Linear and Nonlinear Waves and Solitons

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1 John Scott Russell and The Great Wave of Translation

To the world at large, John Scott Russell is known as the naval architect who designed The Great Eastern, a steamship larger than any built before. But long after the Great Eastern has been forgotten, Russell will be remembered by mathematicians as the man who, despite limited mathematical training and background, was the first person to recognize the highly important mathematical concept now called a *soliton*, or as he referred to it, The Great Wave of Translation. Here is the oft-quoted passage from [30] in which he describes how he first became acquainted with it.

> I was observing the motion of a boat which was rapidly drawn along a narrow channel by a pair of horses, when the boat suddenly stopped—not so the mass of water in the channel which it had put in motion; it accumulated round the prow of the vessel in a state of violent agitation, then suddenly leaving it behind, rolled forward with great velocity, assuming the form of a large solitary elevation, a rounded, smooth and well-defined heap of water, which continued its course along the channel apparently without change of form or diminution of speed. I followed it on horseback, and overtook it still rolling on at a rate of some eight or nine miles an hour, preserving its original figure some thirty feet long and a foot to a foot and a half in height. Its height gradually diminished, and after a chase of one or two miles I lost it in the windings of the channel. Such, in the month of August 1834, was my first chance interview with that singular and beautiful phenomenon which I have called the Wave of Translation

You may feel that there is nothing unusual about what Russell describes here, and indeed many before and since have watched this same scenario play out without remarking anything out of the ordinary. But Russell was very familiar with wave phenomena and had a scientist's keenly observant eye. What struck him was the remakable *stability* of the bow wave as it travelled over a long distance. He knew that if one tried to create a travelling water wave on say a calm lake, it would soon disperse into a train of smaller wavelets—it would *not* just go marching along as a single "heap" over a long distance. Clearly there was something very special about water waves travelling in a narrow and shallow channel.

Russel became fascinated—even a little obsessed with his discovery. He built a wave tank behind his home and proceded to do extensive experiments, recording the results as data and sketches in his notebooks. He found for example that the speed of a soliton depended on its height, and he even was able to discover the correct formula for the speed as a function of height, More surprising still, in Russell's notebooks one finds remarkable sketches of a two-soliton interaction—something that would evoke surprise and amazement when it was rediscovered as a rigorous solution to the KdV Equation (see below) more than a hundred years later.

However, as we shall see, solitons are very much a nonlinear phenomenon, and when some of the best mathematicians of his day, notably Stokes and Airy, tried to understand Russell's observations with the linearized theory of water waves then available, they failed to find any trace of soliton-like behavior and expressed doubts that what Russell had seen was real.

It was only after Russell's death, with the more sophisticated nonlinear mathematical treatment by Boussinesq [6] in 1871 and by Korteweg and de Vries in 1895 [16], that Russell's careful observations and experiments were at last seen to be in complete agreement with mathematical theory. And it took yet another seventy years before the full importance of the Great Wave of Translation, renamed the soliton, was recognized and it became an object of intensive study for the rest of the Twentieth Century. In what follows we will try to recapitulate some of the mathematics that led to a full understanding of what solitons are and what is behind their surprising behavior.

2 Wave Equations

2.1 Introduction

For simplicity we will consider only waves in a one-dimensional space. What we mean by a *wave equation* will be made more precise as we proceed, but to get started we mean a certain kind of equation that specifies how a point u in a certain vector space U evolves as a function of "time", t, by specifying the time derivative, $u_t = \frac{du}{dt}$ as a function of u. That is our equation will be, at least formally, a first order ordinary differential equation in U:

$$(*) \qquad u_t = f(u),$$

where on the right hand side, f is some function $f : U \to U$. To make this precise we must specify the vector space U and what kind of functions f will be permitted. For U we will take the space of smooth (i.e., infinitely differentiable) functions $u(x)$ of a real variable x with values in the vector space $V = \mathbb{R}^n$ or $\mathbb{C}^n$, and f we will be a "partial differential operator", i.e., $f(u)(x)$ will be a smooth function $F(u(x), u_{x_i}(x), u_{x_i x_j}(x), \dots)$ of the values of u and certain of its partial derivatives at x—in fact, the function F will generally be a polynomial. A solution of $(*)$ is then a smooth curve $u(t)$ in U such that, if we write $u(t)(x) := u(x,t)$, then

$$\frac{\partial u}{\partial t}(x,t) = F\left(u(x,t), \frac{\partial u}{\partial x_i}(x,t), \frac{\partial^2 u}{\partial x_i \partial x_j}(x,t), \dots\right).$$

We will study the so-called "Cauchy Problem" for such partial differential equations, i.e., the problem of finding a solution, in the above sense, with $u(x,0)$ some given element $u_0(x)$ of U. So far this should more properly be called simply an "evolution equation", since in general such equations will describe evolving phenomena that are *not* wave-like in character, and only after certain additional assumptions are made concerning F is it appropriate to call it a wave equation.

While we will be interested in obvious questions such as existence, uniqueness, and general properties of solutions of the Cauchy problem, we will be even more concerned with the origin and properties of a certain remarkable class of solutions, the so-called *solitons*, of a very special kind of equation, the "integrable equations" such as the Korteweg de Vries Equation (KdV), the Sine-Gordon Equation (SGE), the Nonlinear Schrödinger Equation (NLS).

In addition to a first order ODE on U, we could also consider second and higher order ODE, but these can easily be reduced to first order ODE by the standard trick of adding more dependent variables. For example, to study the classic wave equation in one space dimension, $w_{tt} = c^2 w_{xx}$, a second order ODE, we can add a new independent variable v and consider instead the first order system $w_t = v$, $v_t = c^2 w_{xx}$, which we can put in the form (*) by writing $u_t = F(u)$, with $u = (w, v)$, $F(w, v) = (v, c^2 w_{xx})$.

