{M}}(E_2)$ is a Banach space of sections of E_2 which is the completion of $C^\infty(E_2)$ in a topology that is dominated by the C^∞ topology. Then for $s \in C^\infty(E_1)$ $dD_s : T(\mathcal{M}(E_1))_s \longrightarrow T(\bar{\mathcal{M}}(E_2))_{Ds}$ is an extension of $\Lambda(D)_s$.

<u>Proof</u>. Immediate from 17.4.

<u>Remark</u>. Note that Corollary 17.5 applies in particular to Theorems 16.10, 16.11, 16.12, 16.13, and 16.15.

We next derive an expression for $\Lambda(D)_s$ in local coordinates. For this purpose we can suppose $M = D^n$, and replace E_1 and E_2 with vector bundle neighborhoods of s and Ds respectively which we can identify with $M \times \mathbb{R}^m$ and $M \times \mathbb{R}^q$. Then a section s of E_1 is given by m real valued functions of $x = (x_1,\ldots,x_n) \in D^n$

$$s(x) = (s_1(x),\ldots,s_m(x))$$

and similarly Ds is given by q-real valued functions of x.

$$Ds(x) = ((Ds)_1(x),\ldots,(Ds)_q(x))$$

and to say $D \in Df_r(E_1,E_2)$ implies that there exist q C^∞ functions $F_j(x,y_i,y_i^\alpha)$ $(i = 1,\ldots m$ and α ranges over n-multi-indices with $|\alpha| \le r)$ such that

$$(Ds)_j(x) = F_j(x,s_i(x),\ D^\alpha s_i(x))$$

The F_j we recall (from Definition 15.8) are called the parametric representation of D. Now $\Lambda(D)_s \in \mathrm{Diff}_r(E_1,E_2)$ so that similarly there are q C^∞ functions $L_j(x,y_i,y_i^\alpha)$ such that if $\sigma = (\sigma_1,\ldots,\sigma_m)$ is a C^∞ section of E_1 then

$$\Lambda(D)_s(\sigma)(x) = (L_1(x,\sigma_i(x),D^\alpha\sigma_i(x)),\ldots,L_q(x,\sigma_i(x),\ D^\alpha\sigma_i(x))).$$

Moreover since $\Lambda(D)_s$ is a linear differential operator, the functions $L_j(x,y_i,y_i^\alpha)$ are linear in (y_i,y_i^α), i.e. there exists C^∞ functions of x, A_i^j and $A_{\alpha,i}^j$ such that

$$L_j(x,y_i,y_i^\alpha) = \sum_i A_i^j(x)y_i + \sum_{\alpha,i} A_{\alpha,i}^j(x)y_i^\alpha$$

and the problem is to express these A_i^j and $A_{\alpha,i}^j$ in terms of the F_j and s. The answer as we shall see is $A_i^j(x) = \frac{\partial F_j}{\partial y_i}(x,s_i(x), D^\alpha s_i(x))$ and $A_{\alpha,i}^j(x) = \frac{\partial F_j}{\partial y_i^\alpha}(x,s_i(x), D^\alpha s_i(x))$. We state this as:

__17.6 Theorem__. Let $D \in Df_r(E_1,E_2)$ be given parametrically by

$$Ds(x) = (F_1(x,s_i(x), D^\alpha s_i(x)),\dots,F_q(x,s_i(x), D^\alpha s_i(x)))$$

as discussed above. Then $\Lambda(D)_s$ is given parametrically by:

$$\Lambda(D)_s(\sigma)(x) = (L_1(x,\sigma_i(x), D^\alpha\sigma_i(x)),\dots,L_q(x,\sigma_i(x), D^\alpha\sigma_i(x)))$$

where:

$$L_j(x,y_i,y_i^\alpha) = \sum_i \frac{\partial F_j}{\partial y_i}(x,s_i(x), D^\alpha s_i(x))y_i + \sum_{\alpha,i} \frac{\partial F_j}{\partial y_i^\alpha}(x,s_i(x), D^\alpha s_i(x))y_i^\alpha .$$

__Proof__. By Theorem 17.2 we have

$$\Lambda(D)_s(\sigma)\ (x) = \frac{d}{dt}\Big|_{t=o} D(s + t\,\sigma)(x), \quad \text{hence}$$

$$L_j(x,\sigma_i(x), D^\alpha\sigma_i(x)) = \frac{d}{dt}\Big|_{t=o} F_j(x,s_i(x) + t\sigma_i(x), D^\alpha s_i(x) + t\, D^\alpha\sigma_i(x))$$

from which the desired result is immediate from the "chain rule". q.e.d.

We next recall the definition of the symbol of a linear differential operator. For details see Chapter IV, §3 of Seminar on the Atiyah-Singer Index Theorem, Annals of Mathematics Studies, No. 57. Let ξ_1 and ξ_2 be C^∞ vector bundles over M and let $L \in \mathrm{Diff}_r(\xi_1,\xi_2)$. Then if v is a cotangent vector of M at x (we write $(v,x) \in T^*(M)$) the symbol of L at (v,x) is a linear map $\sigma_r(L)(v,x)$ of $(\xi_1)_x$ into $(\xi_2)_x$ defined as follows: choose any C^∞ function g on M such that $g(x) = 0$ and $dg_x = v$ and given $e \in (\xi_1)_x$ choose any C^∞ section f of ξ_1 such that $f(x) = e$; then $\sigma_r(L)(v,x)\, e = \frac{1}{r!}\, L(g^r f)(x)$.

If $\pi: T^*(M) \longrightarrow M$ is the canonical projection then we have bundles $\pi^*\xi_1$ and $\pi^*\xi_2$ over $T^*(M)$ whose fibers at (v,x) are respectively the fibers of ξ_1 and ξ_2 at $\pi(v,x)$; namely $(\xi_1)_x$ and $(\xi_2)_x$. Hence $\sigma_r(L) \in \mathrm{Hom}(\pi^*\xi_1,\pi^*\xi_2)$ (of course one must check that $\sigma_r(L)$ is C^∞ in the appropriate sense). Notice that it is immediate from the definition of the symbol that if $L = T_* j_r$ where T is a vector bundle homomorphism of $J^r(\xi_1)$ into ξ_2, then $\sigma_r(L)(v,x)$ depends on T only through its value at x. Indeed $\sigma_r(L)\,(v,x)e = T_x(j_r(\frac{1}{r!}\, g^r f)(x))$ where g and f are as above.

If we identify a neighborhood U of x in M with D^n via some chart and identify ξ_1 and ξ_2 over this neighborhood with $D^n \times \mathbb{R}^m$ and $D^n \times \mathbb{R}^q$ then L is represented locally as $L = \sum_{|\alpha| \le r} A^\alpha(x) D^\alpha$ where the A^α are C^∞ maps of D^n into the space $L(\mathbb{R}^m, \mathbb{R}^q)$ of linear maps of $\mathbb{R}^m$ into $\mathbb{R}^q$. If $s = (s_1,\ldots,s_m)$ is a section of ξ_1 over D^n and $Ls = ((Ls)_1,\ldots,(Ls)_q)$ then $Ls(x) = \sum_{|\alpha| \le r} A^\alpha(x)\, D^\alpha s(x)$ or in terms of components

$(Ls)_i(x) = \sum_{\alpha \le r} \sum_{j=1} A^\alpha_{ij}(x)\, D^\alpha s_j(x)$ where $A^\alpha_{ij}(x)$ is the usual matrix representation of the linear map $A^\alpha(x): \mathbb{R}^m \longrightarrow \mathbb{R}^q$. Now let v be a cotangent vector of M over $p \in U$ and let $v = \sum_{i=1} v_i (dx_i)_p$. Then it follows easily that

$$\sigma_r(L)\,(v,p) = \sum_{|\alpha| = r} v^\alpha A^\alpha(p)$$

where $v^\alpha = v_1^{\alpha_1} v_2^{\alpha_2} \cdots v_n^{\alpha_n}$ (note the sum is over all α with $|\alpha| = r$, <u>not</u> $|\alpha| \le r$). In matrix form

$$\sigma_r(L)(v,p)_{ij} = \sum_{|\alpha|=r} v^\alpha A^\alpha_{ij}(p)$$

Now let us return to our non-linear differential operator $D = F_* j_r \in Df_r(E_1,E_2)$ where $F:J^r(E_1) \longrightarrow E_2$ is a C^∞ fiber bundle morphism. Given $s \in C^\infty(E_1)$ we have the linearization

$\Lambda(D)_s = \delta_{j_r(s)} F \circ j_r \in \mathrm{Diff}_r\ (T_s(E_1), T_{Ds}\ (E_2))$. Then given

$(v,x) \in T^*(M)$ we have $\sigma_r(\Lambda(D)_s)\ (v,x)$ which is a linear map $T_s(E_1)_x = TF(E_1)_{s(x)}$ into $T_{Ds}(E_2')_x = TF(E_2)_{Ds(x)}$. Moreover as we have seen, aside from $(v,x), \sigma_r(\Lambda(D)_s)\ (v,x)$ depends only on $\delta_{j_r(s)}F$ at x, i.e. only on F and $j_r(s)(x)$ or equivalently only on D and $j_r(s)(x)$. Hence it makes sense to define:

$$\sigma_r(D)(j_r(s)_x,\ (v,x)) = \sigma_r(\Lambda(D)_s)(v,x).$$

Then $\sigma_r(D)$ is a function on the fiber product $T^*_r(E_1) = J^r(E_1) \times_M T^*(M)$. We have maps $\pi:T^*_r(E_1) \longrightarrow E_1$ and $\overline{F}: T^*_r(E_1) \longrightarrow E_2$ given by $\pi(j_r(s)_x,\ (v,x)) = s(x)$ and $\overline{F}(j_r(s)_x,\ (v,x)) = F\ j_r(s)_x = Ds(x)$, hence the fibers of the vector bundles $\pi^*TF(E_1)$ and $\overline{F}^*TF(E_2)$ at $(j_r(s)_x,\ (v,x))$ are respectively $TF(E_1)_{s(x)}$ and $TF(E_2)_{Ds(x)}$ so that $\sigma_r(D)$ is a homomorphism of $\pi^*TF(E_1)$ into $\overline{F}^*TF(E_2)$. For reference we restate all this as

<u>17.7 Definition</u>. Let E_1 and E_2 be C^∞ fiber bundles over M and let $D \in Df_r(E_1,E_2)$, say $D = F_* \circ j_r$ where $F:J^r(E_1) \longrightarrow E_2$ is a C^∞ fiber bundle morphism. Let $T^*_r(E_1)$ denote the fiber bundle over M which is the fiber product of $J^r(E_1)$ and $T^*(M)$ and define maps π and $\overline{F}$ of $T^*_r(E_1)$ into E_1 and E_2 respectively by $\pi(j_r(s)_x,(v,x))=s(x)$ and $\overline{F}(j_r(s),(v,x)) = F(j_r(s)_x = Ds(x)$. Then the symbol of D, $\sigma_r(D)$, is defined to be the element of $\mathrm{Hom}(\pi^*TF(E_1), \overline{F}^*\ TF(E_2))$ given by

$$\sigma_r(D)(j_r(s)_x, (v,x)) = \sigma_r(\Lambda_r(D)_s)\ (v,x)$$

where $\Lambda_r(D)_s$ is the linearization of D at s (Theorem 17.2).

We next describe how to compute $\sigma_r(D)$ in "local coordinates".

<u>17.8 Theorem</u>. Let $M = D^n$, $E_1 = M \times \mathbb{R}^m$, $E_2 = M \times \mathbb{R}^q$ and let $D \in Df_r(E_1, E_2)$ be given by:

$$Ds(x) = (F_1(x, s_i(x), D^\alpha s_i(x)), \ldots, F_q(x, s_i(x), D^\alpha s_i(x)))$$

where $F_j(x, y_i, y_i^\alpha)$ are C^∞ functions of $x = (x_1, \ldots, x_n)$, $(y_1, \ldots, y_m)$ and $y_i^\alpha (i=1, \ldots, m$ and α ranges over all n-multi-indices with $1 \leq |\alpha| \leq r)$. Then since each fiber of $TF(E_1)$ is clearly canonically isomorphic to $\mathbb{R}^m$ and each fiber of $TF(E_2)$ is canonically isomorphic to $\mathbb{R}^q$, $\sigma_r(D)(j_r(s)_x, (v,x))$ is given by a $q \times m$ matrix $\sigma_{ij}(x, j_r(s)_x, (v,x))$. If $v = \sum_{i=1}^{n} v_i dx_i$ then this matrix is given explicitly by the formula:

$$\sigma_{ij}(x, j_r(s)_x, (v,x)) = \sum_{|\alpha|=r} v^\alpha \frac{\partial F_j}{\partial y_i^\alpha} (x, s_i(x), D^\alpha s_i(x))$$

where $v^\alpha = v_1^{\alpha_1} \ldots v_n^{\alpha_n}$.

<u>Proof</u>. Immediate from Theorem 17.6 and the remarks above which describe the explicit formula for the symbol of a linear operator in matrix form. q.e.d.

<u>Remark</u>. Let ξ_1 and ξ_2 be C^∞ vector bundles over M and let $D = F_* \circ j_r \in \mathrm{Diff}_r(\xi_1, \xi_2)$ where $F: J^r(\xi_1) \longrightarrow \xi_2$ is a C^∞ vector bundle morphism. Then it is natural to ask what the connection is between $\sigma_r(D) \in \mathrm{Hom}\,(p^*\xi_1, p^*\xi_2)$ (where $p: T^*(M) \longrightarrow M$ is the natural projection) and

$\sigma_r(D) \in \mathrm{Hom}\,(\pi^*TF(\xi_1), \overline{F}^*TF(\xi_2))$, where as above $\pi: T_r^*(\xi_1) \longrightarrow \xi_1$ is the natural map. Now if $q: T_r^*(\xi_1) = J^r(\xi_1) \times_M T^*(M) \longrightarrow T^*(M)$ is the natural projection then $q^*p^*(\xi_1) = \pi^*TF(\xi_1)$ (the fiber of each at $(j_r(s)_x, (v,x))$ is canonically isomorphic to $(\xi_1)_x$) and $q^*p^*(\xi_2) = \overline{F}^*TF(\xi_2)$ so we can regard $\sigma_r(D)$ (when D is considered as a non-linear operator) as an element of $\mathrm{Hom}\,(q^*p^*(\xi_1), q^*p^*(\xi_2))$. Now composition with q maps $\mathrm{Hom}\,(p^*\xi_1, p^*\xi_2)$ into $\mathrm{Hom}\,(q^*p^*(\xi_1), q^*p^*(\xi_2))$ and the formula "$\sigma_r(D) = \sigma_r(D) \circ q$" expresses the relation between the non-linear $\sigma_r(D)$ (on the left) and the linear $\sigma_r(D)$ (on the right). The point is that since D is linear, $\Lambda(D)_s = D$ for all $s \in C^\infty(\xi_1)$, so since $\Lambda(D)_s$ is independent of s, $\sigma_r(D)\ (j_r(s), (v,x)) = \sigma_r(\Lambda(D)_s)\ (v,x)) = \sigma_r(D)\ (v,x)$ does not depend on the $j_r(s)$ component, i.e. it is lifted from $T^*(M)$.

<u>17.9 Definition</u> An element $D \in Df_r(E_1,E_2)$ is called an <u>elliptic</u> differential operator of order r from E_1 to E_2 if for all $(j_r(s), (v,x)) \in T_r^*(E_1)$ with $v \neq 0$, $\sigma_r(D)(j_r(s),(v,x)): T((E_1)_x)_{s(x)} \longrightarrow T((E_2)_x)_{Ds(x)}$ is a linear isomorphism. We will denote the set of all such D by $\mathrm{Elptc}_r(E_1,E_2)$.

<u>Remark</u>. Clearly $\mathrm{Elptc}_r(E_1,E_2)$ is non empty only if E_1 and E_2 have the same fiber dimension.

Recall that if ξ_1 and ξ_2 are C^∞ vector bundles over M then the set $\mathrm{Ell}_r(\xi_1,\xi_2)$ of r^{th} order elliptic linear differential operators from ξ_1 to ξ_2 is the subset of $D \in \mathrm{Diff}_r(\xi_1,\xi_2)$ such that $\sigma_r(D)\ (v,x)$ is a linear isomorphism of $(\xi_1)_x$ onto $(\xi_2)_x$ for all $(v,x) \in T^*(M)_x$ with $v \neq 0$. Clearly $\mathrm{Ell}_r(\xi_1,\xi_2) \subseteq \mathrm{Elptc}_r(\xi_1,\xi_2)$. Moreover

<u>17.10 Theorem</u>. If $D \in \mathrm{Elptc}_r(E_1,E_2)$ then $\Lambda(D)_s \in \mathrm{Ell}_r(T_s(E_1), T_{Ds}(E_2))$ for all $s \in C^\infty(E_1)$. Conversely if each element of $J^r(E_1)$ is the r-jet of a global section of E_1 then $D \in \mathrm{Elptc}_r(E_1,E_2)$ provided each linearization of D is elliptic.

Proof. Immediate from the fact that $\sigma_r(D)(j_r(s)_x,(v,x)) = \sigma_r(\Lambda(D)_s)(v,x)$.

q.e.d.

Recall that $T_r^*(E_1) = J^r(E_1) \times_M T^*(M)$ as well as being a C^∞ fiber bundle over M is a C^∞ vector bundle (under the natural projection) over $J^r(E_1)$ whose fiber over $j_r(s)_x$ is just $T^*(M)_x$ (for indeed one definition of the total space of $T_r^*(E_1)$ is the total space of $p^*(T^*(M))$ where $p: J^r(E_1) \longrightarrow M$ is the natural projection). We will denote by $\hat{T}_r^*(E_1)$ the Thom space of this vector bundle. For convenience we fix a Riemannian metric in M (which induces a Riemannian structure on $T_r^*(E_1)$) and identify $\hat{T}_r^*(E_1)$ with $BT_r^*(E_1)/ST_r^*(E_1)$, i.e. the unit ball bundle of $T_r^*(E_1)$ with the unit sphere bundle identified to a point.

Recall that if ξ_1 and ξ_2 are vector bundles over a "reasonable" space X and there is given a vector bundle isomorphism $f: \xi_1|A \approx \xi_2|A$ where A is a "reasonable" closed subspace then there is defined a "difference element" $d(\xi_1,\xi_2,f) \in K(X,A) = \tilde{K}(X/A)$. (See Seminar on the Atiyah-Singer Index Theorem, Chapter II, §3). Now if $D \in \mathrm{Elptc}_r(E_1,E_2)$ then $\pi^*TF(E_1)$ and $\bar{F}^*TF(E_2)$ are C^∞ vector bundles over $BT_r^*(E_1)$ and, by definition of elliptic, $\sigma_r(D)$ defines an isomorphism of their restrictions to $ST_r^*(E_1)$. Thus we have a difference element

$d(\pi^*TF(E_1), \bar{F}^*TF(E_2), \sigma_r(D)|ST_r^*(E_1)) \in \tilde{K}(\hat{T}_r^*(E_1))$.

Now the natural projection $p: J^r(E_1) \longrightarrow E_1$ is a homotopy equivalence (in fact it can be given the structure of a vector bundle, although not in a natural way) hence, by the covering homotopy theorem, since $T_r^*(E_1) = p^*\, T_o^*(E_1)$, there is an induced homotopy equivalence $p^*:\hat{T}_r^*(E_1) \longrightarrow \hat{T}_o^*(E_1)$ which induces an isomorphism $p^!: \tilde{K}(\hat{T}_o^*(E_1)) \approx \tilde{K}(\hat{T}_r^*(E_1))$.

<u>17.11 Definition.</u> If $D \in \mathrm{Elptc}_r(E_1,E_2)$ then we define an element $\gamma(D) \in \tilde{K}(\hat{T}_o^*(E_1))$ by $(p^!)^{-1}\, d(\pi^*TF(E_1), \bar{F}^*TF(E_2), \sigma_r(D)|ST_r^*(E_1))$.

If $s \in C^\infty(E_1)$ then since $\Lambda(D)_s \in \mathrm{Ell}_r(T_s(E_1), T_{Ds}(E_2))$ it defines an element $\gamma(\Lambda(D)_s) \in \tilde{K}(\hat{T}^*(M))$ by $\gamma(\Lambda(D)_s = d(q^*T_s(E_1), q^*T_{Ds}(E_2), \sigma_r(\Lambda(D)_s)|ST^*(M))$ where $q:T^*(M) \longrightarrow M$. Clearly $\gamma(\Lambda(D)_s) = \tilde{s}^*\gamma(D)$ where $\tilde{s}:\hat{T}^*(M) \longrightarrow \hat{T}_o^*(E_1)$ is induced by the map $(v,x) \longmapsto (s(x), (v,x))$ of $T^*(M)$ into $T_o^*(E_1)$. This follows from the definition of $\sigma_r(D)$ in terms of $\sigma_r(\Lambda(D)_s)$ (cf. Def. 17.7) and the functoriality of the difference construction (S.A.S.I.T. Chapter II, §3, Lemma 1, statement (i)).