2.2 Travelling Waves and Plane Waves

Let us recall the basic intuitive idea of what is meant by "wave motion". Suppose that $u(x,t)$ represents the "strength" or "amplitude" of some physical quantity at the spatial point x and time t. For example, if you think of u as representing the height of water in a canal, then the graph of $u^0(x) = u(x, t_0)$ gives a snapshot of u at time t_0, and we can understand the evolution of u in time as representing the propagation of the shape of this graph. In other words, for t_1 close to t_0, the shape of the graph of $u^1(x) = u(x, t_1)$ near x_0 will be related in some simple way to the shape of u^0 near x_0. Perhaps the simplest example of this is a so-called *travelling wave*, namely a u of the form $u(x,t) = f(x - ct)$, where $f : \mathbb{R} \to V$ defines the wave shape, and c is a real number defining the propagation speed of the wave. If we define the *profile* of the wave at time t to be the graph of the function $x \mapsto u(x,t)$, then the initial profile (at $t = 0$) is just the graph of f, and **at any later t, the profile at time t is obtained by translating each point $(x, f(x))$ of the initial profile ct units to the right to the point** $(x + ct, f(x))$. So, the wave profile of a travelling wave just propagates by rigid translation with velocity c. As we will see below, the general solution of the equation $u_t = cu_x$ is an arbitrary travelling wave moving with velocity c, and the general solution to the equation $u_{tt} = c^2 u_{xx}$ is the sum (or "superposition") of two arbitrary travelling waves, both moving with speed $|c|$, but in opposite directions.

There is a special kind of complex-valued travelling wave, called a *plane wave*, that plays a fundamental rôle in the theory of linear wave equations. The general form of a plane wave is $u(x,t) = Ae^{i\phi}e^{i(kx-\omega t)}$, where A is a positive constant called the *amplitude*, $\phi \in [0, 2\pi)$ is called the *initial phase*, and k and ω are two real parameters called the *wave number* and *angular frequency*. (Note that $\frac{k}{2\pi}$ is the number of waves per unit length, while $\frac{\omega}{2\pi}$ is the number of waves per unit time.) Rewriting u as $u(x,t) = Ae^{i\phi}e^{ik(x-\frac{\omega}{k}t)}$, we see it is indeed a travelling wave of velocity is $\frac{\omega}{k}$.

In studying a wave equation, a first step is to find all travelling wave solutions (if any) it admits. For a constant coefficient linear wave equation we will see that for each wave number k there is a unique angular frequency $\omega(k)$ for which the equation admits a plane wave solution, and the velocity $\frac{\omega(k)}{k}$ of this plane wave as a function of k (the so-called *dispersion relation* of the equation) completely determines the equation, and is crucial for understanding how solutions disperse as time progresses. Also, the fact that there is a unique (up to a multiplicative constant) travelling wave solution $u_k(x,t) = e^{i(kx-\omega(k)t)}$ with wave number k allows us to solve the equation explicitly by representing the general solution as a superposition of these solutions u_k. This in essence is the Fourier method.

For nonlinear wave equations, travelling wave solutions are in general severely restricted. Usually only very special profiles, characteristic of the particular equation, are possible for travelling wave solutions, and in particular they do not normally admit any plane wave solutions.

2.3 Some Model Equations

Perhaps the most familiar of all wave equation is **The Classic Wave Equation** $u_{tt} - c^2 u_{xx} = 0$. As we saw above, we can reduce this to a standard first-order evolution equation by replacing the one-component vector u by a two-component vector (u, v) satisfying $(u, v)_t = (v, c^2 u_{xx})$, i.e., $u_t = v$ and $v_t = c^2 u_{xx}$. To solve the Cauchy problem for the Classic Wave Equation, factor the wave operator, $\frac{\partial^2}{\partial t^2} - c^2\frac{\partial^2}{\partial x^2}$, as a product$(\frac{\partial}{\partial t} - c\frac{\partial}{\partial x})(\frac{\partial}{\partial t} + c\frac{\partial}{\partial x})$, and transform to so-called "characteristic coordinates", $\xi = x - ct$, $\eta = x + ct$. The equation becomes $\frac{\partial^2 u}{\partial\xi\partial\eta} = 0$, that clearly has the general solution $u(\xi, \eta) = F(\xi) + G(\eta)$. Transforming back to "laboratory coordinates" x, t, the general solution is $u(x,t) = F(x-ct) + G(x+ct)$. If the initial shape of the wave is $u(x, 0) = u_0(x)$ and its initial velocity is $u_t(x, 0) = v(x, 0) = v_0(x)$, then an easy algebraic computation gives the following very explicit formula:

$$u(x,t) = \frac{1}{2}[u_0(x-ct)+u_0(x+ct)] + \frac{1}{2c}\int_{x-ct}^{x+ct} v_0(\xi)\, d\xi,$$

known as "D'Alembert's Solution" of the Cauchy Problem for the Wave Equation. Note the geometric interpretation in the important "plucked string" case, $v_0 = 0$; the initial profile u_0 breaks up into the sum of two travelling waves, both with the same profile $u_0/2$, and one travels to the right, and the other to the left, both with speed c. It is an easy excercise to derive D'Alembert's solution using the following hint: since $u_0(x) = F(x) + G(x)$, $u_0'(x) = F'(x) + G'(x)$, while $v_0(x) = u_t(x, 0) = -cF'(x) + cG'(x)$.

Remark 2.3.1. There are a number of important consequences that follow easily from the form of the D'Alembert solution:

a) The solution is well-defined for initial conditions (u_0, v_0) in the space of distributions, and gives a flow on this much larger space.

b) The quantity $\int_{-\infty}^{\infty} |u_x|^2 + (\frac{1}{c})^2|u_t|^2\, dx$ is a "constant of the motion". More precisely, if this integral is finite at one time for a solution $u(x,t)$, then it is finite and has the same value at any other time.

c) The "domain of dependence" of a point (x,t) of space-time consists of the interval $[x-ct, x+ct]$. That is, the value of any solution u at (x,t) depends only on the values u_0 and v_0 in the interval $[x-ct, x+ct]$. Another way to say this is that the "region of influence" of a point x_0 consists of the interior of the "light-cone" with vertex at x_0, i.e., all points (x,t) satisfying $x_0 - ct < x < x_0 + ct$. (These are the points having x_0 in their domain of dependence.) Still a third way of stating this is that the Classical Wave Equation has signal propagation speed c, meaning that the value of a solution at (x,t) depends only on the values of u_0 and v_0 at points x_0 from which a signal propagating with speed c could reach x in time t (i.e., points inside the sphere of radius ct about x.)

(Note: It is easy to prove b) by noting that $|u_x(x,t)|^2 + (\frac{1}{c})^2|u_t(x,t)|^2 = 2(|F'(x-ct)|^2 + |G'(x+ct)|^2)$.)

Our next model equation is **The Linear Advection Equation**, $u_t - cu_x = 0$. Using again the trick of transforming to the coordinates, ξ, η, the equation becomes $\frac{\partial u}{\partial \xi} = 0$, so the general solution is $u(\xi) =$ constant, and the solution to the Cauchy Problem is $u(x,t) = u_0(x-ct)$. As before we see that if u_0 is any distribution then $u(t) = u_0(x-ct)$ gives a well-defined curve in the space of distributions that satisfies $u_t - cu_x = 0$, so that we really have a flow on the space of distributions whose generating vector field is $c\frac{\partial}{\partial x}$. Since $c\frac{\partial}{\partial x}$ is a skew-adjoint operator on $L^2(\mathbb{R})$, it follows that this flow restricts to a one-parameter group of isometries of $L^2(\mathbb{R})$, i.e., $\int_{-\infty}^{\infty} u(x,t)^2\,dx$ is a constant of the motion. It is also easy to prove this directly by showing that $\frac{d}{dt}\int_{-\infty}^{\infty} u(x,t)^2\,dx$ is zero. (Hint: It suffices to show this when u_0 is smooth and has compact support, since these are dense in L^2. For such functions we can rewrite the integral as $\int_{-\infty}^{\infty} \frac{\partial}{\partial t}u(x,t)^2\,dx$ and the result will follow if we can show that $\frac{\partial}{\partial t}u(x,t)^2$ can be written for each t in the form $\frac{d}{dx}h(x)$, where h is smooth and has compact support.)

We next consider the **General Linear Evolution Equation**, $u_t + P(\frac{\partial}{\partial x})u = 0$. Here $P(\xi)$ is a polynomial with complex coefficients. For example, if $P(\xi) = -c\xi$ then we get back the Linear Advection Equation. We will outline the theory of these equations in a separate section below and see that they can analyzed easily and completely using the Fourier Transform. (It will turn out that to qualify as a wave equation, the odd coefficients of the polynomial P should be real and the even coefficients pure imaginary, or more simply, $P(i\xi)$ should be imaginary valued on the real axis. This is the condition for $P(\frac{\partial}{\partial x})$ to be a skew-adoint operator on $L^2(\mathbb{R})$.)

Our next family of model equations is the **The General Conservation Law**, $u_t = (F(u))_x$. Here $F(u)$ can any smooth function of u and its partial derivatives with respect to x. For example, if $P(\xi) = a_1\xi + \cdots + a_n\xi^n$, we get the linear evolution equation $u_t = P(\frac{\partial}{\partial x})u$ by taking $F(u) = a_1 u + \cdots + a_n\frac{\partial^{n-1}u}{\partial x^{n-1}}$. On the other hand, $F(u) = -(\frac{1}{2}u^2 + \delta^2 u_{xx})$ gives the KdV equation $u_t + uu_x + \delta^2 u_{xxx} = 0$ that we consider below. Note that if $F(u(x,t))$ vanishes at infinity then integration gives $\frac{d}{dt}\int_{-\infty}^{\infty} u(x,t)\,dx = 0$, i.e., $\int_{-\infty}^{\infty} u(x,t)\,dx$ is a "constant of the motion", and this is where the name "Conservation Law" comes from. We will be concerned mainly with the case that $F(u)$ is a zero-order operator, i.e., $F(u)(x) = F(u(x))$, where F is a smooth function on $\mathbb{R}$. In this case, if we let $f = F'$, then we can write our Conservation Law in the form $u_t = f(u)u_x$. In particular, taking $f(\xi) = c$ (i.e., $F(\xi) = c\xi$) gives the Linear Advection Equation $u_t = cu_x$, while $F(\xi) = -\frac{1}{2}\xi^2$ gives the important **Inviscid Burgers Equation**, $u_t + uu_x = 0$ that we will meet again later.

There is a very beautiful and highly developed theory of such Conservation Laws, and again we will devote a separate subsection to outlining some of the basic results from this theory. Recall that for the Linear Advection Equation we have an explicit solution for the Cauchy Problem, namely $u(x,t) = u_0(x-ct)$, which we can also write as $u(x,t) = u_0(x - f(u(x,t))t)$, where $f(\xi) = c$. If we are incredibly optimistic we might hope that we could more generally solve the Cauchy Problem for $u_t = f(u)u_x$ by solving $u(x,t) = u_0(x - f(u(x,t))t)$ as an implicit equation for $u(x,t)$. This would mean that we could generalize our algorithm for finding the profile of u at time t from the initial profile as follows: translate each point $(\xi, u_0(\xi))$ of the graph of u_0 to the right by an amount $f(u_0(\xi))t$ to get the graph of $x \mapsto u(x,t)$. This would of course give us a simple method for solving any such Cauchy Problems, and **the amazing thing is that this bold idea actually works**. However, one must be careful. As we shall see, this algorithm, that goes by the name *the method of characteristics*, contains the seeds of its own eventual failure. For a general initial condition u_0 and function f, we shall see that we can predict a positive time T_B (the so-called "breaking time") after which the solution given by the method of characteristics can no longer exist as a smooth, single-valued funtion.