18. THE INDEX OF A NON-LINEAR ELLIPTIC OPERATOR

In this section, given C^∞ fiber bundles E_1 and E_2 over a compact C^∞ manifold M without boundary we will associate to each $D \in \mathrm{Elptc}_r(E_1, E_2)$ an element $i_a(D) \in K(C^\infty(E_1))$ called the analytic index of D. If we choose a "base point" s_o for $C^\infty(E_1)$ then we have a canonical augmentation map $\dim: K(C^\infty(E_1)) \longrightarrow Z$ and $\dim i_a(D)$ turns out to be the usual analytic index of the linearized elliptic operator $\Lambda(D)_{s_o} : C^\infty(T_{s_o}(E_1)) \longrightarrow C^\infty(T_{Ds_o}(E_2))$, i.e. $\dim \ker(\Lambda(D)_{s_o}) - \dim \operatorname{coker}(\Lambda(D)_{s_o})$. Of course if E_1 and E_2 are vector bundles and $D \in \mathrm{Ell}_r(E_1, E_2)$ then $C^\infty(E_1)$ is contractible so $\dim: K(C^\infty(E_o)) \longrightarrow K(pt) = Z$ is an isomorphism and $\Lambda(D)_{s_o} = D$ so in this case there is no point in distinguishing between $i_a(D)$ as an element of $K(C^\infty(E_1))$ and as an element of Z. However in the general case, when $C^\infty(E_1)$ is not homotopically trivial, $i_a(D)$ can carry more information about D than does its "dimension" and it is presumably one of the important homotopy invariants of D. As we shall also see $i_a(D) \in K(C^\infty(E_1))$ is completely determined by the element $\gamma(D) \in \tilde{K}(\hat{T}_o^*(E_1))$ defined in Definition 17.11 and the "index problem" for non-linear elliptic operators is the problem of finding an explicit formula for $i_a(D)$ in terms of $\gamma(D)$ (this is of course solved in the case of linear D by the Atiyah-Singer Index Theorem). As we shall also see the "index problem" for non-linear elliptic operators can be interpreted as a special case of the index problem for what is called a "parameterized family of linear elliptic operators" (provided the parameter space is allowed to be suitably general). The latter problem has been solved, at least with suitable restrictions on the parameter space, by W. Shih, L. Illusie, and others, and it seems likely that their results will lead to a solution of the index problem for non-linear elliptic operators.

First we must explain how K is defined for arbitrary spaces. The basic condition we impose is that K should be a functor from spaces and homotopy classes of maps to abelian groups, which for compact spaces is naturally isomorphic to the Grothendieck group of vector bundles and which is representable, i.e. $K(X)$ should be naturally equivalent to $[X,C]$, the homotopy classes of maps of X into a homotopy abelian H-space C. This uniquely determines C up to homotopy type, namely $C = Z \times B_G$ where B_G means the classifying space of G, and G is either $\varinjlim O(n)$ or $\varinjlim U(n)$ depending on whether we mean K_O or K_U (for the most part we shall mean K_O except when we explicitly complexify).

By a theorem proved independently by K. Jänich (Thesis, Bonn, 1964) and M. F. Atiyah (K Theory, Lectures by M. F. Atiyah, Notes by D. W. Anderson, Mimeo Harvard Math. Dept., 1964) there is a particularly nice choice of C for our purposes. Namely let H be a separable, infinite dimensional Hilbert space (real or complex according as we want K_0 or K_U) and let Fred(H) denote the space of Fredholm operators on H , i.e. bounded linear maps. $T:H \longrightarrow H$ such that ker T and coker T are finite dimensional (T(H) is automatically closed in H so we can take coker $T = (T(H))^{\perp} = \ker(T^*))$. Fred(H) is topologized as a subspace of the space of all bounded operators on H with the usual norm topology $(||T|| = \mathrm{Sup}\,\{||Tx|| \mid ||x|| = 1\})$. If X is a space then a continuous map $f:X \longrightarrow \mathrm{Fred}(H)$ is called admissible if $\ker(f) = \{(v,x) \in H \times X \mid v \in \ker f(x)\}$ and $\mathrm{coker}(f) = \{(v,x) \in H \times X \mid v \in \mathrm{im}\, f(x)^{\perp}\}$ are vector bundles over X under the obvious projection. Then (cf. Atiyah, loc. cit.) if X is a <u>compact</u> space and $\gamma \in [X,\mathrm{Fred}(H)]$ then γ has an admissible representative g and the element $\mathrm{ind}(\gamma) = [\ker(g)] - [\mathrm{coker}(g)]$ of the Grothendieck group K(X) of vector bundles of vector bundles over X is well defined (i.e. independent of the choice of admissible $g \in \gamma$), and $\mathrm{ind}:[X,\mathrm{Fred}(H)] \longrightarrow K(X)$ is a bijection. Finally Fred(H) is homotopy abelian H-space under usual operator composition, making [X,Fred(H)] an abelian group, and ind is even a group isomorphism. Henceforth we will define $K(X) = [X,\mathrm{Fred}(H)]$ for arbitrary spaces X .

Next let X be a paracompact space and let B_1 and B_2 be Hilbert space bundles over X (with GL(H), the general linear group of Hilbert space with the norm topology as structural group). A Hilbert bundle morphism $f:B_1 \longrightarrow B_2$ will be called a <u>Fredholm bundle morphism</u> if $f_x:(B_1)_x \longrightarrow (B_2)_x$ is a Fredholm map for each $x \in X$. In this case we define $\mathrm{ind}(f) \in K(X)$ as follows: by a Theorem of N. Kuiper (Topology, vol. 3 (1964), pp. 19-30) GL(H) is contractible, from which it follows that there exist bundle isomorphisms $g:X \times H \approx B_1$ and $h:B_2 \approx X \times H$. Then $x \longmapsto h_x f_x g_x$ is a map $hfg:X \longrightarrow \mathrm{Fred}(H)$. Moreover, again by the contractibility of GL(H) it follows that g and h are well determined up to homotopy and hence the homotopy class of hfg is a well determined element of $[X,\mathrm{Fred}(H)] = K(X)$ which we denote by ind(f) .

<u>18.1</u> <u>Definition</u>. Let X and Y be C^1 Hilbert manifolds and let $f:X \longrightarrow Y$ be a C^1-map. We say f is a <u>Fredholm map</u> if $df:T(X) \longrightarrow f^*T(Y)$ is a Fredholm bundle morphism over X and in this case we define $\mathrm{ind}(f) \in K(X)$ by $\mathrm{ind}(f) = \mathrm{ind}(df)$. More generally if $j:\Omega \longrightarrow X$ is a continuous map we say f is a <u>Fredholm map relative to</u> j if $df \circ j: j^*T(X) \longrightarrow j^*f^*\,T(Y)$ is a Fredholm bundle morphism (i.e. if $df_{j(\omega)}: T(X)_{j(\omega)} \longrightarrow T(Y)_{fj(\omega)}$ is a Fredholm operator for each $\omega \in \Omega$) and in this case we define $\mathrm{ind}(f,j) \in K(\Omega)$ by $\mathrm{ind}(f,j) = \mathrm{ind}(df \circ j)$.

<u>Remark</u>. If $f:X \longrightarrow Y$ is a Fredholm map then clearly f is a Fredholm map relative to any $j:\Omega \longrightarrow X$ and then $\mathrm{ind}(f,j) = j^*\mathrm{ind}(f)$ where $j^*:K(X) \longrightarrow K(\Omega)$ is the functorial K(j) .

We are now in a position to define the index of a non-linear elliptic differential operator.

18.2 Theorem and Definition. Let M be a compact, n-dimensional C^∞ manifold without boundary, let E_1 and E_2 be C^∞ fiber bundles over M and let $D \in \mathrm{Elptc}_r(E_1,E_2)$. Let $k > \frac{n}{2} + r$, so by Theorem 15.7 $D: C^\infty(E_1) \longrightarrow C^\infty(E_2)$ extends to a C^∞ map of Hilbert manifolds $D_{(k)}: L_k^2(E_1) \longrightarrow L_{k-r}^2(E_2)$. Then $D_{(k)}$ is a Fredholm map relative to the inclusion map $\lambda_k: C^\infty(E_1) \longrightarrow L_k^2(E_1)$ and moreover $\mathrm{ind}(D_{(k)}, \lambda_k) \in K(C^\infty(E_1))$ is independent of k and hence defines an element $i_a(D) \in K(C^\infty(E_1))$ called the analytic index of D.

Proof. Given $s \in C^\infty(E_1)$ let $\xi_1 = T_s(E_1)$ and $\xi_2 = T_{Ds}(E_2)$ so $T(L_k^2(E_1))_s = L_k^2(\xi_1)$, $T(L_{k-r}^2(E_2))_{Ds} = L_{k-r}^2(\xi_2)$, $T(L_{k-r}^2(E_2))_{Ds} = L_{k-r}^2(\xi_2)$, and $d(D_{(k)})_s: L_k^2(\xi_1) \longrightarrow L_{k-r}^2(\xi_2)$ is the continuous extension of $\Lambda(D)_s: C^\infty(\xi_1) \longrightarrow C^\infty(\xi_2)$ by Theorem 17.3. Now by Theorem 17.10 $\Lambda(D)_s \in \mathrm{Ell}_r(\xi_1, \xi_2)$, hence by a standard result (e.g. see Theorem 6 of Chapter XI of Seminar on the Atiyah-Singer Index Theorem, referred to as S.A.S.I.T. below) it is a Fredholm operator and hence by definition $D_{(k)}$ is Fredholm relative to λ_k.

The proof that $\mathrm{ind}(D_{(k)}, \lambda_k) = \mathrm{ind}(D_{(k+1)}, \lambda_{k+1})$ depends on singular integral operators and will only be sketched. By S.A.S.I.T., Chapter XI, Theorem 13 we can choose $L_s^1: C^\infty(\xi_1) \longrightarrow C^\infty(\xi_1)$ an element of $\mathrm{Int}_1(\xi_1, \xi_1)$ such that $\sigma_1(L_s^1)(v,x)e = ||v||e$, and so that L_s^1 maps $L_{k+1}^2(\xi_1)$ isomorphically onto $L_k^2(\xi_1)$. Moreover we can choose L_s^1 continuously with s so that we have a Hilbert bundle isomorphism over $C^\infty(E_1)$

$$L^1 : \lambda_{k+1}^* T(L_{k+1}^2(E_1)) \longrightarrow \lambda_k^* T(L_k^2(E_1)) \quad .$$

Similarly there is a Hilbert bundle isomorphism over $C^\infty(E_1)$

$$L^2 : \lambda_{k+1}^* D_{(k+1)}^* T(L_{k+1-r}^2(E_2)) \longrightarrow \lambda_k^* D_k^* T(L_{k-r}^2(E_2))$$

which for $s \in C^\infty(E_1)$ is given by an $L_{Ds}^2 \in \mathrm{Int}_1(\xi_2, \xi_2)$ whose symbol at (v,x) is multiplication by $||v||$. Now since L^1 and $(L^2)^{-1}$ map into $GL(H)$, which is a contractible subset of $\mathrm{Fred}(H)$, $\mathrm{ind}(L_2^{-1}) = \mathrm{ind}(L_1) = 0$, hence $\mathrm{ind}(L^2)^{-1} \circ dD_{(k)} \circ L^1) = \mathrm{ind}((L^2)^{-1}) + \mathrm{ind}(D_k,\lambda_k) + \mathrm{ind}(L^1) = \mathrm{ind}(D_k,\lambda_k)$

Thus it will suffice to show that $(L^2)^{-1} \circ dD_{(k)} \circ L^1$ and $dD_{(k+1)}$ have the same index, i.e. are homotopic as Fredholm bundle morphisms of $\lambda^*_{k+1} T(L^2_{k+1}(E_1)) \longrightarrow \lambda^*_{k+1} D^*_{(k+1)} T(L^2_{k+1-r}(E_2))$. But over each $s \in C^\infty(E_1)$ their difference is given by an extension of $\Lambda(D)_s - (L^2_{Ds})^{-1} \circ \Lambda(D)_s \circ L^1_s$ which is an element of $\mathrm{Int}_r(\xi_1, \xi_2)$ whose r^{th} order symbol is clearly zero and which hence is an operator of order $r-1$ and therefore defines a compact linear map of $L^2_{k+1}(\xi_2)$. Thus the result follows from the following lemma.

<u>18.3.</u> <u>Lemma.</u> Let $f_0, f_1: X \longrightarrow \mathrm{Fred}(H)$ be continuous maps such that $f_1(x) - f_0(x)$ is a compact operator on H for all $x \in X$. Then f_0 and f_1 are homotopic maps of X into $\mathrm{Fred}(H)$ and hence define the same element of $K(X)$.

<u>Proof</u>. Let $f_t(x) = f_0(x) + t(f_1(x) - f_0(x))$. Clearly $(t,x) \longmapsto f_t(x)$ is a continuous maps of $I \times X$ into the bounded operators on H so it suffices to prove that $f_t(x) \in \mathrm{Fred}(H)$. But $t(f_1(x) - f_0(x))$ is compact and the Lemma follows from S.A.S.I.T., Chapter VII, Corollary 1 of Theorem 2.

q.e.d.

<u>Remark</u>. Actually it can be shown that $D_{(k)}: L^2_k(E_1) \longrightarrow L^2_{k-r}(E_2)$ is a Fredholm map and hence defines an element $\mathrm{ind}(D_{(k)}) \in K(L^2_k(E_1))$. But since the inclusion $\lambda_k: C^\infty(E_1) \longrightarrow L^2_k(E_1)$ is a homotopy equivalence (see the remark following Theorem 13.9) $\lambda^*_k: K(L^2_k(E_1) \longrightarrow K(C^\infty(E_1))$ is an isomorphism, so we lose no information by only considering $\lambda^* \mathrm{ind}(D_{(k)}) = \mathrm{ind}(D_{(k)}, \lambda_k) = i_a(D)$.

As we have remarked $i_a(D) \in K(C^\infty(E_1))$ depends only on $\gamma(D) \in \tilde{K}(\hat{T}_o{}^*(E_1))$. Both in order to establish this fact and to indicate how one might hope to calculate $i_a(D)$ from $\gamma(D)$ (i.e. "solve the index problem") it is convenient to introduce the notion of a "parameterized family of (linear) elliptic operators".

In what follows Ω will be a paracompact space (the parameter space) and M a compact n-dimensional C^∞ manifold without boundary. We shall put $B = \Omega \times M$ (actually most of what we say works also when B is a fiber bundle over Ω with fiber M and structural group the group of diffeomorphisms of M). We put $M^{\omega} = \{\omega\} \times M$ considered as a C^∞ manifold in the obvious way and if ξ is a vector bundle over B we put $\xi^{\omega} = \xi | M^{\omega}$. If η and ξ are C^∞ vector bundles over M then recall that $\mathrm{Hom}(\eta, \zeta) = C^\infty L(\eta, \zeta)$ so that $\mathrm{Hom}(\eta, \zeta)$ has a natural " C^∞ topology" and so does the open subspace $\mathrm{Iso}(\eta, \zeta)$. In particular if U is a space then $U \times \eta$ and $U \times \zeta$ are vector bundles over $U \times M$ and if $\psi: U \times \eta \approx U \times \zeta$ is C^0 vector bundle isomorphism such that $\psi_u: \{u\} \times \eta \approx \{u\} \times \zeta$ is in $\mathrm{Iso}(\eta, \zeta)$ (i.e. is C^∞) for all $u \in U$, then it makes sense to say that ψ defines a continuous map $(u \longmapsto \psi_u)$ of U into $\mathrm{Iso}(\eta, \zeta)$.

18.3. Definition. We define a category $VB_\Omega(M)$ whose objects, called C^∞ families of vector bundles on M parameterized by Ω , are defined by the following given:

1) a vector bundle ξ over $B = \Omega \times M$

2) An open cover $\{\mathcal{O}_\alpha\}_{\alpha \varepsilon A}$ of Ω and for each $\alpha \varepsilon A$ a C^∞ vector bundle η_α over M and a vector bundle isomorphism $\varphi_\alpha : \mathcal{O}_\alpha \times \eta_\alpha \approx \xi|(\mathcal{O}_\alpha \times M)$, such that for each $\alpha, \beta \varepsilon A$, $\varphi_\beta^{-1} \circ \varphi_\alpha : (\mathcal{O}_\alpha \cap \mathcal{O}_\beta) \times \eta_\alpha \approx (\mathcal{O}_\alpha \cap \mathcal{O}_\beta) \times \eta_\beta$ defines a continuous map of $\mathcal{O}_\alpha \cap \mathcal{O}_\beta$ into $\mathrm{Iso}(\eta_\alpha, \eta_\beta)$. If s is a section of ξ then $s \circ \varphi_\alpha^{-1}$ is a section of $\mathcal{O}_\alpha \times \eta_\alpha$ which is the same as a map of $\mathcal{O}_\alpha$ into the sections of η_α . If for each α this is a continuous map of $\mathcal{O}_\alpha$ into $C^\infty(\eta_\alpha)$ (where the latter space has the C^∞ topology) then we say that $s \varepsilon C^\infty_\Omega(\xi)$ and we note that $C^\infty_\Omega(\xi)$ is clearly a vector space.

Remark. As usual with such definitions the "atlas" $\{\varphi_\alpha : \mathcal{O}_\alpha \times \eta_\alpha \approx \xi| (\mathcal{O}_\alpha \times M)\}$ is contained in a unique maximal atlas. It is clear how to define the Whitney sum in the category $VB_\Omega(M)$ and more generally any "C^∞ functor" of objects in $VB_\Omega(M)$. In particular if ξ and η are objects of $VB_\Omega(M)$ we can define $L(\xi, \eta)$ (the bundle whose fiber at b is the space of linear maps of ξ_b into η_b) as an object of $VB_\Omega(M)$. The morphisms of $VB_\Omega(M)$ are of course defined by $\mathrm{Hom}(\xi,\eta) = C^\infty_\Omega(L(\xi,\eta))$. It follows that we have the usual bilinear pairing $\mathrm{Hom}(\xi,\eta) \times C^\infty_\Omega(\xi) \longrightarrow C^\infty_\Omega(\eta)$.

If we have a C^∞ vector bundle ξ over M then we get an object $\xi_\Omega = \Omega \times \xi$ of $VB_\Omega(M)$. such a bundle is called "parametrically trivial". In particular we have the family $T^*_\Omega(M)$ which plays an important role in what follows.

If $\xi \varepsilon VB_\Omega(M)$ then of course each ξ^ω $(\omega \varepsilon \Omega)$ is a C^∞ vector bundle over $M^\omega = \{\omega\} \times M$. It is easy to see how to define $J^k_\Omega(\xi) \varepsilon VB_\Omega(M)$ so that $J^k_\Omega(\xi)^\omega = J^k(\xi^\omega)$. Moreover we then have a k-jet extension map $J^k : C^\infty_\Omega(\xi) \longrightarrow C^\infty_\Omega(J^k_\Omega(\xi))$ such that $j_k(s)|\xi^\omega = j_k(s|\xi^\omega)$.

If $f:\Omega_1 \longrightarrow \Omega_2$ is a continuous map, then there is a canonical map $f^*: VB_{\Omega_2}(M) \longrightarrow VB_{\Omega_1}(M)$. If $\xi \varepsilon VB_{\Omega_2}(M)$ then as a vector bundle $f^*\xi$ is the bundle over $\Omega_1 \times M$ induced from the vector bundle ξ over $\Omega_2 \times M$ by the map $f \times id : \Omega_1 \times M \longrightarrow \Omega_2 \times M$, and it is easy to see how to pull back the extra structure. It is also obvious how morphisms pull back, so that f^* becomes a covariant functor.

In what follows we will write $H^k(\eta)$ to denote the Sobolev Hilbert space $L^2_k(\eta)$ where η is a C^∞ vector bundle over M . For each $\xi \varepsilon VB_\Omega(M)$ we shall now define a Hilbert space bundle $H^k_\Omega(\xi)$ over Ω whose fiber $H^k_\Omega(\xi)_\omega$ over a point $\omega \varepsilon \Omega$ is $H^k(\xi^\omega)$. The Hilbert bundle structure of

$H^k_\Omega(\xi)$ will be defined in terms of the $\varphi_\alpha : \mathcal{O}_\alpha \times \eta_\alpha \approx \xi | (\mathcal{O}_\alpha \times M)$ of Definition 18.3.

Namely, given α and $\omega \in \mathcal{O}_\alpha$ we have a map $\varphi_\alpha(\omega) \in \mathrm{Hom}(\eta_\alpha, \xi^\omega)$ defined by $\varphi_\alpha(\omega)(e)=\varphi_\alpha(\omega,e)$, and by the functoriality of H^k (namely the fact that $H^k = L^2_k$ satisfies Axiom B § 1 of Section 4) $\varphi_\alpha(\omega)$ extends to an isomorphism of topological vector spaces $\varphi_\alpha^{(k)}(\omega) : H^k(\eta_\alpha) \approx H^k(\xi^\omega)$, and we define a trivialization of $H^k(\xi) | \mathcal{O}_\alpha$,

$$\varphi_\alpha^{(k)} : \mathcal{O}_\alpha \times H^k(\eta_\alpha) \approx H^k_\Omega(\xi) | \mathcal{O}_\alpha$$

by $\varphi_\alpha^{(k)}(\omega,s) = \varphi_\alpha^{(k)}(\omega)(s)$. To prove that this defines a Hilbert bundle structure for $H^k_\Omega(\xi)$ we must show that given α and β the map $\omega \longmapsto (\varphi_\beta^{(k)}(\omega))^{-1} \circ \varphi_\alpha^{(k)}(\omega)$ of $\mathcal{O}_\alpha \cap \mathcal{O}_\beta$ into $L(H^k(\eta_\alpha), H^k(\eta_\beta))$ is continuous (in the norm topology). Now by Definition 18.3 $\omega \longmapsto \varphi_\beta^{-1}(\omega)\,\varphi_\alpha(\omega)$ is a continuous map of $\mathcal{O}_\alpha \cap \mathcal{O}_\beta$ into $\mathrm{Hom}(\eta_\alpha, \eta_\beta)$, where $\mathrm{Hom}(\eta_\alpha, \eta_\beta) = C^\infty L(\eta_\alpha, \eta_\beta)$ is given the C^∞ topology. Thus it will suffice to show that the map $f \longmapsto f^{(k)}$ (where $f^{(k)}: H^k(\eta_\alpha) \longrightarrow H^{(k)}(\eta_\beta)$ is the continuous linear map which extends $f_*: C^\infty(\eta_\alpha) \longrightarrow C^\infty(\eta_\beta)$) is continuous from $\mathrm{Hom}(\eta_\alpha, \eta_\beta)$ with the C^∞ topology into $L(H^k(\eta_\alpha), H^k(\eta_\beta))$. But from Corollary 9.7 follows the stronger fact that the map $(f,s) \longmapsto f^{(k)}(s)$ is a continuous bilinear map of $\mathrm{Hom}(\eta_\alpha, \eta_\beta) \times H^k(\eta_\alpha) \longrightarrow H^k(\eta_\beta)$ when $\mathrm{Hom}(\eta_\alpha, \eta_\beta) = C^\infty(L(\eta_\alpha, \eta_\beta))$ is given the topology induced from $H^\ell(L(\eta_\alpha, \eta_\beta))$ provided $\ell > \max(k, \frac{1}{2}\dim M)$.

If $\xi \in VB_\Omega(M)$ then a C^∞ Riemannian structure for ξ is of course given by a C^∞_Ω section of the bundle of symmetric bilinear forms on ξ which is everywhere positive definite. This induces a Riemannian metric on each ξ^ω and hence, given a strictly positive smooth measure on M, an inner product on each $H^0(\xi^\omega)$, which is easily seen to define a C^∞ Riemannian structure on the Hilbert bundle $H^0_\Omega(\xi)$. If for $k > 0$ we define $H^{-k}_\Omega(\xi)$ to be the dual bundle of $H^k_\Omega(\xi)$ (the anti-dual in the complex case) then the Riemannian structure for $H^0_\Omega(\xi)$ together with the embedding $H^k_\Omega(\xi) \subseteq H^0_\Omega(\xi)$ defines an embedding $H^0_\Omega(\xi) \subseteq H^{-k}_\Omega(\xi)$.

<u>Remark</u>. It is here (as A. Douady pointed out to me) where one uses in an essential way that $B = \Omega \times M$ is a product bundle over M, and not just a fiber bundle with fiber M and structural group the group of diffeomorphisms of M. Indeed let $F : \Omega \times M \longrightarrow \Omega \times M$ be a map of the form $F(\omega,y) = F(\omega)(y)$ where F is a continuous map of Ω into the group of diffeomorphisms of M (with the C^∞ topology). Given $\xi \in VB_\Omega(M)$ we can define in an obvious way an $\eta^\omega = F(\omega)^* \xi^\omega$ and there is a natural map $\overline{F}(\omega)_*: H^k(\xi)_\omega \longrightarrow H^k(\eta)_\omega$, namely $s \longmapsto s \circ F(\omega)$. The problem arises because $\overline{F}$ is <u>not</u> a bundle isomorphism. That is, if we represent ξ and η over $\mathcal{O}_\alpha$ as $\mathcal{O}_\alpha \times \xi_\alpha$ and $\mathcal{O}_\alpha \times \eta_\alpha$ then as

a map of $\mathcal{O}_\alpha \longrightarrow L(H^k(\xi_\alpha), H^k(\eta_\alpha))$ $\overline{F}$ is continuous when $L(H^k(\xi_\alpha), H^k(\eta_\alpha))$ is given the "strong operator topology" but not when it is given the norm (also called uniform) topology. Thus the Hilbert bundle structure of $H^k{}_\Omega(\xi)$ depends on the representation of $\Omega \times M$ as a product bundle over Ω, and as a result it is not possible to give a natural Hilbert bundle structure to $\{H^k(\xi^\omega)\}$ when we replace $B = \Omega \times M$ by a fiber bundle with fiber M.

18.4. Lemma. Let M be a compact C^∞ manifold without boundary and with a strictly positive smooth measure and let $\xi \in VB_\Omega(M)$ have a C^∞ Riemannian structure. Then for all $r,k \in Z$, $r > 0$, there is a Hilbert bundle isomorphism $j_r{}^{(k)}{}_\omega : H^k_\Omega(\xi) \longrightarrow H^{k-r}_\Omega(J^r{}_\Omega(\xi))$ such that for each $\omega \in \Omega$ the map $j_r{}^{(k)}{}_\omega : H^k(\xi^\omega) \longrightarrow H^{k-r}(J^r(\xi^\omega))$ is the continuous extension of $j_r : C^\infty(\xi^\omega) \longrightarrow C^\infty(J^r(\xi^\omega))$.

Proof. By choosing a trivialization $\varphi_\alpha : \mathcal{O}_\alpha \times \eta_\alpha \approx \xi | \mathcal{O}_\alpha \times M$ one easily reduces to the case $\xi = \Omega \times \eta$ in which case $H^k{}_\Omega(\xi) = \Omega \times H^k(\eta)$ and $j_r^{(k)} = \text{id} \times \overline{j}_r^{(k)}$ where $\overline{j}_r^{(k)} : H^k(\eta) \longrightarrow H^{k-r}(\eta)$ is the continuous extension of $j_r : C^\infty(\eta) \longrightarrow C^\infty(J^r(\eta))$, which is an isomorphism by the very definition of $H^k(\eta)$ in case $k = r$ and in general by Theorem 5.3 But in this case the lemma is trivial.

q.e.d.

18.5. Lemma. Let M be a compact C^∞ manifold without boundary, $\xi, \eta \in VB_\Omega(M)$ and $T \in \text{Hom}(\xi, \eta)$. Then for all $k \in Z$ there is a Hilbert bundle morphism $T^{(k)} : H^k{}_\Omega(\xi) \longrightarrow H^k(\eta)$ such that for each $\omega \in \Omega$ the map $T^{(k)}_\omega : H^k(\xi^\omega) \longrightarrow H^k(\eta^\omega)$ is the continuous extension of $(T_\omega)_* : C^\infty(\xi^\omega) \longrightarrow C^\infty(\eta^\omega)$, where $T_\omega = T|M^\omega \in \text{Hom}(\xi^\omega, \eta^\omega)$.

Proof. Again we easily reduce to the case $\xi = \Omega \times \zeta_1$, and $\eta = \Omega \times \zeta_2$, so that $\text{Hom}(\xi, \eta)$ can be identified with continuous maps of Ω into $\text{Hom}(\zeta_1, \zeta_2) = C^\infty(L(\zeta_1, \zeta_2))$ (with the C^∞ topology), and the lemma amounts to proving that the natural bilinear map $C^\infty(L(\zeta_1,\zeta_2)) \times C^\infty(\zeta_1) \longrightarrow C^\infty(\zeta_2)$ extends to a continuous bilinear map of $C^\infty(L(\zeta_1,\zeta_2)) \times H^k(\zeta_1) \longrightarrow H^k(\zeta_2)$. But as remarked above Corollary 9.7 implies the stronger fact that it extends to a continuous bilinear pairing of $H^\ell(L(\zeta_1,\zeta_2)) \times H^k(\zeta_1)$ provided $\ell > \max(k, \frac{1}{2}\dim M)$.

q.e.d.

18.6. Definition. Given $\xi, \eta \in VB_\Omega(M)$ we define a subspace $\text{Diff}^\Omega_k(\xi, \eta)$ of the vector space of linear maps of $C^\infty{}_\Omega(\xi)$ into $C^\infty{}_\Omega(\eta)$ as follows: $D : C^\infty{}_\Omega(\xi) \longrightarrow C^\infty{}_\Omega(\eta)$ belongs to

$\mathrm{Diff}^{\Omega}_k(\xi,\eta)$ if and only if it can be factored as $D = F_* \circ j_k$ where $j_k : C^\infty{}_\Omega(\xi) \longrightarrow C^\infty{}_\Omega(J^k{}_\Omega(\xi))$ is the k-jet extension map and $F : J^k{}_\Omega(\xi) \longrightarrow \eta$ is in $\mathrm{Hom}(J^k{}_\Omega(\xi),\eta) = C^\infty{}_\Omega(L(J^k{}_\Omega(\xi),\eta))$. For each $\omega \in \Omega$, $F_\omega = F|M^\omega \in \mathrm{Hom}(J^k(\xi^\omega),\eta^\omega)$ defines $D_\omega = F_\omega \circ j_k : C^\infty(\eta^\omega)$, an element of $\mathrm{Diff}_k(\xi^\omega,\eta^\omega)$. D will be referred to as a family of k^{th} order linear operators from ξ to η and D_ω will be called the member of the family D corresponding to the parameter value ω and we shall often denote D by $\{D_\omega\}_{\omega\in\Omega}$ or simply $\{D_\omega\}$.

Remark. We note that $D \longmapsto F$ is a linear isomorphism of $\mathrm{Diff}^{\Omega}_k(\xi,\eta)$ with $\mathrm{Hom}(J^k{}_\Omega(\xi),\eta)$.

18.7. Theorem. Let M be a compact n-dimensional manifold without boundary and with a strictly positive smooth measure and let $\xi,\eta \in VB_\Omega(M)$ have a C^∞ Riemannian structures. If $D \in \mathrm{Diff}^{\Omega}_r(\xi,\eta)$ then for each $k \in Z$ there is a Hilbert bundle morphism $D^{(k)} : H^k{}_\Omega(\xi) \longrightarrow H^{k-r}{}_\Omega(\eta^\omega)$ such that for each $\omega \in \Omega$ the map $D^{(k)}_\omega(\xi^\omega) \longrightarrow H^{k-r}(\eta^\omega)$ is the continuous extension of the differential operator $D_\omega : C^\infty(\xi^\omega) \longrightarrow C^\infty(\eta^\omega)$.

Proof. Immediate from lemmas 18.4 and 18.5.

Now let ξ,η and D be as in the above theorem and let $T^*{}_\Omega(M) = \Omega \times T^*(M)$. Let $\pi : T^*{}_\Omega(M) \longrightarrow \Omega \times M$ be the natural projection (i.e. $\pi(\omega,(v,x)) = (\omega,x)$). Then $\pi^*\xi$ and $\pi^*\eta$ are C^0 bundles over $T^*{}_\Omega(M)$ whose fibers at $(\omega,(v,x))$ are respectively ξ^ω_x and η^ω_x. We define $\sigma_k(D) \in \mathrm{Hom}(\pi^*\xi,\pi^*\eta)$, called the ($k^{th}$ order) symbol of D as follows:

$$\sigma_k(D)(\omega,(v,x)) = \sigma_k(D_\omega)(v,x) : \xi^\omega_x \longrightarrow \eta^\omega_x .$$

The fact that $\sigma_k(D)$ is really an element of $\mathrm{Hom}(\pi^*\xi,\pi^*\eta) = C^0L(\pi^*\xi,\pi^*\eta)$ follows easily from the fact that if ξ_1 and ξ_2 are two C^∞ fiber bundles over M then the map $\sigma_k : \mathrm{Diff}_k(\xi_1,\xi_2) \longrightarrow \mathrm{Hom}(\pi^*\xi_1,\pi^*\xi_2)$ is continuous when $\mathrm{Diff}_k(\xi_1,\xi_2) \approx C^\infty L(J^k(\xi_1),\xi_2)$ and $\mathrm{Hom}(\pi^*\xi,\pi^*\eta) = C^0L(\pi^*\xi,\pi^*\eta)$ are given the C^∞ and C^0 topologies respectively.

18.8. Definition. If $D \in \mathrm{Diff}^{\Omega}_k(\xi,\eta)$ then $D = \{D_\omega\}$ is called an elliptic family of k^{th} order linear differential operators if $\sigma_k(D)(\omega,(v,x)) : \xi^\omega_x \longrightarrow \eta^\omega_x$ is an isomorphism for $v \neq 0$, or equivalently if for each $\omega \in \Omega$, $D_\omega \in \mathrm{Diff}_k(\xi^\omega,\eta^\omega)$ belongs to $\mathrm{Ell}_k(\xi^\omega,\eta^\omega)$. We denote the set of such D by $\mathrm{Ell}_k^\Omega(\xi,\eta)$. If we give M a Riemannian structure then for $D \in \mathrm{Ell}_k^\Omega(\xi,\eta)$, $\sigma_k(D)|ST^*_\Omega(M)$, the restriction of $\sigma_k(D)$ to the unit sphere bundle of $T^*_\Omega(M)$, is an element of $\mathrm{Iso}(\tilde{\xi},\tilde{\eta})$, where $\tilde{\xi}$ and $\tilde{\eta}$ are respectively the restrictions of $\pi^*\xi$ and $\pi^*\eta$ to $ST^*_\Omega(M)$, hence the difference construction defines an element $\delta(\pi^*\xi,\pi^*\eta,\sigma_k(D)|ST^*_\Omega(M))$ of $\tilde{K}(BT^*_\Omega(M), ST^*_\Omega(M)) = \tilde{K}(\hat{T}^*_\Omega(M))$ where $BT^*_\Omega(M)$ is the unit ball bundle of $T^*_\Omega(M)$ and $\hat{T}^*_\Omega(M) = BT^*_\Omega(M)/ST^*_\Omega(M)$ is the Thom space of $T^*_\Omega(M)$. We denote this element of $\tilde{K}(\hat{T}^*_\Omega(M))$ by $\gamma(D)$.

18.9. Theorem. Let M be a compact C^∞ manifold without boundary and let $\xi,\eta \in VB_\Omega(M)$ have Riemannian structures. If $D \in \mathrm{Ell}_r^\Omega(\xi,\eta)$ then for each $k \in Z$ the Hilbert bundle morphism $D^{(k)}: H^k_\Omega(\xi) \longrightarrow H^{k-r}_\Omega(\eta)$ of Theorem 18.7 is a Fredholm bundle morphism and hence defines an element $\mathrm{ind}(D^{(k)})$ of $K(\Omega)$. Moreover this element of $K(\Omega)$ is independent of k, so D defines an element $i_a(D) \in K(\Omega)$, called the analytic index of D, by $i_a(D) = \mathrm{ind}(D^{(k)})$.

Proof. With some fairly obvious changes the proof is similar to that sketched for Theorem 19.2 and will be omitted.

We now sketch briefly the Index Theorem (or conjecture) for families of elliptic linear differential operators. Much of what follows must be regarded as provisional since there are a great many technical details to be checked and I have not by any means checked them all carefully.

In what follows M is a C^∞ compact n-dimensional Riemannian manifold without boundary. If $\xi \in VB_\Omega(M)$, $\tilde{\xi}$ denotes $p^*\xi$ where p is the projection of the unit sphere bundle of $T^*_\Omega(M)$ onto $\Omega \times M$. As usual S.A.S.I.T. refers to Seminar on the Atiyah-Singer Index Theorem.

First given $\xi,\eta \in VB_\Omega(M)$ one defines $\mathrm{Int}_k^\Omega(\xi,\eta)$, the space of families of k^{th} order integro-differential operators from ξ to η. An element $L = \{L_\omega\}_{\omega\in\Omega}$ is a map $L: C^\infty_\Omega(\xi) \longrightarrow C^\infty_\Omega(\eta)$ such that if $s \in C^\infty_\Omega(\xi)$ then $(Ls)|M^\omega = L_\omega(s|M^\omega)$ where $L_\omega \in \mathrm{Int}_k(\xi^\omega,\eta^\omega)$ (S.A.S.I.T. Chapter XI and Chapter XIV) and moreover $\omega \longmapsto L_\omega$ is "continuous" in the sense that if we represent ξ and η over an open set $\mathcal{O} \subseteq \Omega$ as $\mathcal{O} \times \zeta_1$ and $\mathcal{O} \times \zeta_2$ where ζ_1 and ζ_2 are C^∞ vector bundles over M, then $\omega \longmapsto L_\omega \in \mathrm{Int}_k(\zeta_1,\zeta_2)$ is continuous from $\mathcal{O}$ into $\mathrm{Int}_k(\zeta_1,\zeta_2)$. The topology of $\mathrm{Int}_k(\zeta_1,\zeta_2)$ is the weakest topology making the following maps continuous: (1) the map $\sigma_k: \mathrm{Int}_k(\zeta_1,\zeta_2) \longrightarrow \mathrm{Smbl}_k(\zeta_1,\zeta_2) = \mathrm{Hom}(\tilde{\zeta}_1,\tilde{\zeta}_2)$ where $\tilde{\zeta}_i$ is ζ_i pulled back to $ST^*(M)$ and $\mathrm{Hom}(\tilde{\zeta}_1,\tilde{\zeta}_2) = C^\infty L(\tilde{\zeta}_1,\tilde{\zeta}_2)$ has the compact open topology; (2) and each of the "extension" maps $\mathrm{Int}_k(\zeta_1,\zeta_2) \longrightarrow L(H^\ell(\zeta_1), H^{\ell-k}(\zeta_2))$ which assigns to each $T \in \mathrm{Int}_k(\zeta_1,\zeta_2)$ the continuous extension $T^{(\ell)}: H^\ell(\zeta_1) \longrightarrow H^\ell(\zeta_2)$. It follows that if $L \in \mathrm{Int}_k^\Omega(\xi,\eta)$ then for each $\ell \in Z$ we have a Hilbert bundle morphism $L^{(\ell)}: H^\ell_\Omega(\xi) \longrightarrow H^{\ell-k}_\Omega(\eta)$ such that for each $\omega \in \Omega$, $L_\omega^{(\ell)}: H^\ell(\xi^\omega) \longrightarrow H^{\ell-k}(\eta^\omega)$ is the continuous extension of $L_\omega: C^\infty(\xi^\omega) \longrightarrow C^\infty(\eta^\omega)$. It also follows that we get an element $\sigma_k(L) \in \mathrm{Hom}(\tilde{\xi},\tilde{\eta})$, called the symbol of L, if we define $\sigma_k(L)(\omega,(v,x)) = \sigma_k(L_\omega)(v,x)$ for $(\omega,(v,x)) \in ST^*_\Omega(M) = \Omega \times ST^*(M)$. Of course L is called k^{th} order elliptic if $\sigma_k(L) \in \mathrm{Iso}(\tilde{\xi},\tilde{\eta})$ and we denote the set of all such L by $E_k^\Omega(\xi,\eta)$. Then we have a map $\gamma : E_k^\Omega(\xi,\eta) \longrightarrow \tilde{K}(\hat{T}^*_\Omega(M)) = \tilde{K}(BT^*_\Omega(M), ST^*_\Omega(M))$ given by the difference operation: $\gamma(L) = \delta(\bar{\pi}^*\xi, \bar{\pi}^*\eta,$ where $\bar{\pi} : BT^*_\Omega(M) \longrightarrow \Omega \times M$. Also for $L \in E_k^\Omega(\xi,\eta)$, $L^{(\ell)}$ is Fredholm, and $\mathrm{ind}(L^{(\ell)})$ is independent of ℓ and defines $i_a(L) \in K(\Omega)$. Of course $\mathrm{Diff}_k^\Omega(\xi,\eta) \subseteq \mathrm{Int}_k^\Omega(\xi,\eta)$ if $k \geq 0$ and σ_k extends the symbol map on $\mathrm{Diff}_k^\Omega(\xi,\eta)$ so that $\mathrm{Ell}_k^\Omega(\xi,\eta) = \mathrm{Diff}_k^\Omega(\xi,\eta)$ so that $\mathrm{Ell}_k^\Omega(\xi,\eta) = \mathrm{Diff}_k^\Omega(\xi,\eta) \cap E_k^\Omega(\xi,\eta)$ and moreover the maps $\gamma: E_k^\Omega(\xi,\eta) \longrightarrow \tilde{K}(\hat{T}^*_\Omega(M))$ and $i_a: E_k^\Omega(\xi,\eta) \longrightarrow K(\Omega)$ are extensions of the maps already defined on $\mathrm{Ell}_k^\Omega(\xi,\eta)$. If $L_1, L_2 \in E_k^\Omega(\xi,\eta)$ and $\sigma_k(L_1) = \sigma_k(L_2)$ then for all $\omega \in \Omega$, $\sigma_k(L_1^\omega) = \sigma_k(L_2^\omega)$ hence $L_1^\omega - L_2^\omega$ is in $OP_{k-1}(\xi^\omega,\eta^\omega)$ (i.e. extends to a continuous linear map of $H^\ell(\xi^\omega)$ into $H^{\ell-k+1}(\eta^\omega)$ and hence (S.A.S.I.T. Chapter X §4, Cor. 1 of Theorem 3) to a compact map $H^\ell(\xi^\omega) \longrightarrow H^{\ell-k}(\eta^\omega)$ so by Lemma 18.3 $\mathrm{ind}(L_1^{(\ell)}) = \mathrm{ind}(L_2^{(\ell)})$ or $i_a(L_1) = i_a(L_2)$. In other words for $L \in E_k^\Omega(\xi,\eta)$, $i_a(L) \in K(\Omega)$ depends only on $\sigma_k(L) \in \mathrm{Iso}(\tilde{\xi},\tilde{\eta})$.