The Korteweg-de Vries (or KdV) Equation $u_t + uu_x + \delta^2 u_{xxx} = 0$. If we re-scale the independent variables by $t \to \beta t$ and $x \to \gamma x$, then the KdV equation becomes:

$u_t+\left(\frac{\beta}{\gamma}\right)uu_x+\left(\frac{\beta}{\gamma^3}\right)\delta^2 u_{xxx} = 0$, and by appropriate choice of β and γ we can obtain any equation of the form $u_t + \lambda uu_x + \mu u_{xxx} = 0$, and any such equation is referred to as "the KdV equation". Common choices, convenient for many purposes, are $u_t \pm 6uu_x + u_{xxx} = 0$ and we will use both. This is one of the most important and most studied of all evolution equations. It is over a century since it was shown to govern wave motion in a shallow channel, but less than fifty years since the remarkable phenomenon of soliton interactions was discovered in the course of studying certain of its solutions.

2.4 Linear Wave Equations; Dispersion and Dissipation

Evolution equations that are not only linear but also translation invariant can be solved explicitly using Fourier methods, and are interesting both for their own sake, and also because they serve as a tool for studying nonlinear equations.

The general linear evolution equation has the form $u_t + P(\frac{\partial}{\partial x})u = 0$, where to begin with we can assume that the polynomial P has coefficients that are smooth complex-valued functions of x and t: $P(\frac{\partial}{\partial x})u = \sum_{i=1}^{r} a_i(x,t)\frac{\partial^i u}{\partial x^i}$. For each (x_0, t_0), we have a space-time translation operator $T_{(x_0,t_0)}$ acting on smooth functions of x and t by $T_{(x_0,t_0)}u(x,t) = u(x - x_0, t - t_0)$, and we say that the operator $P(\frac{\partial}{\partial x})$ is *translation invariant* if it commutes with all the $T_{(x_0,t_0)}$. It is an easy excercise to see that the necessary and sufficient condition for $P(\frac{\partial}{\partial x})$ to be translation invariant is that the coefficients a_i of P should be constant complex numbers.

There are at least two excellent reasons to assume that our equation is translation invariant. First, the eminently practical one that in this case we can use Fourier techniques to solve the initial value problem explicitly and investigate the solutions in detail. But there is frequently an even more important physical reason for postulating translation invariance. If we are trying to model the dynamics of a fundamental physical quantity u by an evolution equation of the above type, then x will denote the "place where", and t the "time when" the quantity has the value $u(x,t)$. Now, if our proposed physical law is truly "fundamental", its validity should not depend on where or when it is applied—it will be the same on Alpha Centauri as on Earth, and the same in a million years as it is today—we can even take that as part of the definition of what we mean by fundamental. The way to give a precise mathematical formulation of this principle of space-time symmetry or homogeneity is to demand that our equation should be invariant under some transitive group acting on space and time. In any case, we will henceforth assume that P does in fact have constant complex numbers as coefficients.

If we now substitute $u(x,t) = e^{i(kx-\omega t)}$ into our linear equation, $u_t + P(\frac{\partial}{\partial x})u = 0$, then we find the relation $-i\omega u + P(ik)u = 0$, or $\omega = \omega(k) := \frac{1}{i}P(ik)$. For $u(x,t)$ to be a plane wave solution, we need the angular frequency, ω, to be real, so we will have a (unique) plane wave solution for each real wave number k just when $\frac{1}{i}P(ik)$ is real (i.e., $P(ik)$ is imaginary) for k on the real axis. This just translates into the condition that the odd coefficients of P should be real and the even coefficients pure imaginary, and we assume this in what follows. As we shall see, one consequence will be that we can solve the initial value problem for any initial condition u_0 in L^2, and the solution is a superposition of these plane wave solutions—clearly a strong reason to consider this case as describing honest "wave equations", whatever that term should mean.

The relation $\omega(k) := \frac{1}{i}P(ik)$ relating the angular frequency ω and wave number k of a plane wave solution of a linear wave equation is called the *dispersion relation* for the equation. The propagation velocity of the plane wave solution with wave number k is called the *phase velocity* at wave number k, given by the formula $\frac{\omega(k)}{k} = \frac{1}{ik}P(ik)$ (also sometimes referred to as the dispersion relation of the equation). Note that the dispersion relation is not only determined by the polynomial P defining the evolution equation, but conversely determines it.

Now let u_0 be any initial wave profile in L^2, so $u_0(x) = \int \hat{u}_0(k)e^{ikx}\,dk$, where $\hat{u}_0(k) = \frac{1}{2\pi}\int u_0(x)e^{-ikx}\,dk$ is the Fourier Transform of u. If we define $\hat{u}(k,t) = e^{-P(ik)t}\hat{u}_0(k)$, we see that $\hat{u}(k,t)e^{ikx} = \hat{u}_0(k)e^{ik(x-\frac{\omega(k)}{k}t)}$ is a plane wave solution to our equation with initial condition $\hat{u}_0(k)e^{ikx}$. We now define $u(x,t)$ (formally) to be the superposition of these plane waves: $u(x,t) \sim$

$\int \hat{u}(k,t)e^{ikx}\,dk$. So far we have not used the fact that $P(ik)$ is imaginary for k real, and we now notice that it implies $|e^{-P(ik)t}| = 1$, so $|\hat{u}(k,t)| = |\hat{u}_0(k)|$, hence $\hat{u}(k,t)$ is in L^2 for all t, and in fact it has the same norm as $\hat{u}_0$. It then follows from Plancherel's Theorem that $u(x,t)$ is in L^2 for all t, and has the same norm as u_0, and it is then elementary to see that our formal solution $u(x,t)$ is in fact an honest solution of the Cauchy Problem for our evolution equation, and in fact defines a one-parameter group of unitary transformations of L^2.