Now using property S5 of Int_k (S.A.S.I.T. Chapter XI) and a partition of unity argument one can define a map $\psi_k: \mathrm{Hom}(\tilde{\xi},\tilde{\eta}) \longrightarrow \mathrm{Int}_k^\Omega(\xi,\eta)$ which is a continuous right inverse for σ_k, and by restriction we get $\psi_k: \mathrm{Iso}(\tilde{\xi},\tilde{\eta}) \longrightarrow E_k^\Omega(\xi,\eta)$. It follows that we can define a map $i_a^{(k)}: \mathrm{Iso}(\tilde{\xi},\tilde{\eta}) \longrightarrow K(\Omega)$ such that $i_a(L) = i_a^{(k)}(\sigma_k(L))$ for $L \in E_k^\Omega(\xi,\eta)$. Using the element $L' \in \mathrm{Int}_1^\Omega(\xi,\xi)$ defined in the proof of Theorem 18.2 one shows easily that $i_a^{(k+1)} = i_a^{(k)}$ so we can think of $i_a = i_a^{(k)}$ being defined on $\mathrm{ISO}(\xi,\eta)$. Moreover if σ_0 and σ_1 are "homotopic" elements of $\mathrm{Iso}(\xi,\eta)$ (i.e. there is a continuous arc $\sigma_t \in \mathrm{Iso}(\xi,\eta)$ $t \in [0,1]$ having σ_0 and σ_1 as endpoints) then $\psi_k(\sigma_t)$ is a continuous arc in $E_k^\Omega(\xi,\eta)$ and $\psi_k(\sigma_t)^\ell : H^\ell_\Omega(\xi) \longrightarrow H^{\ell-k}_\Omega(\eta)$ is a continuous arc of Fredholm bundle morphism so $i_a(\sigma_0) = \mathrm{ind}(\psi_k(\sigma_0)^\ell = \mathrm{ind}(\psi_k(\sigma_1)^\ell = i_a(\sigma_1)$ i.e. $i_a: \mathrm{Iso}(\tilde{\xi},\tilde{\eta}) \longrightarrow K(\Omega)$ is constant on homotopy classes (arc components) .

The final reduction is to show that i_a can actually be defined on $\tilde{K}(\hat{T}^*_\Omega(M))$. This means that every element of $\tilde{K}(T_\Omega^*(M))$ is of the form $\gamma(L)$ for some $L \in E_k^\Omega(\xi,\eta)$ (for some choice of ξ and η) and that if $\gamma(L_1) = \gamma(L_2)$ for $L_1 \in E_{k_1}^\Omega(\xi_1, \eta_1)$ and $L_2 \in E_{k_2}^\Omega(\xi_2, \eta_2)$ then $i_a(L_1) = i_a(L_2)$ and hence we can define a map $i_t: \tilde{K}(T_\Omega^*(M)) \longrightarrow K(\Omega)$ such that if $L \in E_k(\xi,\eta)$ then $i_a(L) = i_t(\gamma(L))$. The arguments here are generalizations of Chapter XV of S.A.S.I.T. The function $\tilde{i}_t: K(T^*_\Omega(M)) \longrightarrow K(\Omega)$ is called the topological index and the "index problem" for families of elliptic operators is to give a purely topological description of this map. Weishu Shih has indicated such a description, at least for the case of Ω a finite CW complex in a Research Announcement in the Bulletin of the A.M.S. [Vol. 72, No. 6, Nov. 1966, pp. 984-991].

We will end this chapter with a description of how the index problem for non-linear elliptic differential operators can be reduced to the index problem for families, so that a complete solution of the latter implies a solution of the former, at least in a formal sense.

Let E_1 and E_2 be two C^∞ fiber bundles over a compact C^∞ manifold with boundary M and let $\Omega = C^\infty(E_1)$. Given $D \in Df_k(E_1,E_2)$ we will define two elements $\xi,\eta \in VB_\Omega(M)$ and $\Lambda(D) \in \mathrm{Diff}_k^\Omega(\xi,\eta)$. Moreover $\Lambda(D) \in \mathrm{Ell}_k{}^\Omega(\xi,\eta)$ if and only if $D \in \mathrm{Elptc}_k(E_1,E_2)$ and in the case $i_a(D) = i_a(\Lambda(D))$. Finally when $D \in \mathrm{Elptc}_k(E_1,E_2)$ we shall see how $\gamma(D) \in \tilde{K}(T_0^*(E_1))$ determines $\gamma(\Lambda(D)) \in \tilde{K}(T^*{}_\Omega(M))$.

If $s \in \Omega = C^\infty(E_1)$ then $\xi^s = T_s(E_1) = s^* TF(E_1)$. For the indexing set of the open covering of Ω we take the set of all vector bundle neighborhoods η of E_1 and we define $\mathcal{O}_\eta = C^\infty(\eta)$, an open set in $C^\infty(E_1)$. For $s \in \mathcal{O}_\eta$ we have a canonical isomorphism of η with $T_s(E_1) = T_s(\eta)$ which defines the isomorphism $\varphi_\eta: \mathcal{O}_\eta \times \eta \approx \xi|(\mathcal{O}_\eta \times M)$. If we denote the resulting $\xi \in VB_\Omega(M)$ by $T_{C^\infty(E_1)}(E_1)$ then we have a corresponding $T_{C^\infty(E_2)}(E_2) \in VB_{C^\infty(E_2)}(M)$. Since $D: C^\infty(E_1) \longrightarrow C^\infty(E_2)$ is a continuous map, we have an induced element $\eta \in VB_\Omega(M)$ defined by $\eta = D^* T_{C^\infty(E_2)}(E_2)$. Of course $\eta^s = (T_{C^\infty(E_2)}(E_2))^{Ds} = T_{Ds}(E_2)$. Needless to say $\Lambda(D)_s: C^\infty(\xi^s) \longrightarrow C^\infty(\eta^s)$ is just the linearization of D at s. If $D = F_* \circ j_k$ where $F: J^k(E_1) \longrightarrow E_2$ is a C^∞ fiber bundle morphism, then $\Lambda(D) = \delta F_* \circ j_k$ where $\delta F: J^k{}_\Omega(\xi) \longrightarrow \eta$ is the morphism of the category $VB_\Omega(M)$ defined by $\delta F_s = \delta_s F: J^k(\xi^s) \longrightarrow \eta^s$, so $\Lambda(D) \in \mathrm{Diff}_k{}^\Omega(\xi,\eta)$. The definitions of ellipticity for both D and $\Lambda(D)$ are the same, namely that $\Lambda(D)_s \in \mathrm{Ell}_k(\xi^s, \eta^s)$ for all $s \in \Omega$. It is also clear that when $D \in \mathrm{Elptc}_k(E_1,E_2)$ the definitions of $i_a(D)$ and $i_a(\Lambda(D))$ coincide.

Finally there remains the question of relating $\gamma(D) \in \tilde{K}(\hat{T}_0{}^*(E_1))$ and $\gamma(\Lambda(D)) \in \tilde{K}(T_\Omega{}^*(M))$ when $D \in \mathrm{Elptc}_k(E_1,E_2)$. We claim that in fact $\gamma(\Lambda(D)) = j^*\gamma(D)$ where $j: \hat{T}_\Omega{}^*(M) \longrightarrow T_0^*(E_1)$ is the map induced by the map $(s,(v,x)) \longmapsto (s(x),(v,x))$ of $(BT^*{}_\Omega(M), ST^*{}_\Omega(M))$ into $(BT_0^*(M), ST_0^*(M))$. If $i_t: \tilde{K}(T^*{}_\Omega(M)) \longrightarrow K(\Omega)$ is the "topological index" map defined above then we can define a topological index map $i_t: \tilde{K}(T_0^*(E_1)) \longrightarrow K(\Omega)$ by $i_t = i_t \circ j^*$, so that if E_2 is any C^∞ fiber bundle over M and $D \in \mathrm{Elptc}_k(E_1,E_2)$ then $i_a(D) = i_t(\gamma(D))$. The "index problem" for non-linear elliptic operators is of course to find an explicit topological description of $i_t: \tilde{K}(T_0^*(E_1)) \longrightarrow K(C^\infty(E_1))$.

19. THE CALCULUS OF VARIATIONS

In this section M will denote a compact C^∞ n-dimensional manifold with boundary and with a strictly positive smooth measure μ, and E will denote a C^∞ fiber bundle over M. As usual $\mathbb{R}_M$ will denote the product one dimensional vector bundle $M \times \mathbb{R}$ over M.

<u>19.0</u>. <u>Definition</u>. By a k^{th} order Lagrangian on E we mean an element of $Df_k(E, \mathbb{R}_M)$. We denote the vector space of k^{th} order Lagrangians on E by $Lgn_k(E)$, and $Lgn_k^{W;\ell}(E)$ will denote the vector subspace $Df_k^{W;\ell}(E, \mathbb{R}_M)$ (see Definition 16.1).

If $L \in Lgn_k(E)$ we define a map $J^L\colon C^\infty(E) \longrightarrow \mathbb{R}$ by $J^L(s) = \int L(s)(x)d\mu(x)$. If $\mathcal{M}$ is a section functor satisfying (B§2) and (B§5) we say that $L \in Lgn_k(E)$ is $\mathcal{M}$-smooth if L extends to a C^∞ map of $\mathcal{M}(E)$ into $L_0^1(\mathbb{R}_M)$.

<u>19.1</u>. <u>Theorem</u>. If $L \in Lgn_k(E)$ is $\mathcal{M}$-smooth then J^L extends to a C^∞ map $J\colon \mathcal{M}(E) \longrightarrow \mathbb{R}$ and for $s \in C^\infty(E)$ $dJ_s(\sigma) = \int \Lambda(L)_s(\sigma)(x)d\mu(x)$ for $\sigma \in T_s(E)$.

<u>Proof</u>. If we define $\ell\colon L_0^1(\mathbb{R}_M) \longrightarrow \mathbb{R}$ by $\ell(s) = \int s(x)d\mu(x)$ then ℓ is a continuous linear functional on $L_0^1(\mathbb{R}_M)$ so that if F is a C^∞ map of a Banach manifold X into $L_0^1(\mathbb{R}_M)$, $\ell \circ F$ is a C^∞ map of X into $\mathbb{R}$ and $d(\ell \circ F_x) = \ell \circ dF_x$. Since $J^L = \ell \circ L$ the theorem follows from Theorem 17.3.

<u>19.2</u>. <u>Theorem.</u> If $L \in Lgn_k(E)$ and $\mathcal{M}$ satisfies (B§2) and (B§5) then L is $\mathcal{M}_{(k)}$ smooth.

<u>Proof</u>. By Theorem 15.7 L extends to a continuous map of $\mathcal{M}_{(k)}(E)$ into $\mathcal{M}(\mathbb{R}_M)$, and by (B§5) there is a continuous inclusion map of $\mathcal{M}(\mathbb{R}_M)$ into $C^0(\mathbb{R}_M)$. Finally the inclusion of $C^0(\mathbb{R}_M)$ into $L_0^1(\mathbb{R}_M)$ is continuous.

q.e.d.

<u>19.3</u>. <u>Corollary</u>. If $L \in Lgn_k(E)$ then L is C^k-smooth.

<u>19.4</u>. <u>Corollary</u>. If $L \in Lgn_k(E)$ and $r > \frac{n}{p}$ then L is L_{k+r}^p-smooth.

19.5. Corollary. If $k > \frac{n}{p} + \ell$ and $w \le pk$ then any $L \in \mathrm{Lgn}_k^{w;\ell}(E)$ is L_k^p-smooth. In particular if p and k are positive integers and $k > \frac{n}{p}$ and $L \in \mathrm{Lgn}_k^{pk}(E) = \mathrm{Lgn}_k^{pk;o}(E)$ then L is L_k^p-smooth.

Proof. Corollary 16.11.

19.6. Definition. Let $\mathcal{M}$ satisfy (B§2) and (B§5) and let $f \in \mathcal{M}(E)$. We define a subset $\mathcal{M}_{\partial f}(E)$ of $\mathcal{M}(E)$; namely $\mathcal{M}_{\partial f}(E)$ is the closure in $\mathcal{M}(E)$ of the set of $g \in \mathcal{M}(E)$ such that for some neighborhood U of ∂M (depending on g) $f|U = g|U$.

19.7. Theorem. If $\mathcal{M}$ satisfies (B§2) and (B§5) then for $f \in \mathcal{M}(E)$, $\mathcal{M}_{\partial f}(E)$ is a closed C^∞ submanifold of $\mathcal{M}(E)$. In fact if $s_o \in \mathcal{M}_{\partial f}(E)$ and ξ is any VBN of s_o in E then $\mathcal{M}(\xi) \cap \mathcal{M}_{\partial f}(E) = s_o + \mathcal{M}^o(\xi)$, where, as in section 6, $\mathcal{M}^o(\xi)$ is the closed linear subspace of $\mathcal{M}(\xi)$ obtained by taking the closure of $C_o^\infty(\xi) = \{s \in C^\infty(\xi) |$ support s is disjoint from $\partial M\}$.

Proof. Clearly $s_o|\partial M = f|\partial M$ and it follows that if $h \in C^\infty(M)$ is identically one in a neighborhood of ∂M and vanishes outside a sufficiently small neighborhood of ∂M then $g = hf + (1-h)s_o \in \mathcal{M}(\xi)$. Now $g = f$ in a neighborhood of ∂M and it follows that $\mathcal{M}_{\partial g}(E) = \mathcal{M}_{\partial f}(E)$, hence we can suppose that $f \in \mathcal{M}(\xi)$. Since $\mathcal{M}(\xi)$ is open in $\mathcal{M}(E)$ it follows that $s \in \mathcal{M}(\xi) \cap \mathcal{M}_{\partial f}(E)$ if and only if $s = \lim_n s_n$ where $s_n \in \mathcal{M}(\xi)$ and $s_n - f = 0$ in a neighborhood U_n of ∂M, i.e. if and only if $s \in f + X$ where X is the closure of the set of $\sigma \in \mathcal{M}(\xi)$ such that $\sigma = 0$ in a neighborhood $U(\sigma)$ of ∂M. Given such a σ there is by (B§5) a sequence $\sigma_n \in C^\infty(\xi)$ such that $\sigma_n \longrightarrow \sigma$ in $\mathcal{M}(\xi)$. If $h \in C^\infty(M)$ is identically one on $M - U(\sigma)$ and has support disjoint from ∂M then $h\sigma_n \longrightarrow \sigma$ in $\mathcal{M}(\xi)$ and $h\sigma_n \in C_o^\infty(\xi)$, proving that $X = \mathcal{M}^o(\xi)$, so $\mathcal{M}(\xi) \cap \mathcal{M}_{\partial f}(E) = f + \mathcal{M}^o(\xi)$. Since $s_o \in \mathcal{M}(\xi) \cap \mathcal{M}_{\partial f}(E)$, $f - s_o \in \mathcal{M}^o(\xi)$ so $\mathcal{M}(\xi) \cap \mathcal{M}_{\partial f}(E) = s_o + (f-s_o) + \mathcal{M}^o(\xi) = s_o + \mathcal{M}^o(\xi)$.

q.e.d.

19.8. Corollary. If $g \in \mathcal{M}_{\partial f}(E)$ then $\mathcal{M}_{\partial f}(E) = \mathcal{M}_{\partial g}(E)$.

We can now explain the generalized "Dirichlet Problem" in the calculus of variations. Given a Lagrangian $L \in \mathrm{Lgn}_k(E)$ which is $\mathcal{M}$-smooth and hence defines a C^∞ map $\mathcal{M}(E) \longrightarrow \mathbb{R}$, namely

$s \longmapsto \int L(s)d\mu$, and given $f \in \mathcal{M}(E)$ we get by restriction of J to $\mathcal{M}_{\partial f}(E)$ a C^∞ map $J: \mathcal{M}_{\partial f}(E) \longrightarrow \mathbb{R}$. The dirichlet problem with this data is to describe the critical locus of J . There are a number of subproblems which are often considered:

1) Existence: Prove that (with suitable assumptions on L) J assumes an absolute minimum on $\mathcal{M}_{\partial f}(E)$ and/or on each component of $\mathcal{M}_{\partial f}(E)$. More generally prove that J satisfies the conclusions of Lusternik-Schnirelman theory, e.g. that on each component of $\mathcal{M}_{\partial f}(E)$ J has at least as many critical points as the Lusternik-Schnirelman category of that component (i.e. the smallest integer n such that the component can be covered by n closed sets each contractible in that component, or ∞ if there is no such n). If $\mathcal{M} = L^2_r$ for some r , so that $\mathcal{M}_{\partial f}(E)$ is a Hilbert manifold then one may try to find hypotheses on L so that J is a "Morse function" i.e. so that for "almost all" (in some appropriate sense) f , J has only non-degenerate critical points whose type numbers satisfy the Morse inequalities.

2) Uniqueness problems. Prove (again of course with suitable hypotheses on L) that the solutions of the Dirichlet problem (i.e. the critical points of J) are unique in some sense. For example that the only critical point of J in a given component of $\mathcal{M}_{\partial f}(E)$ is a single absolute minimum. Or that the critical locus of J is discrete (local uniqueness of solutions of the Dirichlet problem), if not for all choices of f then perhaps at least for "almost all" f .

3) Smoothness Problems. Prove (again with suitable hypotheses on L) that the solutions of the Dirichlet problem have a given degree of smoothness provided f has some other degree of smoothness. For example one might try to prove that if $f \in \mathcal{M}_r(E)$ then any solution of the Dirichlet problem (i.e. any critical point of J in $\mathcal{M}_{\partial f}(E)$) must also be in $\mathcal{M}_r(E)$. If this is true for all r when $\mathcal{M} = L^p_m$ then by the Sobolev theorems it follows that if $f \in C^\infty(E)$ then any critical point of J is also C^∞ . This is the analogue of the classical "Weyl Lemma" in the classical Dirichlet problem.

Dirichlet type problems are perhaps the most important, or at least the most studied, of Calculus of Variations problems. However before going on to consider them in more detail we will stop to mention some of the more important other types of problems.

<u>Free boundary problems</u>. Here we look for the critical points of $s \longmapsto \int L(s)d\mu$ on the entire manifold $\mathcal{M}(E)$. Note that in case $\partial M = \emptyset$ there is no distinction between the Dirichlet problem and the free boundary problem. However as we shall see it is necessary to make minor modifications of the treatment of the Dirichlet problem in this case.

<u>"End manifold" problems</u>. Here we are given a closed C^∞ sub-bundle G of $E|\partial M$ and we define $\mathcal{M}_G(E) = \{s \in \mathcal{M}(E) \mid s(\partial M) \subseteq G\}$. If $s \in \mathcal{M}_G(E)$ then we can find a VBN ξ of s in E such that

$G \cap (\xi|\partial M) = \gamma$ is a VBN of $s|\partial M$ in G. Then $\mathcal{M}_G(E) \cap \mathcal{M}(\xi) = \{s \in \mathcal{M}(\xi) | s(\partial M) \subseteq \gamma\}$ which clearly is a closed linear subspace of $\mathcal{M}(\xi)$ (since γ is closed in ξ) which proves that $\mathcal{M}_G(E)$ is a C^∞ submanifold of $\mathcal{M}(E)$ (and also makes its tangent space clear). The problem now is to find the critical points of $s \longmapsto \int L(s)d\mu$ on $\mathcal{M}_G(E)$.

Non-linear eigenvalue problems. Let $\mathcal{M}$ denote either $\mathcal{M}_{\partial f}(E)$, $\mathcal{M}(E)$, or $\mathcal{M}_G(E)$ as above and let $J: \mathcal{M} \longrightarrow \mathbb{R}$ be the function $s \longmapsto \int L(s)d\mu$. Suppose that in addition to $L \varepsilon \mathrm{Lgn}_k(E)$ we are given a second Lagrangian $K \varepsilon \mathrm{Lgn}_k(E)$ which is also $\mathcal{M}$-smooth and hence defined a C^∞ map $F: \mathcal{M} \longrightarrow \mathbb{R}$ given by $s \longrightarrow \int K(s)d\mu$. This is the data for what is classically called a "Lagrange-multiplier" problem or a "Variational problem with side-condition". Recently F. Browder and some of his students have considered problems of this sort under the name of "Non-linear elliptic eigenvalue problems" (see - e.g. Bull. Amer. Math. Soc. 71 (1961) pp. 176-183 and Annals of Math., 82 no. 3 Nov. 1965 pp. 459-477). The problem is to find extremals of the function J subject to the side condition $K = c$. We assume that $c \varepsilon \mathbb{R}$ is a regular value of K, i.e. that K has no critical points on $\mathcal{M}^c = K^{-1}(c)$. Then $\mathcal{M}^c$ is a C^∞ submanifold of $\mathcal{M}$ and $J^c = J|\mathcal{M}^c$ is a C^∞ map of $\mathcal{M}^c$ into $\mathbb{R}$ and the problem is to find the critical points of J^c. Now clearly if $s \varepsilon \mathcal{M}^c$ then $T(\mathcal{M}^c)_s = \{\sigma \varepsilon T(\mathcal{M})_s \mid dF_s(\sigma) = 0\}$, while since $dJ^c_s = dJ_s|T(\mathcal{M}^c)_s$ the condition for s to be a critical point of J^c is that $dJ_s(\sigma) = 0$ whenever $dF_s(\sigma) = 0$. Since s is not a critical point of F, $dF_s \neq 0$, and since the null space of a non-zero linear functional on a vector space determines the linear functional up to a scalar multiple, it follows that for some scalar λ (the "Lagrange multiplier") $dJ_s = \lambda dF_s$, or equivalently $d(J - \lambda F)_s = 0$. Thus if we can find all the critical points on $\mathcal{M}$ for the functions $\mathcal{J}^\lambda = J - \lambda F$ (which are associated to the Lagrangians $\mathcal{J}^\lambda = L - \lambda K$) then the solutions of our non-linear eigenvalue problem follow; they are precisely the points $s \varepsilon \mathcal{M}$ which are simultaneously critical points of some $\mathcal{J}^\lambda$ and solutions of $F(s) = c$. This is of course the standard Lagrange-multiplier reduction of a variational problem with side condition to a parameterized family of variational problems without side condition. If $\overline{L}$ and $\overline{K}$ are the Euler-Lagrange operators for the Lagrangians L and K (differential operators of order $2k$ defined below) then to find the critical points of $\mathcal{J}^\lambda$ is the same as to find solutions of $\overline{L}(s) = \lambda \overline{K}(s)$, which explains why this is called a non-linear eigenvalue problem.

We shall now give a fairly detailed discussion of a special but important case of the Dirichlet problem. We suppose p and k are positive integers with $k > \frac{n}{p}$ and we let $L \varepsilon \mathrm{Lgn}^{pk}_k(E)$. For $f \varepsilon L^p_k(E)$ we will put $\Omega(f) = (L^p_k)_{\partial f}(E)$. Then as we have seen above $s \longmapsto \int L(s)d\mu$ extends to a C^∞ map of $\Omega(f)$ into $\mathbb{R}$ which we denote by $J^{(f)}$ or simply $J: \Omega(f) \longrightarrow \mathbb{R}$.

We first compute dJ_s. Since this is a local question we replace E by a VBN ξ so that by Theorem 19.7 if we choose $s_o \in L^p_k(\xi) \cap \Omega(f)$ then locally we can identify $\Omega(f)$ with $s_o + L^p_k(\xi)^o$ and $T(\Omega(f))_s$ is canonically identified with $L^p_k(\xi)^o$, and by Theorem 19.1 for $\sigma \in L^p_k(\xi)^o$

$$dJ_s(\sigma) = \int \Lambda(L)_s(\sigma)\, d\mu \quad .$$

Classically this is what is called the "first variation of J in integrated form". To get the more familiar "Euler-Lagrange" form of the first variation we proceed as follows. The Lagrangian L is given by $L = F_* \circ j_k$ where $F: J^k(\xi) \longrightarrow \mathbb{R}_M$ is a C^∞ bundle morphism and $j_k: C^\infty(\xi) \longrightarrow C^\infty(J^k(\xi))$ is the k-jet extension map. As usual let δF denote the vertical differential of F. Note that δF is a C^∞ bundle homomorphism of $J^k(\xi)$ into $L(J^k(\xi), \mathbb{R}_M) = J^k(\xi)^*$. Then by Theorem 17.2 $\Lambda(L)_s = (\delta_{j_k(s)}F)_* \circ j_k$ where $\delta_{j_k(s)}F: J^k(\xi) \longrightarrow \mathbb{R}_M$ is the vector bundle homomorphism (C^∞ if s is C^∞) defined by $(\delta_{j_k(s)}F)(x) = \delta_{j_k(s)(x)}F$. Next we choose Riemannian structures for ξ and $J^k(\xi)$ (and we emphasize that the Euler-Lagrange form of the first variation of J will depend on these choices). Then we have a canonical C^∞ vector bundle isomorphism $\ell \longrightarrow \tilde{\ell}$ of $J^k(\xi)^*$ with $J^k(\xi)$ given of course by $\ell(j_k(\sigma)(x) = < \tilde{\ell}, j_k(\sigma)(x) >$. If we compose δF with this isomorphism we get a C^∞ fiber bundle homomorphism $\nabla F: J^k(\xi) \longrightarrow J^k(\xi)$ which we call the vertical gradient of F. For $e \in J^k(\xi)$ we define $\nabla_e F \in J^k(\xi)$ by $\nabla_e F = \nabla F(e)$ so that we have

$$\Lambda(L)_s(\sigma)(x) = (\delta_{j_k(s)(x)}F)\ (j_k(\sigma)(x) = < \nabla_{j_k(s)(x)}F,\ j_k(\sigma)(x) >$$

and we may write for $\sigma \in L^p_k(\xi)^o$:

$$dJ_s(\sigma) = \int < \nabla_{j_k(s)}F,\ j_k(\sigma) > d\mu$$

[Note that since $\sigma \longmapsto j_k(\sigma)$ maps $L^p_k(\xi)^o$ isomorphically to $L^p_o(\xi)$, $\nabla_{j_k(s)}F \in L^{\bar{p}}_o(\xi)$, where $\frac{1}{p} + \frac{1}{\bar{p}} = 1$ and in fact $s \longmapsto \nabla_{j_k(s)}F$ is a C^∞ map of $s_o + L^p_k(\xi)^o$ into $L^{\bar{p}}_o(\xi)$ (this can be seen another way: since $L \in Df^{pk}_k(\xi, \mathbb{R}_M)$ it follows easily that $s \longmapsto \nabla_{j_k(s)}F$ is in $Df^{pk-k}_k(\xi, J^k(\xi))$, then use Corollary 16.12)].

We are now able to do the usual integration by parts. In the present set-up this is accomplished as follows. Since $j_k: C^\infty(\xi) \longrightarrow C^\infty(J^k(\xi))$ is a k^{th} order linear differential operator and since we have specified Riemannian structures on both ξ and $J^k(\xi)$ and a measure on M there is a uniquely determined "adjoint" linear differential operator of order k, $j_k^*: C^\infty(J^k(\xi)) \longrightarrow C^\infty(\xi)$ which is characterized by the identity

$$\int < \lambda, j_k(\sigma) > d\mu = \int < j_k^* \lambda, \sigma > d\mu$$

for $\lambda \in C^\infty(J^k(\xi))$ and $\sigma \in C^\infty(\xi)$ provided at least one of λ and σ has support disjoint from ∂M. By continuity this continues to hold provided $\lambda \in L_0^{\overline{p}}(J^k(\xi))$ and σ is in the closure of $C_0^\infty(\xi)$ in $L_k^p(\xi)$, namely in $L_k^p(\xi)^\circ$. Thus we can write dJ_s in the Euler-Lagrange form:

$$dJ_s(\sigma) = \int < j_k^* (\nabla_{j_k(s)}F), \sigma > d\mu$$

The map $s \longmapsto j_k^*(\nabla_{j_k(s)}F)$ of $C^\infty(\xi)$ into $C^\infty(\xi)$ is called the Euler-Lagrange operator for the problem. Since $s \longmapsto \nabla_{j_k(s)}F$ is in $Df_k^{pk-k}(\xi,J^k(\xi))$ and $j_k^* \in \mathrm{Diff}_k(J^k(\xi),\xi)$ it follows from Definition 16.4 that the Euler-Lagrange operator is in $\mathrm{Div}_{2k}^{pk}(\xi,\xi)$ and hence by Theorem 16.15 that it extends to a C^∞ map of $s_0 + L_k^p(\xi)^\circ$ into $L_{-k}^{\overline{p}}(\xi)$. Since $L_{-k}^{\overline{p}}(\xi)$ is the dual space of $L_k^p(\xi)^\circ = T(\Omega(f))_s$ it follows that s is a critical point of J (i.e. $dJ_s = 0$) if and only if $j_k^* (\nabla_{j_k(s)}F) = 0$, i.e. if and only if s satisfies the Euler-Lagrange equation of the problem.

Let us next see what the Euler-Lagrange operator looks like in "local coordinates". We assume that local coordinates in an open set U of M are chosen so as to be "unimodular" with respect to μ, i.e. the coordinate representation for $d\mu$ is $dx_1dx_2\dots dx_n$. This is always possible locally. Over U we represent ξ as $U \times V$ and $J^k(\xi)$ as $U \times \bigoplus_{|\alpha|\leq k} V$. For simplicity we shall suppose that V has an orthogonal structure and that over U the Riemannian structures for ξ and $J^k(\xi)$ are derived from it via the above local product representations. A section of ξ is given over U by $s: U \longrightarrow V$, $j_k(s)$ is given by $\{D^\alpha s\}_{0\leq|\alpha|\leq k}$ and $L(s) = F(x,D^\alpha s(x))$ where $F: U \times_{|\alpha|\leq k}^{\oplus} V \longrightarrow \mathbb{R}$ is C^∞. Then (see § 17 following Corollary 17.5)

$$\Lambda(D)_s(\sigma)(x) = \sum_{|\beta|\leq k} \frac{\partial F(x,D^\alpha s(x))}{\partial p^\beta} D^\beta \sigma(x)$$

from which it follows that

$$\Lambda_{j_k(s)(x)}F = \{\frac{\partial F(x,D^\alpha s(x))}{\partial p^\beta}\}_{0\le|\beta|\le k} \quad \text{and}$$

$$dJ_s(\sigma) = \int \sum_{|\beta|\le k} < \frac{\partial F(x,D^\alpha s(x))}{\partial p^\beta}, \; D^\beta\sigma(x) > dx_1\ldots dx_n$$

and integrating by parts (or equivalently using the fact that the formal adjoint of D^β is $(-1)^{|\beta|}D^\beta$)

$$dJ_s(x) = \int < \sum_{|\beta|\le k} (-1)^{|\beta|} D^\beta \frac{\partial F(x,D^\alpha s(x))}{\partial p^\beta}, \quad \sigma(x) > dx_1\ldots dx_n$$

so that the Euler-Lagrange operator is $\sum_{|\beta|\le k} (-1)^{|\beta|} D^\beta (\frac{\partial F(x,D^\alpha s(x))}{\partial p^\beta})$

We are going to treat the global questions of existence, uniqueness, and smoothness of critical points of J in a somewhat more specialized (and more tractable) setting. Namely we shall assume that our C^∞ fiber bundle E is a product bundle $M \times W$ and that the fiber W is a closed C^∞ submanifold of an orthogonal vector space V. Thus E is a closed C^∞ sub bundle of the Riemannian vector bundle $\eta = M \times V$ and sections of E (respectively η) are maps of M into W (respectively V). We will also assume that $L \in Lgn_k^{pk}(E)$ is the restriction of an $\mathcal{L} \in Lgn_k^{pk}(\eta)$ which has a specialized form (spelled out below) in terms of the linear structure of η. While this is a substantial reduction of the generality we have been considering up until this point, it does include a number of important special cases and leads to some fairly strong consequences in those situations to which it applies.

We begin with some remarks on the analytical consequences of these new assumptions. In the first place if X is a C^∞ vector field on M (i.e. $X \in C^\infty(T(M))$) we can regard X as an element of $Diff_1(\eta, \eta)$ as follows: if $s \in C^\infty(\eta)$ then s is a C^∞ map of M into V so for $x \in M$ we have $ds_x\colon T(M)_x \longrightarrow V$ and we define $Xs \in C^\infty(\eta)$ by $(Xs)(x) = ds_x(X_x)$. We note then $L(\eta, \eta) = M \times L(V,V)$ is a product Riemannian vector bundle over M so that we may also regard X as an element of $Diff_1(L(\eta,\eta), L(\eta,\eta))$. We also have a bilinear map $(H,\sigma) \longmapsto H\sigma$ of $C^\infty(L(\eta,\eta)) \times C^\infty(\eta) \longrightarrow C^\infty(\eta)$ given by $(H\sigma)(x) = H(x)\sigma(x)$ and clearly $X(H\sigma) = (XH)\sigma + H(X\sigma)$.

Recall that there is a natural norm $\| \;\|_{L_0^q}$ on $L_0^q(\eta)$, namely

$$\|s\|_{L_0^q} = (\int \|s(x)\|^q \, d\mu(x))^{1/q}$$

We will define specific norms on each of the Banach spaces $L^q_\ell(\eta)$ (and $L^q_\ell(L(\eta,\eta))$). Choose a finite set of smooth vector fields on M, say $X_1, \ldots, X_r$ which are ample (i.e. each $v \in T(M)_x$ can be written as a linear combination of the $(X_i)_x$). Then we make the

19.8. Definition.

$$\|s\|_{L^q_\ell} = \left(\sum_{j=0}^{\ell} \sum_{i_1,\ldots,i_j=1}^{r} \|X_{i_1} \cdots X_{i_j} s\|^q_{L^q_0} \right)^{1/q}$$

It is easily seen that this is an admissible norm for $L^q_\ell(\eta)$ (and even a Hilbert space norm when $q = 2$).

We note that as an immediate consequence of the definition:

19.9. Theorem. If $s \in L^q_{\ell+1}(\eta)$ then

$$\|s\|_{L^q_{\ell+1}} = \left(\|s\|^q_{L^q_0} + \sum_{i=1}^{r} \|X_i s\|^q_{L^q_\ell} \right)^{1/q}$$

19.10. Lemma. Let $t > \frac{n}{p}$ and $0 \leq s \leq t$. Then the bilinear map $(H,\sigma) \longmapsto H\sigma$ of $C^\infty(L(\eta,\eta)) \times C^\infty(\eta) \longrightarrow C^\infty(\eta)$ extends to continuous bilinear maps

$$L^p_t(L(\eta,\eta)) \times L^p_s(\eta) \longrightarrow L^p_s(\eta)$$

and

$$L^p_s(L(\eta,\eta)) \times L^p_t(\eta) \longrightarrow L^p_s(\eta)$$

Proof. In case $\eta = M \times \mathbb{R}$ this is just Corollary 9.7. The general case is an easy consequence of this special case.

q.e.d.

Next let $T: \eta \longrightarrow L(\eta,\eta)$ be a C^∞ fiber bundle morphism. For $s \in C^\infty(\eta)$ let $T(s) \in C^\infty(L(\eta,\eta))$ denote the map $x \longmapsto T(s(x))$. Then as we know if $\mathcal{M}$ is a section functor satisfying (B§2) and (B§5) the map $s \longmapsto T(s)$ extends to a C^∞ map (namely $\mathcal{M}(T)$) of $\mathcal{M}(\eta)$ into $\mathcal{M}(L(\eta,\eta))$, which we continue to denote by $s \longmapsto T(s)$ in the present instance. In particular taking $\mathcal{M} = L^p_k$ (where as usual $k > \frac{n}{p}$) we get a C^∞ map $s \longmapsto T(s)$ of $L^p_k(\eta)$ into $L^p_k(L(\eta,\eta))$.

19.11. Lemma. Let $T:\eta \longrightarrow L(\eta,\eta)$ be a C^∞ fiber bundle morphism and let $k > n/p$. If $\{s_i\}$ is any bounded sequence in $L^p_k(\eta)$ then passing to a suitable subsequence we can suppose that for all C^∞ vector fields X on M

$$(X(T(s_i)))(s_j - s_i) + (T(s_i) - T(s_j))(Xs_j)$$

tends to zero in $L^p_{k-1}(\eta)$.

Proof. Choose $0 < \varepsilon < 1$ so that $k - \varepsilon > n/p$. Then by the corollary of Theorem 9.1 the inclusion of $L^p_k(\eta)$ into $L^p_{k-\varepsilon}(\eta)$ is completely continuous so passing to a subsequence we can suppose that $s_i \longrightarrow s_o$ in $L^p_{k-\varepsilon}(\eta)$. Since as we have just remarked $s \longmapsto T(s)$ is C^∞ and hence continuous from $L^p_{k-\varepsilon}(\eta)$ to $L^p_{k-\varepsilon}(L(\eta,\eta))$ it follows that $T(s_i) \longrightarrow T(s_o)$ in $L^p_{k-\varepsilon}(L(\eta,\eta))$ and hence that $||T(s_i) - T(s_j)||_{L^p_{k-\varepsilon}} \longrightarrow 0$, and of course $||s_i - s_j||_{L^p_{k-\varepsilon}} \longrightarrow 0$ also. Now by Lemma 19.10 (taking $t = k-\varepsilon$ and $s = k-1$)

$$||(X(T(s_i)))(s_j - s_i) + (T(s_i) - T(s_j))(Xs_j)||_{L^p_{k-1}}$$

is less than a constant times

$$||X(T(s_i))||_{L^p_{k-1}} \, ||s_i - s_j||_{L^p_{k-\varepsilon}} + ||T(s_i) - T(s_j)||_{L^p_{k-\varepsilon}} \, ||Xs_j||_{L^p_{k-1}}$$

and by Theorem 19.9 the latter is less than a constant times

$$||T(s_i)||_{L^p_k} \, ||s_i - s_j||_{L^p_{k-1}} + ||Ts_i - Ts_j||_{L^p_{k-\varepsilon}} \, ||s_j||_{L^p_k} .$$

Since $||s_i||_{L^p_k}$ is bounded by hypothesis it will suffice to prove that $||Ts_i||_{L^p_k}$ is bounded so the following lemma completes the proof.

19.12. Lemma. If $T:\xi_1 \longrightarrow \xi_2$ is a C^∞ fiber bundle morphism of vector bundles over M then, for $k > \frac{n}{p}$, T maps bounded sets in $L^p_k(\xi_1)$ into bounded sets in $L^p_k(\xi_2)$.

Proof. We reduce easily to the case $M = D^n$, $\xi_1 = M \times \mathbb{R}^r$, $\xi_2 = M \times \mathbb{R}$. Then referring to the proof of Lemma 19.9 we see it is sufficient to prove that if $\varphi:\xi_1 \longrightarrow \xi_2$ is a C^∞ fiber bundle morphism then φ maps a bounded set B of $C^0(\xi_1)$ into a bounded set of $C^0(\xi_2)$. But there is a compact set $K \subseteq \xi_1$ such that $\mathrm{im}(s) \subseteq K$ if $s \in B$ and φ is bounded on K and the lemma follows.

q.e.d.

For each $w \in W$ let $q(w)$ denote the orthogonal projection of $V = T(V)_w$ onto $T(W)_w$. Then q is a C^∞ map of W into the vector space $L(V,V)$ and since W is a closed C^∞ submanifold of V, it extends to a C^∞ map of V into $L(V,V)$. Then if we define $Q(x,v) = (x,q(v))$, Q is a C^∞ fiber bundle morphism of $\eta = M \times V$ into $L(\eta,\eta) = M \times L(V,V)$ and justifies the following definition.

19.13. Definition. We denote by $Q:\eta \longrightarrow L(\eta,\eta)$ a C^∞ fiber bundle morphism such that for $e \in E_x$ $Q(e)$ is the orthogonal projection of $T(\eta_x)_e$ onto $T(E_x)_e$.

19.14. Theorem. If $s \in L^p_k(E)$ then there is a continuous linear projection P_s of $L^p_k(\eta)$ onto $T(L^p_k(E))_s = \{\sigma \in L^p_k(\eta) \mid \sigma(x) \in T(E_x)_{s(x)}\}$ given explicitly by $P_s = Q(s)$, i.e. $(P_s(\sigma))(x) = Q(s(x))\sigma(x)$. If $s \in \Omega(f)$ then P_s restricts to a continuous linear projection of $L^p_k(\eta)^0$ onto $T(\Omega(f))_s = \{\sigma \in L^p_k(\eta)^0 \mid \sigma(x) \in T(E_x)_{s(x)}\}$. Moreover the map $s \longmapsto P_s$ is a C^∞ map of $L^p_k(E)$ into the Banach space $L(L^p_k(\eta), L^p_k(\eta))$ of bounded linear maps of $L^p_k(\eta)$ into itself which takes $L^p_k(\eta)$ bounded sets to bounded sets.

Proof. That $P_s = Q(s)$ is well defined and that $s \longmapsto P_s$ is C^∞ follows from Lemma 19.10 and the remarks preceding Lemma 19.11. Recalling that $Q(s(x))$ is the orthogonal projection of V on $T(E_x)_{s(x)}$ it follows that $P_s^2 = P_s$ and that $P_s\sigma = \sigma$ if and only if $\sigma(x) \in T(E_x)_{s(x)}$ for all x, i.e. if and only if $\sigma \in T(L^p_k(E))_s$, so that P_s is a projection on $T(L^p_k(E))_s$. Clearly if σ vanishes in a neighborhood of ∂M so does $P_s(\sigma)$ and by continuity it follows that P_s maps $L^p_k(E)^0$ into itself, hence if $s \in \Omega(f)$ then P_s projects $L^p_k(\eta)^0$ onto $T(\Omega(f))_s$.