We now consider briefly what can happen if we drop the condition that the odd coefficients of P are real and the even coefficients pure imaginary. Consider first the special case of the Heat (or Diffusion) Equation, $u_t - \alpha u_{xx} = 0$, with $\alpha > 0$. Here $P(x) = -\alpha X^2$, so $|e^{-P(ik)t}| = |e^{-k^2 t}|$. Thus, when $t > 0$, $|e^{-P(ik)t}| < 1$, and $|\hat{u}(k,t)| < |\hat{u}_0(k)|$, so again $u(k,t)$ is in L^2 for all t, but now $\|u(x,t)\|_{L^2} < \|u_0(x)\|_{L^2}$. Thus our solution is not a unitary flow on L^2, but rather a contracting, positive semi-group. In fact, it is easy to see that for each initial condition $u_0 \in L^2$, the solution tends to zero in L^2 exponentially fast as $t \to \infty$, and in fact it tends to zero uniformly too. This so-called *dissipative* behavior is clearly not very "wave-like" in nature, and the Heat Equation is not considered to be a wave equation.

2.5 Conservation Laws

We now return to the consideration of a conservation law

$$(CL) \qquad u_t + f(u)u_x = 0.$$

We will usually assume that $f'(u) \geq 0$, so that f is a non-decreasing function. This is satisfied in most of the important applications.

Example 2.5.1. Take $F(u) = cu$, so $f(u) = c$ and we get once again the Linear Advection Equation $u_t - cu_x = 0$. The Method of Characteristics below will give yet another proof that the solution to the Cauchy Problem is $u(x,t) = u_0(x - ct)$.

Example 2.5.2. Take $F(u) = \frac{1}{2}u^2$, so $f(u) = u$ and we get the important Inviscid Burgers Equation, $u_t + uu_x = 0$.

We next explain how to solve the Cauchy Problem for such a Conservation Law, using the so-called Method of Characteristics. We look for smooth curves $(x(s), t(s))$ in the (x,t)-plane along which the solution to the Cauchy Problem is constant. Suppose that $(x(s_0), t(s_0)) = (x_0, 0)$, so that the constant value of $u(x,t)$ along this so-called characteristic curve is $u_0(x_0)$. Then $0 = \frac{d}{ds}u((x(s),t(s)) = u_x x' + u_t t'$, and hence

$$\frac{dx}{dt} = \frac{x'(s)}{t'(s)} = -\frac{u_t}{u_x} = f(u(x(s),t(s)) = f(u_0(x_0)),$$

so the characteristic curve is a straight line of slope $f(u_0(x_0))$, i.e., u has the constant value $u_0(x_0)$ along the line $\Gamma_{x_0} : x = x_0 + f(u_0(x_0))t$. Note the following geometric interpretation of this last result: as we promised to show earlier, **to find the wave profile at time t (i.e., the graph of the map $x \mapsto u(x,t)$), translate each point $(x_0, u_0(x_0))$ of the initial profile to the right by the amount $f(u_0(x_0))t$.** The analytic statement of this geometric fact is that the solution $u(x,t)$ to our Cauchy Problem must satisfy the implicit equation $u(x,t) = u_0(x - tf(u(x,t)))$. Of course the above is heuristic—how do we know that a solution exists?—but it isn't hard to work backwards and make the argument rigorous. The idea is to first define "characteristic coordinates" (ξ, τ) in a suitable strip $0 \leq t < T_B$ of the (x,t)-plane. We define $\tau(x,t) = t$ and $\xi(x,t) = x_0$ along the characteristic Γ_{x_0}, so $t(\xi,\tau) = \tau$ and $x(\xi,\tau) = \xi + f(u_0(\xi))\tau$. But of course, for this to make sense, we must show that there is a unique Γ_{x_0} passing through each point (x,t) in the strip $t < T_B$. The easiest case is $f' = 0$, say $f = c$, giving the Linear Advection Equation, $u_t + cu_x = 0$. In this case, all characteristics have the same slope, $1/c$, so that no two characteristics intersect, and there is clearly exactly one characteristic through each point, and we can define $T_B = \infty$.

From now on we will assume that the equation is "truly nonlinear", in the sense that $f'(u) > d > 0$, so that f is a strictly increasing function. If u_0' is everywhere positive, then $u_0(x)$ is strictly increasing, and hence so is $f(u_0(x))$. In this case we can again take $T_B = \infty$. For, since the slope of the characteristic Γ_{x_0} issuing from $(x_0, 0)$ is $\frac{1}{f(u_0(x))}$, it follows that if $x_0 < x_1$ then Γ_{x_1} has smaller slope than Γ_{x_0}, and hence these two characteristics cannot intersect for $t > 0$, and again every point (x,t) in the upper half-plane lies on at most one characteristic Γ_{x_0}.

Finally the interesting general case: suppose u_0' is somewhere negative. In this case we define T_B to be the infimum of $[-u_0'(x)f'(u_0(x))]^{-1}$, where the inf is taken over all x with $u_0'(x) < 0$. For reasons that will appear shortly, we call T_B the *breaking time*. As we shall see, T_B is the largest T for which the Cauchy Problem for (CL) has a solution with $u(x,0) = u_0(x)$ in the strip $0 \leq t < T$ of the (x,t)-plane. It is easy to construct examples for which $T_B = 0$; this will happen if and only if there exists a sequence $\{x_n\}$ with $u_0'(x_n) \to -\infty$. In the following we will assume that T_B is positive, and that in fact there is a point x_0 where $T_B = \frac{-1}{u_0'(x_0)f'(u_0(x_0))}$. In this case, we will call Γ_{x_0} a *breaking characteristic*.

Now choose any point x_0 where $u_0'(x_0)$ is negative. For x_1 slightly greater than x_0, the slope of Γ_{x_1} will be greater than the slope of Γ_{x_0}, and it follows that these two characteristics will meet at the point (x,t) where $x_1 + f(u_0(x_1))t = x = x_0 + f(u_0(x_0))t$, namely when $t = -\frac{x_1 - x_0}{f(u_0(x_1)) - f(u_0(x_0))}$. It is now an easy excercise to show that T_B is the least t for which any two characteristics intersect at some point (x,t) with $t \geq 0$, and also that there is always at least one characteristic curve passing through any point (x,t) in the strip $0 \leq t < T_B$. (It is also not hard to construct counterexamples to these statements if u_0' is not required to be continuous).

Thus the characteristic coordinates (ξ, τ) are well-defined in the strip $0 \leq t < T_B$ of the (x,t)-plane. Note that since $x = \xi + f(u_0(\xi))\tau$, $\frac{\partial x}{\partial \xi} = 1 + f'(u_0(\xi))u_0'(\xi)\tau$, and $\frac{\partial x}{\partial \tau} = f(u_0(\xi))$, while $\frac{\partial t}{\partial \xi} = 0$ and $\frac{\partial t}{\partial \tau} = 1$. It follows that the Jacobian of (x,t) with respect to (ξ,τ) is $\frac{\partial x}{\partial \xi} = 1 + f'(u_0(\xi))u_0'(\xi)$, which is positive in $0 \leq t < T_B$, so that (ξ,τ) are smooth coordinates in this strip. On the other hand, if Γ_{x_0} is a breaking characteristic, then then the Jacobian aproaches zero along Γ_{x_0} as t approaches T_B, confirming that the characteristic coordinates cannot be extended to any larger strip.

By our heuristics above, we know that the solution of the Cauchy Problem for (CL) with initial value u_0 should be given in characteristic coordinates by the explicit formula $u(\xi,\tau) = u_0(\xi)$, and so we define a smooth function u in $0 \leq t < T_B$ by this formula. Since the map from (x,t) to (ξ,τ) is a diffeomophism, this also defines u as a smooth function of x and t, but it will be simpler to do most calculations in characteristic coordinates. In any case, since a point (x,t) on the characteristic Γ_ξ satisfies $x = \xi + f(u_0(\xi))t$, we see that $u = u(x,t)$ is the solution of the implicit equation $u = u_0(x - tf(u))$. It is now obvious that $u(x,0) = u_0(x)$. Next use the chain-rule: $u_x = u_\xi \frac{\partial \xi}{\partial x}$ and $u_t = u_\xi \frac{\partial \xi}{\partial t}$ to compute the partial derivatives u_x and u_t as functions of ξ and τ:

$$u_t(\xi,\tau) = -\frac{u_0'(\xi)f(u_0(\xi))}{1 + u_0'(\xi)f'(u_0(\xi))\tau}$$

and

$$u_x(\xi,\tau) = \frac{u_0'(\xi)}{1 + u_0'(\xi)f'(u_0(\xi))\tau}$$

and and it follows from this that u actually satisfies the equation (CL) in $0 \leq t < T_B$, and so solves the Cauchy Problem.

To see how things go wrong at the breaking time T_B, we can check easily that along a breaking characteristic Γ_{x_0}, the value of u_x at the point $x = x_0 + f(u_0(x_0))t$ is given by $\frac{u_0'(x_0)T_B}{T_B - t}$. (Note that this is just the slope of the wave profile at time t over the point x.) This allows us to see a qualitative but very precise picture of how u develops a singularity as t approaches the breaking time T_B, a process usually referred to as *shock formation* or *steepening and breaking of the wave profile*.

Namely, let Γ_{x_0} be a breaking characteristic and consider an interval I around x_0 where u_0 is decreasing. Let's follow the evolution of that part of the wave profile that is originally over I. Recall our algorithm for evolving the wave profile: each point $(x, u_0(x))$ of the initial profile moves to the right with a constant velocity $f(u_0(x))$, so at time t it is at $(x + f(u_0(x))t, u_0(x))$. Thus, the higher part of the wave profile, to the left, will move faster than the lower part to the right, so the profile will bunch up and become steeper, until it eventually becomes vertical or "breaks" at time T_B when the slope of the profile actually becomes infinite over the point $x_0 + f(u_0(x_0))T_B$. (In fact, the above formula shows that the slope goes to $-\infty$ like a constant times $\frac{1}{t - T_B}$.) Note that the values of u remain bounded as t approaches T_B. In fact, it is clearly possible to continue the wave profile past $t = T_B$, using the same algorithm. However, for $t > T_B$ there will be values x^* where the vertical line $x = x^*$ meets the wave profile at time t in two distinct points (corresponding to two characteristics intersecting at the point (x^*, t)), so the profile is no longer the graph of a single-valued function.

For certain purposes it is interesting to know how higher derivatives u_{xx}, $u_{xxx}, \ldots$ behave as t approaches T_B along a breaking characteristic, (in particular, in the next section we will want to compare u_{xxx} with uu_x). These higher partial derivatives can be estimated in terms of powers of u_x using $\frac{\partial}{\partial x} = \frac{\partial}{\partial \xi}(\frac{\partial x}{\partial \xi})^{-1}$, and $\frac{\partial x}{\partial \xi} = 1 + f'(u_0(\xi))u_0'(\xi)\tau$. from which it follows easily that along a breaking characteristic Γ_{x_0}, as $t \to T_B$, $u_{xx} = O(u_x^3) = O((t - T_B)^{-3})$, and $u_{xxx} = O(u_x^5) = O((t - T_B)^{-5})$.

2.6 Split-Stepping

We now return to the KdV equation, say in the form $u_t = -uu_x - u_{xxx}$. If we drop the nonlinear term, we have left the dispersive wave equation $u_t = -u_{xxx}$, that we considered in the section on linear wave equations. Recall that we can solve its Cauchy Problem, by using the Fourier Transform.