The final remark follows from Lemma 19.12.

q.e.d.

We are now able to state and prove the result we have been leading up to. While it looks a little technical, it expresses what I believe is a very important property of the way $L_k^p(E)$ is embedded in $L_k^p(\eta)$ and it plays a crucial role in what follows. It is due to Karen Uhlenbeck.

<u>19.15</u>. <u>Theorem</u>. Given any sequence $\{s_i\}$ in $L_k^p(E)$ which is bounded in $L_k^p(\eta)$, by passing to a sub-sequence we can suppose that $(I - P_{s_i})(s_i - s_j)$ tends to zero in $L_k^p(\eta)$.

<u>Proof</u>. Since $Q(s_i(x))$ is an orthogonal projection in V , $||(I - Q(s_i(x)))\sigma(x)|| \leq ||\sigma(x)||$, hence by definition of $|| \ ||_{L_o^q}$ it is clear that $||(I - P_{s_i})\sigma||_{L_o^p} < ||\sigma||_{L_o^p}$ and in particular $||(I - P_{s_i})(s_i - s_j)||_{L_o^p} \leq ||s_i - s_j||_{L_o^p}$. Now by Theorem 9.2 the inclusion $L_k^p(\eta) \subseteq L_o^p(\eta)$ is completely continuous so by passing to a subsequence of $\{s_i\}$ we can suppose that $\{s_i\}$ is L_o^p-Cauchy and so $||(I - P_{s_i})(s_i - s_j)||_{L_o^p} \longrightarrow 0$. Hence by Lemma 19.9 it will suffice to show that by passing to a subsequence we can suppose for all C^∞ vector fields X on M that $X((I - P_{s_i})(s_i - s_j))$ tends to zero in $L_{k-1}^p(\eta)$. This in turn will follow from Lemma 19.11 if we establish the identity

$$X((I - P_{s_i})(s_i - s_j)) = (X(Q(s_i)))(s_j - s_i) + (Q(s_i) - Q(s_j))\,(Xs_j) \quad .$$

Now if $s \in C^\infty(E)$ and $\sigma \in C^\infty(\eta)$ then $X(P_s\sigma) = X(Q(s)\sigma) = (X(Q(s)))\sigma + Q(s)(X\sigma)$ and since both sides are (as functions of s and σ) continuous and in fact C^∞ from $L_k^p(E) \times L_k^p(\eta)$ into $L_{k-1}^p(\eta)$ by Lemma 19.10 and the remarks that follow it, the same continues to hold for $s \in L_k^p(E)$ and $\sigma \in L_k^p(\eta)$, hence the two sides of the proposed identity are easily seen to differ by

$$(Xs_i - Q(s_i)(Xs_i)) - (Xs_j - Q(s_j)(Xs_j))$$

so it will suffice to show that $Xs = Q(s)(Xs)$ for $s \in L_k^p(E)$. Now since $s \longmapsto Xs$ is in $\mathrm{Diff}_1(\eta,\eta)$, it is a continuous linear map of $L_k^p(\eta)$ to $L_{k-1}^p(\eta)$, hence both $s \longmapsto Xs$ and $s \longmapsto Q(s)x(s)$ are by Lemma 19.10 continuous linear maps of $L_k^p(E)$ into $L_{k-1}^p(\eta)$ and it will suffice to prove they agree on the dense subspace $C^\infty(E)$. But if $s \in C^\infty(E)$ then s is a C^∞ map of M into W so ds_x maps $T(M)_x$ into $T(W)_{s(x)} = T(E_x)_{s(x)}$ so $(Xs)(x) = ds_x(X_x) \in T(E_x)_{s(x)}$. Then since $Q(s(x))$ is a projection on $T(E_x)_{s(x)}$ the equality $Xs = Q(s)(Xs)$ is immediate.

q.e.d.

For the definition and basic properties of Finsler manifolds we refer to "Lusternik-Schnirelman theory on Banach manifolds" , Topology, Vol. 5 (1966), pp. 115-132 which in the sequel we refer to as [LSTBM]. In particular the following result follows from the Corollary of Theorem 3.6 of that paper.

<u>19.16</u>. <u>Theorem</u>. For $k > \frac{n}{p}$ the Banach manifold $L_k^p(E)$ and its closed submanifold $\Omega(f)$ are closed submanifolds of the Banach space $L_k^p(\eta)$ and hence are complete Finsler manifolds in the Finsler metric induced from the flat Finsler structure on $L_k^p(\eta)$.

Now if X is a Finsler manifold then there is a Finsler structure on its cotangent bundle $T^*(X)$, the norm in $T^*(X)_p$ being given of course by $||\ell|| = \mathrm{Sup}\,\{\ell(v) | v \in T(X)_p \,,\, ||v|| = 1\}$. If Y is a submanifold of X with the induced Finsler structure and $\ell' \in T^*(Y)_p$ is the restriction of $\ell \in T^*(X)_p$ to $T(Y)_p$ then clearly $||\ell'|| \le ||\ell||$. In particular if $F:X \longrightarrow \mathbb{R}$ is a C^1 map and $f = F|Y$ then $df_p = dF_p|T(Y)_p$ so $||df_p|| \le ||dF_p||$. We shall use the above freely in what follows.

<u>19.17</u>. <u>Theorem</u>. Let $\mathcal{J}:L_k^p(\eta) \longrightarrow \mathbb{R}$ be a C^1 map such that $d\mathcal{J}: L_k^p(\eta) \longrightarrow L_k^p(\eta)^*$ maps bounded sets to bounded sets. Let $J:L_k^p(E) \longrightarrow \mathbb{R}$ denote the restriction of $\mathcal{J}$ and let $\{s_i\}$ be an $L_k^p(\eta)$ bounded sequence in $L_k^p(E)$ such that $||dJ_{s_i}|| \longrightarrow 0$. Then passing to a sub-sequence we can suppose

$$d\mathcal{J}_{s_i}(s_i - s_j) \longrightarrow 0$$

The same holds if we replace $L_k^p(E)$ by $\Omega(f)$.

<u>Proof</u>. $d\mathcal{J}_{s_i}(s_i - s_j) = d\mathcal{J}_{s_i}(P_{s_i}(s_i - s_j)) + d\mathcal{J}_{s_i}((I-P_{s_i})(s_i - s_j)) =$ $dJ_{s_i}(P_{s_i}(s_i - s_j)) + d\mathcal{J}_{s_i}((I-P_{s_i})(s_i - s_j))$. Hence $|d\mathcal{J}_{s_i}(s_i - s_j)| \le ||dJ_{s_i}||\cdot||P_{s_i}||\cdot||s_i - s_j||_{L_k^p}$ $+ ||d\mathcal{J}_{s_i}|| \cdot ||(I-P_{s_i})(s_i - s_j)||_{L_k^p}$. Now by hypothesis $||s_i||_{L_k^p}$ (and hence $||s_i - s_j||_{L_k^p}$) and

$||d\mathcal{J}_{s_i}||$ are bounded, and by Theorem 19.14 so also is $||P_{s_i}||$. Since $||dJ_{s_i}|| \longrightarrow 0$ and since by Theorem 19.15 we can suppose (by passing to a sub-sequence) that $||(I-P_{s_i})(s_i - s_j)||_{L^p_k}$ tends to zero the result follows for $L^p_k(E)$. If we replace $L^p_k(E)$ by $\Omega(f)$ the proof is the same once we note that the difference of two elements of $\Omega(f)$ lies in $L^p_k(\eta)^o$, for it then follows that $P_{s_i}(s_i - s_j) \in T(\Omega(f))_{s_i}$ and hence again that $d\mathcal{J}_{s_i}(P_{s_i}(s_i - s_j)) = dJ_{s_i}(P_{s_i}(s_i - s_j))$.

q.e.d.

Our existence theorems for solutions of Dirichlet problems is based on the following concept introduced by the author and S. Smale.

19.18. Definition. Let X be a C^1 Finsler manifold. A C^1 map $F:X \longrightarrow \mathbb{R}$ is said to satisfy condition (C) if given any subset S of X such that $|F|$ is bounded on S but $||dF||$ is not bounded away from zero on S , there is a critical point of F adherent to S .

19.19. Theorem. Let X be a complete C^2 Finsler manifold and let $F:X \longrightarrow \mathbb{R}$ be a C^2 function which is bounded below and satisfies condition (C) . Then F assumes a minimum on each component of X . In fact F satisfies the conclusions of Lusternik-Schnirelman theory [LSTBM, Theorem 7.1] and in particular on each component of X there are at least as many critical points of F as the Lusternik-Schnirelman category of that component. If X is Riemannian and the critical points of f are all non-degenerate than F satisfies the conclusions of Morse Theory and in particular the type numbers of F and the betti-numbers of X satisfy the Morse inequalities.

The meaning of the last statement of the above theorem is explained in "Morse Theory on Hilbert Manifolds", Topology, vol. 2 (1963), pp. 299-340, referred to as [MTHM] in the sequel. In that paper it is assumed that X and F are C^3, but N. Kuiper has recently extended the results to the C^2 case.*

What we shall now do is give a general condition on a C^2 function $\mathcal{J}:L^p_k(\eta) \longrightarrow \mathbb{R}$ that insures that the function $J:\Omega(f) \longrightarrow \mathbb{R}$ given by $J = \mathcal{J}|\Omega(f)$ satisfies the conditions imposed in Theorem 19.19 on $F:X \longrightarrow \mathbb{R}$. We will then discuss the question of what conditions on $\mathcal{L} \in \mathrm{Lgn}^{pk}_k(\eta)$ will insure that $\mathcal{J}(s) = \int \mathcal{L}(s)d\mu$ will have the desired properties.

* This also follows from results in Smale's "Morse Theory and a non-linear generalization of the Dirichlet Problem", Annals of Math. (80), Sept. 1964, pp. 382-396.

19.20. Theorem. Let $\mathcal{J} : L_k^p(\eta) \longrightarrow \mathbb{R}$ be a C^2 function such that $d\mathcal{J} : L_k^p(\eta) \longrightarrow L_k^p(\eta)^*$ maps bounded sets to bounded sets. Assume in addition that for $s_1, s_2 \in f + L_k^p(\eta)^o$

$$(*) \qquad (d\mathcal{J}_{s_1} - d\mathcal{J}_{s_2})\ (s_1 - s_2) \geq ||s_1 - s_2||_{L_k^p}\ \varphi(||s_1 - s_2||_{L_k^p})$$

where φ is a strictly monotone map of $\mathbb{R}^+$ to itself satisfying $\lim_{t\to 0} \varphi(t) = 0$ and $\lim_{t\to\infty} \varphi(t) = \infty$. Then the restriction of $\mathcal{J}$ to $\Omega(f)$ is a C^2 function $J:\Omega(f) \longrightarrow \mathbb{R}$ which satisfies the hypotheses of Theorem 19.19, i.e. J is bounded below and satisfies condition (C).

Proof. We note that $\Omega(f) \subseteq f + L_k^p(\eta)^o$, and in fact recalling that $\Omega(f) = (L_k^p)_{\partial f}(E)$ it is clear from the argument of proof of Theorem 19.7 that $\Omega(f) = (f + L_k^p(\eta)^o) \cap L_k^p(E)$. Since $f + L_k^p(\eta)^o$ is convex it follows the convex hull of $\Omega(f)$ is included in $f + L_k^p(\eta)^o$. Thus if we choose $a \in \Omega(f)$ and for any other $s \in \Omega(f)$ put $\sigma(t) = a + t(s-a)$ then $\sigma(t) \in f + L_k^p(\eta)^o$ for $0 \leq t \leq 1$ so

$$(d\mathcal{J}_{\sigma(t)} - d\mathcal{J}_a)\ (t(s-a)) \geq t||s-a||_{L_k^p}\ \varphi(t||s-a||_{L_k^p}) \quad .$$

Thus

$$\begin{aligned}
\mathcal{J}(s) &= \mathcal{J}(a) + \int_0^1 d\mathcal{J}_{\sigma(t)}\ (s-a)\ dt \\
&= \mathcal{J}(a) + d\mathcal{J}_a(s-a) + \int_0^1 \frac{1}{t}\ (d\mathcal{J}_{\sigma(t)} - d\mathcal{J}_a)\ (t(s-a))\ dt \\
&\geq \mathcal{J}(a) + d\mathcal{J}_a(s-a) + ||s-a||_{L_k^p} \int_0^1 \varphi\ (t||s-a||_{L_k^p})\ dt \\
&\geq \mathcal{J}(a) + (\int_0^1 \varphi\ (t||s-a||_{L_k^p})\ dt - K)\ ||s-a||_{L_k^p}
\end{aligned}$$

where K is any positive number greater than $||d\mathcal{J}_a||$. Since $\lim_{t\to\infty} \varphi(t) = \infty$ we can choose $r > 0$

so that $\varphi(\frac{1}{2}r) > 2(1+\varepsilon)K$ where $\varepsilon > 0$. Then since φ is positive and monotonically increasing for $||s-a||_{L^p_k} \geq r$ we have $\int_0^1 \varphi(t||s-a||_{L^p_k})dt > \int_{\frac{1}{2}}^1 2(1+\varepsilon)Kdt = (1+\varepsilon)K$. Thus for $||s-a||_{L^p_k} \geq r$ we have $\mathcal{J}(s) \geq \mathcal{J}(a) + \varepsilon K\, ||s-a||_{L^p_k}$, while for $||s-a||_{L^p_k} < r$ we have $\mathcal{J}(s) \geq \mathcal{J}(a) - K\,||s-a||_{L^p_k} > \mathcal{J}(a)\text{-}Kr$, so that J is bounded below by J(a) - Kr. Now if S is a subset of $\Omega(f)$ on which J is bounded, say by C, then for $s \in S$ we have

$$||s-a||_{L^p_k} \leq \frac{C - \mathcal{J}(a)}{\varepsilon K}$$

and hence $||s||_{L^p_k} \leq ||a||_{L^p_k} + \frac{C - \mathcal{J}(a)}{\varepsilon K}$ provided $||s-a||_{L^p_k} > r$, while of course $||s||_{L^p_k} < ||a||_{L^p_k} + r$ if $||s-a||_{L^p_k} < r$. Hence S is an $L^p_k(\eta)$ bounded set. We now assume in addition that $||dJ||$ is not bounded away from zero on S and choose $\{s_i\}$ in S with $||dJ_{s_i}|| \longrightarrow 0$. By Theorem 19.17 we have after passing to a subsequence that $d\,\mathcal{J}_{s_i}(s_i - s_j) \longrightarrow 0$ and hence $(d\,\mathcal{J}_{s_i} - d\,\mathcal{J}_{s_j})\,(s_i - s_j) \longrightarrow 0$ so that by (*)

$$||s_i - s_j||_{L^p_k}\, \varphi\,(||s_i - s_j||_{L^p_k}) \longrightarrow 0$$

Given $\varepsilon > 0$, since $\lim\limits_{t \to 0} \varphi(t) = 0$ and φ is strictly monotone, it follows that $\varepsilon\,\varphi(\varepsilon) > 0$ and moreover if $t\,\varphi(t) < \varepsilon\,\varphi(\varepsilon)$ then $t < \varepsilon$. Thus if we choose N so that $i, j > N$ implies $||s_i - s_j||_{L^p_k}\, \varphi\,(||s_i - s_j||_{L^p_k}) < \varepsilon\,\varphi(\varepsilon)$ it also implies that $||s_i - s_j||_{L^p_k} < \varepsilon$. Hence $\{s_i\}$ is L^p_k-Cauchy and, since $\Omega(f)$ is closed in $L^p_k(\eta)$, $s_i \longrightarrow s \in \Omega(f)$. Moreover, since $||dJ||$ is continuous, $||dJ_s|| = \lim ||dJ_{s_i}|| = 0$ so s is a critical point of J which is adherent to S and condition (C) is satisfied.

q.e.d.

19.21. Theorem. If the fiber of E is compact then in Theorem 19.20 we can replace the condition $(*)$ by the following weaker condition:

$$(**) \qquad (d\mathcal{J}_{s_1} - d\mathcal{J}_{s_2})(s_1 - s_2) \geq \|s_1 - s_2\|_{L^p_k} \varphi(\|s_1 - s_2\|_{L^p_k}) - \|s_1 - s_2\|_{L^p_o} \psi(\|s_1 - s_2\|_{L^p_o})$$

where φ as in Theorem 19.20 and $\psi : \mathbb{R}^+ \longrightarrow \mathbb{R}^+$ maps bounded sets to bounded sets.

Proof. Since E has compact fiber it follows that $\Omega(f)$ (and in fact $L^p_k(E)$) is an $L^p_o(\eta)$ bounded set and hence so also is its convex hull. Choose $B > 0$ so that $\psi(t) < B$ for $t < 2 \operatorname{Sup} \{\|s\|_{L^p_o} \mid s$ is in the convex hull of $\Omega(f)\}$.

Since $\|s\|_{L^p_o} \leq \|s\|_{L^p_k}$, for s_1 and s_2 in the convex hull of $\Omega(f)$ we have $(d\mathcal{J}_{s_1} - d\mathcal{J}_{s_2})(s_1 - s_2) \geq \|s_1 - s_2\|_{L^p_k} (\varphi \|s_1 - s_2\|_{L^p_k} - B)$. Then if we replace K by $K + B$ in the first part of the proof of Theorem 19.20 (where it is proved that J is bounded below and that J bounded sets are $L^p_k(\eta)$ bounded) we get the same results in the present case. In proving that condition (C) is satisfied, the sequence $\{s_i\}$ which we must show has a L^p_k -Cauchy subsequence is known to be L^p_k -bounded and hence has an L^p_o -Cauchy subsequence by Rellich's Theorem, and hence we may assume that $\|s_i - s_j\|_{L^p_o} \psi(\|s_i - s_j\|_{L^p_o})$ tends to zero. Since as before we know that $(d\mathcal{J}_{s_i} - d\mathcal{J}_{s_j})(s_i - s_j)$ tends to zero for a suitable subsequence we can by $(**)$ assume that $\|s_i - s_j\|_{L^p_k} \varphi(\|s_i - s_j\|_{L^p_k})$ tends to zero and the remainder of the proof is the same.

q.e.d.

19.22. Definition. Let $U_1, \ldots, U_r$ be orthogonal vector spaces and let $\zeta_i = M \times U_i$ be the corresponding product Riemannian vector bundles over M. If $A_i \in \operatorname{Diff}_k(\eta, \zeta_i)$ $i = 1, \ldots, r$ then we say that $\{A_i\}$ is an <u>ample</u> family of k^{th} order linear operators for η provided there exist constants

C_1 and C_2 such that for all $U \in L_k^p(\eta)^o$

$$||U||^p_{L^p_k} \leq C_1 \sum_{i=1}^{r} ||A_i U||^p_{L^p_o} + C_2 ||U||^p_{L^p_o}$$

and we shall say that $\{A_i\}$ is strongly ample if we can choose $C_2 = 0$.

Remark. In the inequality defining ampleness we can replace $||U||_{L^p_o}$ by $||U||_{L^p_{k-1}}$ and the resulting condition is equivalent. In one direction this is clear and in the other it follows from the well known fact (sometimes called the Poincare inequality) that given $\varepsilon > 0$ there exists $C > 0$ such that $||U||_{L^p_{k-1}} \leq \varepsilon ||U||_{L^p_k} + C||U||_{L^p_o}$.

19.23. Theorem. If $\{A_i\}$ is an ample family of k^{th} order linear operators for η then a nessary and sufficient condition that it be strongly ample is that the only $u \in L_k^p(\eta)^o$ for which all $A_i u = 0$ is $u = 0$.

Proof. Necessity is clear. Conversely if $\{A_i\}$ is not strongly ample then we can find a sequence $\{u_n\}$ in $L_k^p(\eta)^o$ with $||u_n||_{L^p_k} = 1$ and $A_i u_n \longrightarrow 0$ in $L_k^p(\eta)$ for all i . Passing to a sub-sequence we can suppose $\{u_n\}$ converges in the weak topology of $L_k^p(\eta)^o$ to an element u . Since $A_i: L_k^p(\eta)^o \longrightarrow L_o^p(\eta)$ is linear and continuous, it is also continuous when both spaces have their weak topologies, and since $A_i u_n$ converges strongly (hence weakly) to zero in $L_o^p(\eta)$ it follows that $A_i u = 0$ for all i . Hence to prove sufficiency of the condition it will suffice to prove that $u \neq 0$. But by Rellich's Theorem we can (passing to a subsequence) suppose $\{u_n\}$ converges (strongly) in $L_o^p(\eta)$ to a limit u^* . Since u_n converges to u weakly in $L_k^p(\eta)$ and hence a fortiori weakly in $L_o^p(\eta)$ it follows that $u = u^*$. Then from the ampleness inequality and the fact that $||u_n||_{L^p_o} \longrightarrow ||u||_{L^p_o}$ while $||A_i u_n||_{L^p_o} \longrightarrow 0$ it follows that $||u||^p_{L^p_o} \geq \frac{1}{c^2} > 0$, hence $u \neq 0$.

q.e.d.