On the other hand, if we drop the linear term, we are left with the inviscid Burgers Equation, $u_t = -uu_x$, and as we know it exhibits steepening and breaking of the wave profile, causing a shock singularity to develop in finite time T_B for any non-trivial initial condition u_0 that vanishes at infinity. Up to this breaking time, T_B, we can again solve the Cauchy Problem, either by the method of characteristics, or by solving the implicit equation $u = u_0(x - ut)$ for u as a function of x and t.

Now, in [5] it is proved that KdV defines a global flow on the Sobolev space $H^4(\mathbb{R})$ of functions $u : \mathbb{R} \to \mathbb{R}$ having derivatives of order up to four in L^2, so it is clear that dispersion from the linear u_{xxx} term must be counteracting the peaking from the nonlinear uu_x term, preventing the development of a shock singularity. In order to understand this balancing act better, it would be useful to have a method for taking the two flows defined by $u_t = -u_{xxx}$ and $u_t = -uu_x$ and combining them to define the flow for the full KdV equation. (In addition, this would give us a method for solving the KdV Cauchy Problem numerically.)

In fact there is a very general technique that applies in such situations. In the pure mathematics community it is usually called the Trotter Product Formula, while in the applied mathematics and numerical analysis communities it is called split-stepping. Let me state it in the context of ordinary differential equations. Suppose that Y and Z are two smooth vector fields on $\mathbb{R}^n$, and we know how to solve each of the differential equations $dx/dt = Y(x)$ and $dx/dt = Z(x)$, meaning that we know both of the flows ϕ_t and ψ_t on $\mathbb{R}^n$ generated by X and Y respectively. The Trotter Product Formula is a method for constructing the flow θ_t generated by $Y + Z$ out of ϕ and ψ; namely, letting $\Delta t = \frac{t}{n}$, $\theta_t = \lim_{n \to \infty}(\phi_{\Delta t}\psi_{\Delta t})^n$. The intuition behind the formula is simple. Think of approximating the solution of $dx/dt = Y(x) + Z(x)$ by Euler's Method. If we are currently at a point p_0, to propagate one more time step Δt we go to the point $p_0 + \Delta t\,(Y(p_0) + Z(p_0))$. Using the split-step approach on the other hand, we first take an Euler step in the $Y(p_0)$ direction, going to $p_1 = p_0 + \Delta t\, Y(p_0)$, then take a second Euler step, but now from p_1 and in the $Z(p_1)$ direction, going to $p_2 = p_1 + \Delta t\, Z(p_1)$. If Y and Z are constant vector fields, then this gives exactly the same final result as the simple full Euler step with $Y + Z$, while for continuous Y and Z and small time step Δt it is a good enough approximation that the above limit is valid. The situation is more delicate for flows on infinite dimensional manifolds, nevertheless it was shown by F. Tappert in [26] that the Cauchy Problem for KdV can be solved numerically by using split-stepping to combine methods for $u_t = -uu_x$ and $u_t = -u_{xxx}$.

Split-stepping suggests a way to understand the mechanism by which dispersion from u_{xxx} balances shock formation from uu_x in KdV. Namely, if we consider wave profile evolution under KdV as made up of a succession of pairs of small steps (one for $u_t = -uu_x$ and the one for $u_t = -u_{xxx}$), then when u, u_x, and u_{xxx} are not too large, the steepening mechanism will dominate. But recall that as the time t approaches the breaking time T_B, u remains bounded, and along a breaking characteristic u_x only blows up like $(T_B - t)^{-1}$ while u_{xxx} blows up like $(T_B - t)^{-5}$. Thus, near breaking in time and space, the u_{xxx} term will dwarf the nonlinearity and will disperse the incipient shock. In fact, computer simulations do show just such a scenario playing out.

3 The Korteweg-de Vries Equation

We have just seen that the Korteweg-de Vries equation,

$$\text{(KdV)} \qquad u_t + 6uu_x + u_{xxx} = 0,$$

expresses a balance between dispersion from its third-derivative term and the shock-forming tendency of its nonlinear term, and in fact many models of one-dimensional physical systems that exhibit mild dispersion and weak nonlinearity lead to KdV as the controlling equation at some level of approximation.

As mentioned earlier, KdV first arose as the equation modelling solitary gravity waves in a shallow canal. Such waves are rare and not easy to produce, and as discussed in the first section, they were apparently first noticed by Russell in 1834, and early attempts by Stokes and Airy to model them mathematically seemed to indicate that they could not be stable—and their very existence was at first a matter of debate. Later work by Boussinesq and Rayleigh corrected errors in this earlier theory, and finally a paper in 1894 by Korteweg and de Vries [16] settled the matter by giving a convincing mathematical argument that wave motion in a shallow canal is governed by KdV, and showing by explicit computation that their equation admitted travelling-wave solutions that had exactly the properties described by Russell, including the relation of height to speed that Russell had determined experimentally in a wave tank he had constructed.

But it was only much later that further remarkable properties of the KdV equation became evident. In 1954, Fermi, Pasta and Ulam (FPU) used one of the very first digital computers to perform numerical experiments on an elastic string with nonlinear restoring force, and their results contradicted the then current expectations of how energy should distribute itself among the normal modes of such a system [10]. A decade later, Zabusky and Kruskal re-examined the FPU results in a famous paper [29], where they showed that the FPU string was well approximated by the KdV equation. They then did their own computer experiments, solving the Cauchy Problem for KdV with initial conditions corresponding to those used in the FPU experiments. In the results of these simulations they observed the first example of a "soliton", a term that they coined to describe a remarkable particle-like behavior (elastic scattering) exhibited by certain KdV solutions. Zabusky and Kruskal showed how the coherence of solitons explained the anomalous results observed by Fermi, Pasta, and Ulam. But in solving that mystery they had uncovered a larger one; the behavior of KdV solitons was unlike anything seen before in applied mathematics, and the search for an explanation of their remarkable behavior led to a series of discoveries that changed the course of applied mathematics for the next thirty years. We next fill in some of the mathematical details behind the above sketch, beginning with a discussion of explicit solutions to the KdV equation.