Now let $\{A_i\}$ be as in Definition 19.22 and let $A = A_1 \oplus \dots \oplus A_r \in \mathrm{Diff}_k(\eta, \zeta)$ where $\zeta = \zeta_1 \oplus \dots \oplus \zeta_r = M \times U$ where $U = U_1 \oplus \dots \oplus U_r$. Note that

$||Au(x)||^2 = \sum_{i=1}^{r} ||A_i u(x)||^2$ so that

$$\frac{1}{r} \sum_{i=1}^{r} ||A_i u(x)||^p \leq ||Au(x)||^p \leq \sum_{i=1}^{r} ||A_i u(x)||^p$$

and integrating over M

$$\frac{1}{r} \sum_{i=1}^{r} ||A_i u||^p_{L^p_o} \leq ||Au||^p_{L^p_o} \leq \sum_{i=1}^{r} ||A_i u||^p_{L^p_o}$$

from which it is immediate that $\{A_i\}$ is ample if and only if $\{A\}$ is ample, and since $Au = 0$ if and only if all $A_i u = 0$ the same holds for "strongly ample". In other words we can always replace an ample family by a single ample operator. Note also that $\sigma_k(A) = \sigma_k(A_1) \oplus \dots \oplus \sigma_k(A_r)$ so that $\sigma_k(A)(v,x)$ is injective if and only if the intersection of the kernels of $\sigma_k(A_i)(v,x): \eta_x \longrightarrow (\zeta_i)_x$ is zero.

<u>19.25</u>. <u>Theorem</u>. If $A_i \in \mathrm{Diff}_k(\eta, \zeta_i)$ are as in Definition 19.22 then a necessary and sufficient condition that $\{A_i\}$ be an ample family of k^{th} order linear operators for η is that for each non-zero cotangent vector $(v,x) \in T^*(M)_x$ the intersection of the kernels of the linear symbol maps $\sigma_k(A_i)(v,x): \eta_x \longrightarrow (\zeta_i)_x$ is zero.

<u>Proof</u>. By the preceding remark we can suppose $\{A_i\} = \{A\}$ so our condition is that $\sigma_k(A)(v,x): \eta_x \longrightarrow \zeta_x$ is injective for each non-zero cotangent vector (v,x), and the condition for ampleness is then just the usual "a priori estimate" for such overdetermined elliptic operators. We sketch the proof. Given an injective linear map $T: V \longrightarrow U$ let P_T be the orthogonal projection of U on $\mathrm{im}(T)$ and define $F(T): V \longrightarrow U$ by $F(T)(u) = T^{-1}(P_T u)$. Then it is easily seen that F is a C^∞ map of the open submanifold of $L(V,U)$ consisting of injective maps to $L(U,V)$ and it follows that if we define $\sigma(v,x) = F(\sigma_k(A)(v,x))$ then σ is a symbol of order $-k$ so we can find a pseudo differential operator B from ζ to η such that $\sigma_{-k}(B) = \sigma$. Clearly $\sigma_o(BA) = \sigma_{-k}(B)\, \sigma_k(A) = \sigma_o(I)$,

hence $BA = I + S$ where S is a pseudo differential operator of order -1 . Then

$$\|u\|_{L^p_k} \le \|BAu\|_{L^p_k} + \|Su\|_{L^p_k} \le a_1\|Au\|_{L^p_o} + a_2\|u\|_{L^p_{k-1}}$$

and then using the remark following Definition 19.22 the ampleness of A is immediate.

q.e.d.

Examples. It follows from Definition 19.8 that if $X_1,\ldots,X_r$ is an ample set of C^∞ vector fields on M (considered as elements of $\mathrm{Diff}_1(\eta,\eta)$) then $(X_{i_1}X_{i_2}\ldots X_{i_k}) \in \mathrm{Diff}_k(\eta,\eta)$, with the indices i_j varying independently from 1 to r, forms an ample family of k^{th} order linear operators for η. However by the preceding theorem it follows that there are much smaller ample sub-families. For example $X_1^k, X_2^k, \ldots, X_r^k$ is ample. Indeed if (v,x) is a cotangent vector of M then it is easily seen that $\sigma_1(X_i)(v,x)$ is multiplication by $v(X_i)$, hence $\sigma_k(X_i^k)(v,x)$ is multiplication by $v(X_i)^k$. Since $(X_1)_x, \ldots, (X_r)_x$ spans the tangent space of M at x it follows that if $v \ne 0$ then some $v(X_i) \ne 0$ so $\sigma_k(X_i^k)(v,x)$ is injective. Also since $\sigma_k(X_1^k + \ldots + X_r^k)(v,x)$ is multiplication by $\sum_{i=1}^r v(X_i)^k$ it follows that for k even the single k^{th} order operator $(X_1^k + \ldots + X_r^k)$ is an ample k^{th} order linear operator for η as is $(X_1^2 + \ldots + X_r^2)^{\frac{k}{2}}$ whose symbol at (v,x) is multiplication by $(\sum_i v(X_i)^2)^{\frac{k}{2}}$. Also in the case of even k we can choose as a single ample k^{th} order linear operator $\Delta^{k/2}$ where Δ is the Laplace-Beltrami operator of any Riemannian metric for M ; for $\sigma_k(\Delta^{k/2})(v,x)$ is multiplication by $\|v\|^k$.

If $\mathcal{L} \in \mathrm{Lgn}_k(\eta) = \mathrm{Df}_k(\eta,\mathbb{R}_M)$, then $\mathcal{L}(s) = F_*(j_k(s))$ where $F: J^k(\eta) \longrightarrow \mathbb{R}_M$ is a C^∞ fiber bundle morphism. Then for each $x \in M$ the restriction of F to $J^k(\eta)_x$ is a C^∞ real valued function F^x on the vector space $J^k(\eta)_x$ and for any $j_k(s)_x$ in $J^k(\eta)_x$ we can therefore form the second differential $(d^2F^x)_{j_k(s)(x)} = \delta^2 F_{j_k(s)(x)}$ which is a symmetric bilinear form on $J^k(\eta)_x$.

The following definition and theorem are essentially due to Karen Uhlenbeck, with some small modifications by the author.

19.26. Definition. If $\mathcal{L} \in \mathrm{Lgn}_k(\eta)$, say $\mathcal{L} = F_* \circ j_k$, then we say that $\mathcal{L}$ is (strongly) p-coercive if there exists a (strongly) ample family $\{A_i\}$ of k^{th} order linear operators for η such that for all $s, u \in C^\infty(\eta)$

$$\delta^2 F_{j_k(s)}(j_k u, j_k u) \geq \sum_{i=1}^{r} ||A_i s||^{p-2}||A_i u||^2 \quad .$$

19.27. Lemma. If U is an orthogonal vector space the given $m > 0$ there is a $c > 0$ such that for all $x, y \in U$

$$\int_0^1 ||x + ty||^m dt \geq c||y||^m \quad .$$

Proof. We can assume $y \neq 0$ so dividing by $||y||^m$ we must prove that $F(x,y) = \int_0^1 ||x + ty||^m dt \geq c > 0$ for all y on the unit sphere. If $||x|| > 2$ then $||x + ty|| \geq 1$ so $F(x,y) \geq 1$. On the other hand F is a continuous positive function on the compact $\{x \in U| \; ||x|| \leq 2\} \times \{y \in U| \; ||y|| = 1\}$ and hence has a positive lower bound there.

q.e.d.

19.28. Theorem. If $\mathcal{L} \in Lgn_k(\eta)$ is p-coercive then there exist $c_1 > 0$ and $c_2 \geq 0$ such that for all $s_1, s_2 \in C^\infty(\eta)$ we have

$$\int_M (\Lambda(\mathcal{L})_{s_1} - \Lambda(\mathcal{L})_{s_2})(s_1 - s_2)d\mu \geq c_1||s_1 - s_2||^p_{L^p_k} - c_2||s_1 - s_2||^p_{L^p_o} \quad .$$

Moreover if $\mathcal{L}$ is strongly p-coercive we may take $c_2 = 0$.

Proof. Let $\mathcal{L} = F_* \circ j_k$ where F satisfies the inequality of Definition 19.26 and the A_i satisfy the inequality of Definition 19.22. Put
$a = j_k(s_1)(x)$, $b = j_k(s_2)(x)$, $c(t) = b + t\,(a-b)$, $x_i = A_i s_2(x)$, and $y_i = A_i(s_1 - s_2)(x)$.
Then by Definition 19.26 we have

$$\delta^2 F_{c(t)}(a-b,a-b) \geq \sum_{i=1}^{r} ||x_i + ty_i||^{p-2}||y_i||^2$$

Now $\int_0^1 \delta^2 F_{c(t)}(a-b,a-b)dt = (\delta F_a - \delta F_b)(b-a)$, so that using Lemma 19.27 and recalling that $\Lambda(\mathcal{L})_s(u)(x) = \delta F_{j_k s_1(x)}(j_k u(x))$ we find $(\Lambda(\mathcal{L})_{s_1} - \Lambda(\mathcal{L})_{s_2})(s_1 - s_2) \geq c \sum_{i=1}^{r} ||A_i(s_1 - s_2)||^p$..

If we now integrate over M with respect to the measure μ and use Definition 19.22 we get the desired result.

q.e.d.

We now come to our main existence theorem for a class of Dirichlet variational problems. For convenience we restate a number of the definitions we have been using as part of the hypotheses of the theorem.

19.29. Theorem. Let M be a compact n-dimensional C^∞ manifold (possibly with boundary) with a smooth measure μ. Let V be an orthogonal vector space and W a closed submanifold (without boundary) of V. Let η denote the product Riemannian bundle $M \times V$ and E the closed C^∞ sub-bundle $M \times W$. Let k be a positive integer, $p \geq 2$ an integer with $pk > n$ and let $\mathscr{L} \in \mathrm{Lgn}_k^{pk}(\eta) = \mathrm{Df}_k^{pk}(\eta, \mathbb{R}_M)$ be p-coercive, and if W is not compact assume $\mathscr{L}$ is strongly p-coercive. Define a C^∞ map $\mathscr{J}: L_k^p(\eta) \longrightarrow \mathbb{R}$ by $\mathscr{J}(s) = \int \mathscr{L}(s) d\mu$ (Corollary 19.5) and for $f \in L_k^p(E)$ let J denote the restriction of $\mathscr{J}$ to $\Omega(f) = (f + L_k^p(\eta)^\circ \cap L_k^p(E) =$ the closure in $L_k^p(E)$ of those $g \in L_k^p(E)$ which agree with f in a neighborhood of ∂M. Then, regarding $\Omega(f)$ as a Finsler manifold in the Finsler structure induced from the flat Finsler structure of $L_k^p(\eta)$, $J:\Omega(f) \longrightarrow \mathbb{R}$ is C^∞ map which is bounded below and satisfies condition (C). Hence (Theorem 19.19) J assumes an absolute minimum on each component of $\Omega(f)$, and on each component of $\Omega(f)$ J has at least as many critical points as the Lusternik-Schnirelman category of that component.

Proof. Let $\mathscr{L} = F \circ j_k$ where $F:J^k(\eta) \longrightarrow \mathbb{R}_M$ is a C^∞ fiber bundle morphism. Then $\delta F:J^k(\eta) \longrightarrow J^k(\eta)^*$ is a C^∞ fiber bundle morphism which defines an element of $\mathrm{Df}_k^{pk-k}(\eta, J^k(\eta)^*)$ and by Corollary 16.12 a C^∞ map $L_k^p(\eta) \longrightarrow L_o^{\bar{p}}(J^k(\eta)^*)$ where $\frac{1}{p} + \frac{1}{\bar{p}} = 1$. By the argument of Lemma 19.12 the latter map takes bounded sets to bounded sets. Since $d\mathscr{J}_s(u) = \int \delta F_{j_k(s)}(j_k u) d\mu$ it follows that $d\mathscr{J}: L_k^p(\eta) \longrightarrow L_k^p(\eta)^*$ maps bounded sets to bounded sets. Since for $s \in C^\infty(\eta)$ we have $\Lambda(\mathscr{L})_s = \delta F_{j_k(s)} \circ j_k$ and since $d\mathscr{J}_s(u) = \int \Lambda(\mathscr{L})_s(u)$ we have from Theorem 19.28

$$(d\mathscr{J}_{s_1} - d\mathscr{J}_{s_2})(s_1 - s_2) \geq c_1 \| s_1 - s_2 \|_{L_k^p}^p - c_2 \| s_1 - s_2 \|_{L_o^p}^p$$

(where if W is not compact we can take $c_2 = 0$) at least for $s_1, s_2 \in C^\infty(\eta)$. However both sides of the above inequality are continuous functions of s_1 and s_2 in the L^p_k topology so the inequality in fact holds for all $s_1, s_2 \in L^p_k(\eta)$. The Theorem now follows from Theorems 19.20 and 19.21.

q.e.d.

We will now show how to find a rich supply of p-coercive Lagrangians in $\mathrm{Lgn}^{pk}_k(\eta)$.

19.30. Lemma. Let m be a positive integer, $p = 2m$ and let $T:V \longrightarrow U$ be a linear map of a finite dimensional real vector space into an orthogonal vector space. Then $f:V \longrightarrow \mathbb{R}$ defined by $f(x) = ||Tx||^p$ is a polynomial map of degree p and

$$df_s(u,u) \geq m||Ts||^{p-2}||Tu||^2$$

Proof. Since $f(s) = (Ts,Ts)^m$ and the inner product is bilinear f is a polynomial of degree $2m = p$ and $df_s(u) = m(Ts,Ts)^{m-1}(Ts,v)$ so that

$d^2f_s(u,v) = m(Ts,Ts)^{m-1}(Tu,Tv) + m(m-1)(Ts,Ts)^{m-2}(Ts,Tu)(Ts,Tv)$ and hence

$$d^2f_s(u,u) = m||Ts||^{p-2}||Tu||^2 + m(m-1)||Ts||^{p-4}(Ts,Tu)^2 = m||Ts||^{p-2}||Tu||^2\left(1 + (m-1)\,\frac{(Ts,Tu)^2}{||Ts||^2||Tu||^2}\right).$$

q.e.d.

19.31. Theorem. If $\{A_i\}$ is a (strongly) ample set of k^{th} order linear operators for η and we define $\mathcal{L}(u) = \sum_i ||A_iu||^p$ where p is an even positive integer, then $\mathcal{L}$ is in $\mathrm{Lgn}^{pk}_k(\eta)$ and is (strongly) p-coercive.

Proof. That $\mathcal{L} \in \mathrm{Lgn}^{pk}_k(\eta)$ is immediate from Theorem 16.7. If we write $A_i = T_i \circ j_k$ where $T_i:J^k(\eta) \longrightarrow \zeta_i$ is a C^∞ vector bundle homomorphism ($\zeta_i = M \times U_i$) then for each $x \in M$ we have a linear map $T^x_i:J^k(\eta)_x \longrightarrow (\zeta_i)_x = U_i$ defining a map $f^x_i:J^k(\eta)_x \longrightarrow \mathbb{R}$ by

$f^x_i(j_k(s)(x)) = ||T^x_i\, j_k(s)(x)||^p = ||A_is(x)||^p$. Then by Lemma 19.30

$d^2(f^x_i)_{j_k(s)(x)}(j_ku(x),j_ku(x)) \geq m||T^x_ij_k(s)(x)||^{p-2}||T^x_ij_k(u)(x)||^2 = m||A_is(x)||^{p-2}$. Now

$F|J^k(\xi)_x = \sum_i f^x_i$ so $\delta^2F_{j_k(s)(x)} = \sum_i (d^2f^x_i)_{j_k(s)(x)}$ so we have $\delta^2F_{j_k(s)}(j_ku,j_ku) \geq m||A_is||^{p-2}||A_iu||^2$

which by Definition 19.26 completes the proof. q.e.d.

There is an important generalization of the siutation of Theorem 19.31 in the case $p = 2$. Let $\{A_i\}$ be a (strongly) ample set of k^{th} order operators for η and let $\mathscr{J}(s) = \sum_i ||A_i s||^2_{L^2_0}$. Given $f \in L^2_k(E)$ let us subtract the essentially irrelevant constant $\mathscr{J}(f)$ getting

$$\tilde{\mathscr{J}}(s) = \mathscr{J}(s) - \mathscr{J}(f) = \mathscr{J}((s-f)+f) - \mathscr{J}(f) = \sum_i (A_i(s-f), A_i(s-f)) + 2\sum_i (A_i f, A_i(s-f))$$

Then for $s \in f + L^2_k(\eta)^\circ$ we may write $\tilde{\mathscr{J}}(s) = (L(s-f),(s-f)) + 2(Lf,(s-f))$ where $L \in \mathrm{Diff}_{2k}(\eta,\eta)$ is the self adjoint, strongly elliptic, non-negative operator $L = \sum_i A_i^* A_i$. The strong ellipticity of L is trivially equivalent to the ampleness of $\{A_i\}$ and L is strictly positive if and only if the $\{A_i\}$ are strongly ample. This suggests the following generalization of the results implied by Theorems 19.29 and 19.31.

<u>19.32</u>. <u>Theorem</u>. Let M, η, E be as in Theorem 19.29 and let $k > \frac{n}{2}$. Let $L \in \mathrm{Diff}_{2k}(\eta,\eta)$ be a strongly elliptic, self adjoint operator, and if W (the fiber of E) is not compact assume in addition that L is positive. Given $f \in L^2_k(E)$ define a C^∞ real valued function $\mathscr{J}: f + L^2_k(\eta)^\circ \longrightarrow \mathbb{R}$ by $\mathscr{J}(s) = (Ls,s) - (Lf,f) = (L(s-f),(s-f)) + 2(Lf,(s-f))$. Let J denote the restriction of $\mathscr{J}$ to the closed submanifold $\Omega(f) = L^2_k(E) \cap (f + L^2_k(\eta)^\circ)$ of the Hilbert space $L^2_k(\eta)$. Then regarding $\Omega(f)$ as a Riemannian manifold in the Riemannian structure induced from $L^2_k(\eta)$, $J:\Omega(f) \longrightarrow \mathbb{R}$ is bounded below and satisfies condition (C).

<u>Proof</u>. Since L extends to a continuous linear map of $L^2_k(\eta)$ into $L^2_{-k}(\eta) = (L^2_k(\eta)^\circ)^*$ (Theorem 6.5) and since the inner product $(\ ,\ )$ in $L^2_0(\eta)$ extends to a continuous bilinear pairing of $L^2_k(\eta) \times L^2_{-k}(\eta) \longrightarrow \mathbb{R}$ (i.e. L^2 is self-dual) it follows that $\mathscr{J}$ is in fact a quadratic polynomial on $f + L^2_0(\eta)$. Thus $\mathscr{J}$ is C^∞ and $d\mathscr{J}_s(u) = (Ls,u)$ for $s \in f + L^2_k(\eta)^\circ$ and $u \in T(f + L^2_k(\eta)^\circ)_s = L^2_k(\eta)^\circ$, i.e. $d\mathscr{J} = L$ which is bounded from $L^2_k(\eta)$ to $L^2_{-k}(\eta)$. The second condition of Theorem 19.20 is met because if $s_1, s_2 \in f + L^2_k(\eta)^\circ$ then $s_1 - s_2 \in L^2_k(\eta)^\circ$ so by Gårding's inequality

$$\begin{aligned}(d\mathscr{J}_{s_1} - d\mathscr{J}_{s_2})(s_1 - s_2) &= (L(s_1 - s_2), s_1 - s_2) \\ &\geq c_1||s_1 - s_2||^2_{L^2_k} - c_2||s_1 - s_2||^2_{L^2_0}\end{aligned}$$

where $c_1 > 0$, $c_2 \geq 0$ and if L is positive we can take $c_2 = 0$. Then the theorem follows from Theorems 19.20 and 19.21 (the assumption in 19.20 that $\mathcal{J}$ was defined on all of $L_k^p(\eta)$ rather than only $f + L_k^p(\eta)^\circ$ was clearly unnecessary).

q.e.d.

Essentially the above theorem was proved by John Saber in his thesis (Brandeis 1965), except that Saber assumed that L was "scalar" in the sense that $\sigma_{2k}(L)(v,x)$ was multiplication by a scalar.

We are now in a position to give a number of interesting applications of Theorem 19.29.

First let us consider the case $n = 1$. In this case we have either $M = I = [0,1]$ or $M = S^1 = \mathbb{R}/Z$. Choose p as small as possible, namely $p = 2$ (so we are in the situation of Hilbert manifolds) we must have $2k = pk > n = 1$ so again choosing k as small as possible we have $k = 1$. An obvious choice for an ample 1^{st} order linear operator is $A = \frac{d}{dx} \varepsilon \operatorname{Diff}_1(\eta, \eta)$. Then the solutions of $u' = Au = 0$ are just the constant maps of M into V. Now if $M = I$ then clearly $L_1^2(\eta)^\circ$ consists of all $u \varepsilon L_1^2(\eta)$ which vanish at 0 and 1, hence there is no non-zero $u \varepsilon L_1^2(\eta)^\circ$ satisfying $Au = 0$, while if $M = S^1$ then $\partial M = \emptyset$ so $L_1^2(\eta)^\circ = L_1^2(\eta)$ and the kernel of $A|L_1^2(\eta)^\circ$ is one-dimensional. Hence A is strongly ample in case $M = I$, but not in the case $A = S^1$ so in the latter case we must assume W (the fiber of E) is compact. Following 19.31 we define $\mathcal{L} \varepsilon \operatorname{Lgn}_1^2(\eta)$ by $\mathcal{L}(s) = ||As||^2 = ||s'||^2$ and $\mathcal{J}: L_1^2(\eta) \longrightarrow \mathbb{R}$ by $\mathcal{J}(s) = \int_0^1 ||s'||^2 dt$. If $f \varepsilon L_1^2(E)$ then (for $M = I$) $\Omega(f) = \{s \varepsilon L_1^2(E) | s(0) = f(0)$ and $s(1) = f(1)\}$ while for $M = S^1$, $\Omega(f) = L_1^2(E)$ so that by Theorem 13.9 up to homotopy type $\Omega(f)$ is the loop space of W when $M = I$, while in the case $M = S^1$ $\Omega(f)$ is a fiber space over W (the projection being $s \longmapsto s(0)$) whose fiber is the loop space of W. Now $\Lambda(\mathcal{L})_s(u) = 2(As, Au) = 2(s', u')$ so $d\mathcal{J}_s(u) = 2\int_0^1 (s'(t), u'(t))dt$. If s is a critical point of J then s is C^∞ (see [MTHM, § 14, Theorem 5] or the general smoothness theorem below) hence if $dJ_s = 0$ then for all $u \varepsilon T(\Omega(f))_s \subseteq L_1^2(\eta)^\circ$ we can integrate by parts and (since $u(0) = u(1) = 0$) $dJ_s(u) = -2\int_0^1 (s''(t), u(t))dt = 0$. Since $T(\Omega(f))_s$ consists of all $u \varepsilon L_1^2(\eta)^\circ$ such that $u(t)$ is tangent to W at $s(t)$ it follows easily that $s''(t)$ is orthogonal to W at $s(t)$ and conversely if $s \varepsilon \Omega(f)$ is C^2 and satisfies this condition then s is a critical point of J. By a well-known result of differential geometry it follows that the critical points of J are exactly the closed geodesics of W (if $M = S^1$) or the geodesics of W joining $f(0)$ to $f(1)$ (if $M = I$) in both cases the parameter being proportional to arc length. Also it follows easily from Schwartz's inequality (see [MTHM p. 334]) that at a point of $\Omega(f)$ where J assumes its minimum on a given component, so does the length function $s \longmapsto \int_0^1 ||s(t)|| dt$. It now follows from Theorem 19.29 that in every homotopy class of C^1 paths in W from p to q there is one (a geodesic) of minimal length and (if W is compact) that in each free homotopy class of maps of S^1 into W there is one (a geodesic) of minimal length. Note that the last statement is clearly false for the surface of revolution of $y = e^x$ so that compactness of W is really a necessary assumption in general.

We can also regard the above as a special case of Theorem 19.32, where for $L \in \mathrm{Diff}_2(\eta, \eta)$ we take $\frac{d^2}{dx^2}$, the Laplace (-Beltrami) operator for M.

This suggests natural generalizations of the geodesic theory for W. The interval (circle) is replaced by D^n (S^n) (in the latter case W must be compact) and the operator L of Theorem 19.32 is taken to be Δ^k, where Δ is the Laplace-Beltrami operator on D^n (or S^n) and $k > \frac{n}{2}$. More generally we can take instead of D^n or S^n any compact Riemannian manifold M of dimension n and let $L = \Delta^k$ where Δ is the Laplace-Beltrami operator for M (provided in case W is not compact $\Delta^k u = 0$ has no non-trivial solutions in $L^2_k(\eta)^0$). Strictly speaking $\Delta^k \in \mathrm{Diff}_{2k}(\mathbb{R}_M, \mathbb{R}_M)$, but if $e_1, \ldots, e_m$ is a basis for V then any $s \in C^\infty(\eta) = C^\infty(M \times V)$ can be written uniquely in the form $s = f_1 e_1 + \ldots + f_m e_m$ where $f_i \in C^\infty(\mathbb{R}_M)$ and we define $\Delta^k s = (\Delta^k f_1)e_1 + \ldots + (\Delta^k s)e_m$. This defines Δ^k as an element of $\mathrm{Diff}_{2k}(\eta, \eta)$ and the definition is independent of the choice of $e_1, \ldots, e_m$. One important difference should be noted between this and the geodesic case. In the geodesic case $J(s)$ is essentially $\int ||s'(t)||^2 dt$, which depends only on the intrinsic metric properties of W (for if $s \in \Omega(f)$ then $s'(t) \in T(W)_{s(t)}$), however in the general case above when $n > 1$ J depends on the embedding of W in V!

If we are willing to go outside the Hilbert category there are other natural generalizations of the geodesic theory. The controlling inequality is $pk > n$. As n gets large if we want to keep $p = 2$ we must let k get large. Alternatively we can keep k small if we let p get large. If we take $p = 2m$ then we can take $\mathcal{J}(s) = \int \sum_i ||A_i s||^p d\mu$, where $\{A_i\}$ is any ample set of k^{th} order linear operators for η. It seems to me that an interesting case to consider seriously would be $k = 1$ and for $\{A_i\}$ the single operator $d \in \mathrm{Diff}_1(\eta, T^*(M) \otimes \eta)$. Note that $T^*(M) \otimes \eta$ has a natural Riemannian structure coming from those of $T(M)$ and η. Also $\sigma_1(d)(v,x)e = v \otimes e$ so $\sigma_1(d)(v,x)$ is indeed injective for $v \neq 0$, and hence d is ample. Finally $du = 0$ if and only if u is a constant map, hence if $\partial M \neq \emptyset$ then $du = 0$ has no non-trivial solution in $L^p_1(\eta)^0$, so in this case d is strongly ample and W need not be compact (but if $\partial M = \emptyset$ we must assume W compact). What makes this case particularly interesting is that J depends only on the Riemannian structure of M and W and not on the embedding of W in V, so the critical points of J should have intrinsic geometric meaning, just as in the geodesic case. To make the situation even closer to the geodesic case take $M = D^n$. $\mathcal{J}(s) = \int \left(\sum_{i=1}^n \left(\frac{\partial s}{\partial x_i}\right)^2\right)^{p/2} dx_1 \ldots dx_n$ and it is an easy exercise to derive the Euler-Lagrange equations explicitly.

We now take up the question of smoothness theorems, i.e. theorems to the effect that (under certain assumptions on the Lagrangian of the Dirichlet problem $J:\Omega(f) \longrightarrow \mathbb{R}$) if f, which a priori is only in $L^p_k(E)$, should happen to be in $L^p_{k+r}(E)$ then all the critical points of J are likewise in $L^p_{k+r}(E)$, so in particular if $f \in C^\infty(E)$ (or if $\partial M = \emptyset$) then all critical points of J are C^∞. I believe such a theorem always holds under the hypotheses of Theorem 19.32, however at present we need the additional hypothesis that the strongly elliptic operator L which defines the problem is "quasi-scalar" in the sense defined below.

An element D of $\mathrm{Diff}_k(\mathbb{R}_M, \mathbb{R}_M)$ defines an element D^V of $\mathrm{Diff}_k(\eta, \eta)$ for any product bundle $\eta = M \times V$ (which is characterized by $D^V(fv) = (Df)v$ for $f \in C^\infty(\mathbb{R}_M)$, $v \in V$) and $\sigma_k(D^V)(v,x):V \longrightarrow V$ is multiplication by the scalar $\sigma_k(D)(v,x)$. Such an operator D^V is called a scalar operator .

19.33. Definition. If η is a C^∞ vector bundle over M and $D \in \mathrm{Diff}_k(\eta, \eta)$ then D is called quasi-scalar if for each $(v,x) \in T^*(M)$ $\sigma_k(D)(v,x):\eta_x \longrightarrow \eta_x$ is multiplication by a scalar.

19.34. Lemma. Let $\eta = M \times V$ be a product vector bundle over M and let $D \in \mathrm{Diff}_k(\eta, \eta)$ be a quasi-scalar operator. If $X_1, \ldots, X_r$ is an ample family of vector fields for M then there exist C^∞ real valued functions $c_{i_1 i_2 \ldots i_k}$ on M $(i_j = 1, \ldots, r)$ such that

$$D - \Sigma\, c_{i_1 i_2 \ldots i_k} X_{i_1} X_{i_2} \cdots X_{i_k}$$

is in $\mathrm{Diff}_{k-1}(\eta, \eta)$.

Proof. Cover M with a finite family of open sets $\mathcal{O}_j$ such that in $\mathcal{O}_j$, $X_{j(1)}, \ldots, X_{j(n)}$ is a basis of sections for $T(M)|\,\mathcal{O}_j = T(\mathcal{O}_j)$, and let $\{\varphi_j\}$ be a C^∞ partition of unity for M subordinate to $\{\mathcal{O}_j\}$. Then for $(v,x) \in T^*(M)|\,\mathcal{O}_j$, $\sigma_k(D)(v,x)$ is multiplication by a scalar which is a homogeneous polynomial of degree k in v with coefficients C^∞ functions of x , say

$$\sigma_k(D)(v,x) = \Sigma\, c^j_{i_1 \ldots i_k}(x)\, v(X_{j(i_1)}) \cdots v(X_{j(i_k)})$$

Then $\Sigma\, \varphi_j(x)\, c^j_{i_1\ldots i_k}(x)\, X_{j(i_1)}\cdots X_{j(i_k)}$ has the same symbol as D and hence differs from D by an element of $\mathrm{Diff}_{k-1}(\eta, \eta)$.

q.e.d.

In what follows $\eta = M \times V$, $E = M \times W$ as usual and P and Q are as defined in 19.13 and 19.14.

19.35. Lemma. If $k > \frac{n}{2}$ and we choose $0 > \varepsilon > 1$ so that $k-\varepsilon > \frac{n}{2}$ then given any r vector fields on M $(r \leq 2k)$ the map

$$s \longmapsto P_s(X_1\ldots X_r s) - (X_1\ldots X_r s)$$

of $C^\infty(E)$ into $C^\infty(\eta)$ extends to a continuous map of $L^2_k(E)$ into $L^2_{k-r+\varepsilon}(\eta)$.

Proof. For $r = 1$ and $s \in C^\infty(E)$ we have $Xs(x) = ds(X_x) \in T(W)_{s(x)}$ hence $P_s(Xs)(x) = Q(s(x))\,(Xs(x))$ = orthogonal projection of $Xs(x)$ on $T(W)_{s(x)} = Xs(x)$, so $P_s(Xs) = Xs$ and the case $r = 1$ is trivial. We proceed by induction. For $s \in C^\infty(E)$ we have $X_1((Q \circ s)(X_2\ldots X_r s)) = (X_1(Q \circ s))(X_2\ldots X_r s) + (Q \circ s)(X_1\ldots X_r s)$ and since $P_s\sigma = (Q \circ s)\sigma$ it follows that

$$P_s(X_1\ldots X_r s) - (X_1\ldots X_r s) = X_1(P_s(X_2\ldots X_r s) - (X_2\ldots X_r s))$$

$$- (X_1(Q \circ s))(X_2\ldots X_r s)$$

Since $X_1 \in \mathrm{Diff}_1(\eta, \eta)$ it follows from the inductive hypothesis and Theorem 6.5 that $s \longmapsto X_1(P_s(X_2\ldots X_r s) - (X_2\ldots X_r s))$ extends to a continuous map of $L^2_k(E)$ into $L^2_{k-r+\varepsilon}(\eta)$. Since $Q\colon \eta \longrightarrow L(\eta, \eta)$ is a fiber bundle morphism $s \longmapsto Q(s)$ extends to a C^∞ map (namely $L^2_k(Q)$) of $L^2_k(\eta)$ into $L^2_k(L(\eta, \eta))$ and (by restriction) to a C^∞ map of $L^2_k(E)$ into $L^2_k(L(\eta, \eta))$. Hence again by Theorem 6.5 $s \longmapsto X_1(Q \circ s)$ extends to a C^∞ map of $L^2_k(E)$ into $L^2_{k-1}(L(\eta, \eta))$ while (once more by Theorem 6.5) $s \longmapsto (X_2\ldots X_r s)$ extends to a C^∞ map of $L^2_k(E)$ into $L^2_{k-r+1}(\eta)$. Thus it will suffice to prove that the map $(T,\sigma) \longrightarrow T\sigma$ of $C^\infty(L(\eta, \eta)) \times C^\infty(\eta)$ into $C^\infty(\eta)$ extends to a continuous map $L^2_{k-1}(L(\eta, \eta)) \times L^2_{k-r+1}(\eta)$ into $L^2_{k-r+\varepsilon}(\eta)$ provided $2 \leq r \leq 2k$. By choosing a basis for V and writing out what this means in terms of matrices we see it is equivalent to prove this in the case $V = \mathbb{R}$, i.e. that multiplication extends from a bilinear map of $C^\infty(\mathbb{R}_M) \times C^\infty(\mathbb{R}_M)$ into $C^\infty(\mathbb{R}_M)$ to a continuous bilinear map of $L^2_{k-1}(\mathbb{R}_M) \times L^2_{k-r+1}(\mathbb{R}_M)$ into $L^2_{k-r+\varepsilon}(\mathbb{R}_M)$, i.e. of $L^2_{k-\varepsilon-(1-\varepsilon)}(\mathbb{R}_M) \times L^2_{k-\varepsilon-(r-1-\varepsilon)}(\mathbb{R}_M)$ into $L^2_{k-\varepsilon-(r-2\varepsilon)}(\mathbb{R}_M)$. Since $k-\varepsilon > \frac{n}{2}$, $1-\varepsilon \geq 0$, $r-1-\varepsilon \geq 0$, and

$(1-\varepsilon) + (r-1-\varepsilon) = r-2\varepsilon \geq 2\,(k-\varepsilon)$ this is a special case of Theorem 9.16.

q.e.d.

19.36. Lemma. If $k > \frac{n}{2}$ and we choose $0 < \varepsilon < 1$ so that $k-\varepsilon > \frac{n}{2}$ then for any $D \in \mathrm{Diff}_{2k}(\eta, \eta)$ which is quasi-scalar the map $s \longmapsto P_s Ds - Ds$ of $C^\infty(E)$ into $C^\infty(\eta)$ extends to a continuous map of $L^2_\ell(E)$ into $L^2_{\ell-2k+\varepsilon}(\eta)$ provided $\ell \geq k$.

Proof. Since as noted above $s \longmapsto Q(s)$ is a continuous map of $L^2_\ell(E)$ into $L^2_\ell(L(\eta, \eta))$ and since by Theorem 9.13 it follows that $(T,\sigma) \longmapsto T\sigma$ is a continuous bilinear map of $L^2_\ell(L(\eta, \eta)) \times L^2_{\ell-2k+1}(\eta)$ into $L^2_{\ell-2k+1}(\eta) \subseteq L^2_{\ell-2k+\varepsilon}(\eta)$, the lemma is clear if $D \in \mathrm{Diff}_{2k-1}(\eta, \eta)$. It is also true for D if it is true for two operators in $\mathrm{Diff}_{2k}(\eta, \eta)$ whose sum is D, so the lemma follows from 19.34 and 19.35 (where in 19.35 we take the k of that lemma to be ℓ and the r of that lemma to be twice the k of the present lemma).

q.e.d.

We next recall the smoothness theorem for strongly elliptic operators.

19.37. Theorem. Let M be a compact n-dimensional manifold, possibly with boundary, with a strictly positive smooth measure. Let η be a C^∞ Riemannian vector bundle over M and $L \in \mathrm{Diff}_{2k}(\eta, \eta)$ a strongly elliptic operator. If $g \in L^2_k(\eta)^0$ and if for some $\rho > 0$ $Lg \in L^2_{-k+\rho}(\eta)$ then $g \in L^2_{k+\rho}(\eta)$.

Proof. See for example S. Agmon's recent "Lectures on Elliptic Boundary Problems", van Nostrand 1965. Theorem 9.7 covers the case $\rho \geq k$ while the case $0 < \rho \leq k$ follows from Lemma 9.5.

19.38. Smoothness Theorem. Let the notation and assumptions be as in Theorem 19.32 and assume in addition that L is quasi-scalar. Then if $f \in L^2_{k+r}(E)$ for some positive r it follows that all the critical points of $J:\Omega(f) \longrightarrow \mathbb{R}$ lie in $L^2_{k+r}(E)$. In particular if $\partial M = \emptyset$ or if $f \in C^\infty(E)$ then all the critical points of J are C^∞.

Proof. Choose the ε of Lemma 19.36 so that for some integer m, $m\,\varepsilon = r$. We will show that if s is a critical point of J and $g = s-f$ then if $g \in L^2_{k+j\varepsilon}(\eta)$ where $j = 0, 1, \ldots, (m-1)$ then it follows that $g \in L^2_{k+(j+1)\varepsilon}(\eta)$. Since we know a priori that $g \in L^2_k(\eta)$ (because both f and s lie in $\Omega(f) \subseteq L^2_k(E) \subseteq L^2_k(\eta)$) it will follow by a finite induction that $g \in L^2_{k+m\varepsilon}(\eta) = L^2_{k+r}(\eta)$. Since by assumption $f \in L^2_{k+r}(\eta)$ it will further follow that $s = g + f \in L^2_{k+r}(\eta)$ and hence $s \in L^2_{k+r}(\eta) \cap L^2_k(E) = L^2_{k+r}(E)$ as was to be proved.

Now since $s \in \Omega(f) \subseteq f + L^2_k(\eta)^o$, $g = (s-f) \in L^2_k(\eta)^o$. Thus the implication $g \in L^2_{k+j\varepsilon}(\eta) \Longrightarrow g \in L^2_{k+(j+1)\varepsilon}$ $(j = 0, 1, \ldots, (m-1))$ will follow from Theorem 19.37 if we can prove the implication $g \in L^2_{k+j\varepsilon}(\eta) \Longrightarrow Lg \in L^2_{-k+(j+1)\varepsilon}$ $(j = 0, 1, \ldots, (m-1))$. Now since $Lg = Ls - Lf$ and since $Lf \in L^2_{-k+r}(\eta) \subseteq L^2_{-k+(j+1)\varepsilon}$ $(j = 0, 1 \ldots (m-1))$ it will in turn suffice to show that $g \in L^2_{k+j\varepsilon}(\eta) \Longrightarrow Ls \in L^2_{-k+(j+1)\varepsilon}(\eta)$ $(j=0, 1 \ldots (m-1))$. Finally since $f \in L^2_{k+r}(\eta) \subseteq L^2_{k+j\varepsilon}(\eta)$, $g \in L^2_{k+j\varepsilon}(\eta) \Longrightarrow s = g + f - L^2_{k+j\varepsilon}(\eta)$, so it will suffice to show that if $s \in L^2_{k+j\varepsilon}(E)$ then $Ls \in L^2_{-k+(j+1)\varepsilon}(\eta)$. Now recall that by Lemma 19.36 $s \longmapsto Ls - P_s Ls$ is a continuous map of $L^2_{k+j\varepsilon}(E)$ into $L^2_{-k+(j+1)\varepsilon}(\eta)$, hence it will suffice to prove that when s is a critical point of J, $P_s Ls = 0$. Now a priori $P_s Ls \in L^2_{-k}(\eta) = (L^2_k(\eta)^o)^*$ so what we must show is that for all $u \in L^2_k(\eta)^o$ we have $(P_s Ls, u) = 0$. Since P_s is a self-adjoint zero order operator this is equivalent to showing $(Ls, P_s u) = 0$ for $u \in L^2_k(\eta)^o$. By Theorem 19.14 P_s maps $L^2_k(\eta)^o$ onto $T(\Omega(f))_s$. On the other hand since $\mathcal{J}(\sigma) = (L\sigma,\sigma) - (Lf,f)$ we have $d\mathcal{J}_s(u) = (Ls,u)$ and hence, since $J = \mathcal{J}|\Omega(f)$, $dJ_s(u) = d\mathcal{J}_s(u) = (Ls,u)$ for all $u \in T(\Omega(f))_s$. Since $dJ_s = 0$ it follows that $(Ls,u) = 0$ for all $u \in T(\Omega(f))_s$, hence $(Ls, P_s u) = 0$ for all $u \in L^2_k(\eta)^o$.

q.e.d.

Remark. If in Theorem 19.32 we take, instead of the J given there, the function defined by $s \longmapsto \int \sum_i ||A_i s(x)||^2 \, d\mu(x)$ where $\{A_i\}$ is an ample family of k^{th} order scalar operators for η, then we still get the same smoothness theorem as above. For up to an additive constant (which does not change the critical points) this function is equal to the one given in Theorem 19.32 if for L we take $\sum_i A_i^* A_i$.

Essentially the above theorem was proved by John Saber in his thesis (Brandeis, 1965).

Added in Proof: In her thesis, Karen Uhlenbeck has proved Theorem 19.38 without the assumption that L is quasi-scalar. Also, she has shown that if J is as in Theorem 19.29 where $\mathcal{L}(u) = ||Au||^p$, $A \in \mathrm{Diff}_k(\eta,\eta)$ being a scalar elliptic operator, then each critical point of J belongs to $C^{k+\frac{1}{p-1}}$ in the interior of M.