To find the travelling wave solutions of KdV is straightforward; if we substitute a travelling wave $u(x,t) = f(x - ct)$ into KdV we obtain the ODE $-cf' + 6ff' + f'''$, and adding as boundary condition that f should vanish at infinity, a routine computation leads to the two-parameter family of travelling-wave solutions:

$$u(x,t) = 2a^2 \operatorname{sech}^2(a(x - 4a^2 t + d)).$$

These are the solitary waves seen by Russell, and they are now usually referred to as the 1-soliton solutions of KdV. Note that their amplitude, $2a^2$, is just half their speed, $4a^2$, while their "width" is proportional to a^{-1}; i.e., taller solitary waves are thinner and move faster.

Next, following Toda [27], we will "derive" [1] the 2-soliton solutions of KdV. Rewrite the 1-soliton solution as $u(x,t) = 2\frac{\partial^2}{\partial x^2} \log\cosh(a(x - 4a^2 t + \delta)$, or $u(x,t) = 2\frac{\partial^2}{\partial x^2} \log K(x,t)$, where $K(x,t) := (1 + e^{2a(x-4a^2t+\delta)})$. We now try to generalize, looking for solutions of the form $u(x,t) = 2\frac{\partial^2}{\partial x^2} \log K(x,t)$, with $K(x,t) := 1 + A_1 e^{2\eta_1} + A_2 e^{2\eta_2} + A_3 e^{2(\eta_1+\eta_2)}$, where $\eta_i = a_i(x - 4a_i^2 t + d_i)$, and we are to choose the A_i and d_i by substituting in KdV and seeing what works. It is easy to see that KdV is satisfied for $u(x,t)$ of this form and arbitrary $A_1, A_2, a_1, a_2, d_1, d_2$, provided that we define $A_3 = \left(\frac{a_2 - a_1}{a_1 + a_2}\right)^2 A_1 A_2$, and solutions of KdV arising this way are called the KdV 2-soliton solutions.

[1] This is a complete swindle! Only knowledge of the form of the solutions allows us to make the clever choice of K.

It can now be shown that for these choices of a_1 and a_2,

$$u(x,t) = 12\frac{3 + 4\cosh(2x - 8t) + \cosh(4x - 64t)}{[\cosh(3x - 36t) + 3\cosh(x - 28t)]^2}.$$

In particular $u(x,0) = 6\,\mathrm{sech}^2(x)$, for t large and negative, $u(x,t)$ is asymptotically equal to $2\,\mathrm{sech}^2(x - 4t - \phi) + 8\,\mathrm{sech}^2(x - 16t + \frac{\phi}{2})$, while for t large and positive, $u(x,t)$ is asymptotically equal to $2\,\mathrm{sech}^2(x - 4t + \phi) + 8\,\mathrm{sech}^2(x - 16t - \frac{\phi}{2})$, where $\phi = \log(3)/3$. (For details see [27], Chapter 6.)

Note what this says. If we follow the evolution from $-T$ to T (where T is large and positive), we first see the superposition of two 1-solitons; a larger and thinner one to the left of and overtaking a shorter, fatter, and slower-moving one to the right. Around $t = 0$ they merge into a single lump (with the shape $6\,\mathrm{sech}^2(x)$), and then they separate again, with their original shapes restored, but now the taller and thinner one is to the right. It is almost as if they had passed right through each other—the only effect of their interaction is the pair of phase shifts—the slower one is retarded slightly from where it would have been, and the faster one is slightly ahead of where it would have been. Except for these phase shifts, the final result is what we might expect from a linear interaction. It is only if we see the interaction as the two solitons meet that we can detect its highly nonlinear nature. (Note that at time $t = 0$, the maximum amplitude, 6, of the combined wave is actually less than the maximum amplitude, 8, of the taller wave when they are separated.) But of course the really striking fact is the resilience of the two individual solitons—their ability to put themselves back together after the collision. Not only is no energy radiated away, but their actual shapes are preserved. (Remarkably, page 384 of Russell's 1844 paper has a sketch of a 2-soliton interaction experiment that Russell had carried out in his wave tank!)

Now back to the computer experiment of Zabusky and Kruskal. For numerical reasons, they chose to deal with the case of periodic boundary conditions—in effect studying the KdV equation $u_t + uu_x + \delta^2 u_{xxx} = 0$ (which they label (1)) on the circle instead of on the line. For their published report, they chose $\delta = 0.022$ and used the initial condition $u(x,0) = \cos(\pi x)$. With the above background, it is interesting to read the following extract from their 1965 report, containing the first use of the term "soliton":

> (I) Initially the first two terms of Eq. (1) dominate and the classical overtaking phenomenon occurs; that is u steepens in regions where it has negative slope. (II) Second, after u has steepened sufficiently, the third term becomes important and serves to prevent the formation of a discontinuity. Instead, oscillations of small wavelength (of order δ) develop on the left of the front. The amplitudes of the oscillations grow, and finally *each* oscillation achieves an almost steady amplitude (that increases linearly from left to right) and has the shape of an individual solitary-wave of (1). (III) Finally, each "solitary wave pulse" or *soliton* begins to move uniformly at a rate (relative to the background value of u from which the pulse rises) which is linearly proportional to its amplitude. Thus, the solitons spread apart. Because of the periodicity, two or more solitons eventually overlap spatially and interact nonlinearly. Shortly after the interaction they reappear virtually unaffected in size or shape. In other words, solitons "pass through" one another without losing their identity. *Here we have a nonlinear physical process in which interacting localized pulses do not scatter irreversibly.*

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