Lecture 13
The Fundamental Forms of a Surface

In the following we denote by $\mathcal{F} : \mathcal{O} \to \mathbf{R}^3$ a parametric surface in $\mathbf{R}^3$, $\mathcal{F}(u,v) = (x(u,v), y(u,v), z(u,v))$. We denote partial derivatives with respect to the parameters u and v by subscripts: $\mathcal{F}_u := \frac{\partial \mathcal{F}}{\partial u}$ and $\mathcal{F}_v := \frac{\partial \mathcal{F}}{\partial u}$, and similarly for higher order derivative. We recall that if $p = (u_0, v_0) \in \mathcal{O}$ then $\mathcal{F}_u(p)$ and $\mathcal{F}_v(p)$ is a basis for $T\mathcal{F}_p$, the tangent space to $\mathcal{F}$ at p, the unit normal to $\mathcal{F}$ at p is $\overrightarrow{\nu}(p) := \frac{\mathcal{F}_u(p) \times \mathcal{F}_v(p)}{\|\mathcal{F}_u(p) \times \mathcal{F}_v(p)\|}$ and that we call the map $\overrightarrow{\nu} : \mathcal{O} \to \mathbf{S}^2$ the Gauss map of the surface $\mathcal{F}$.

13.1 Bilinear and Quadratic Forms

There are two important pieces of data associated to any surface, called its First and Second Fundamental Forms.

The First Fundamental Form encodes the "intrinsic data" about the surface—i.e., the information that you could discover by wandering around on the surface and making measurements **within** the surface.

The Second Fundamental Form on the other hand encodes the information about how the surface is embedded into the surrounding three dimensional space—explicitly it tells how the normal vector to the surface varies as one moves in different directions on the surface, so you could say it tells how the surface is curved in the embedding space.

These two "fundamental forms" are invariant under congruence, and moreover, they are a complete set of invariants for surfaces under congruence, meaning that if two surfaces have the same first and second fundamental forms then they are congruent. This latter fact is part of the Fundamental Theorem of Surfaces. But, it turns out that, unlike the curvature and torsion of a curve, not every **apparently** possible choice of First Fundamental Form and Second Fundamental Form for a surface can be realized by an actual surface. For this to be the case, the two forms must satisfy certain differential identities called the *Gauss-Codazzi Equations* and this fact is also part of the Fundamental Theorem of Surfaces.

Before considering the definitions of the fundamental forms on a surface, we make a short detour back into linear algebra to consider the general notions of bilinear and quadratic forms on a vector space.

13.1.1 Definition. Let V be a real vector space. A real-valued function $B : V \times V \to \mathbf{R}$ is called a *bilinear form* on V if it is linear in each variable separately when the other variable is held fixed. The bilinear form B is called *symmetric* (respectively *skew-symmetric*) if $B(v_1, v_2) = B(v_2, v_1)$ (respectively $B(v_1, v_2) = -B(v_2, v_1)$) for all $v_1, v_2 \in V$.

▷ **13.1—Exercise 1.** Show that every bilinear form on a vector space can be decomposed uniquely into the sum of a symmetric and a skew-symmetric bilinear form.

13.1.2 Definition. A real-valued function Q on a vector space V is called a *quadratic form* if it can be written in the form $Q(v) = B(v, v)$ for some symmetric bilinear form B on V. (We say that Q is *determined by B*.)

▷ **13.1—Exercise 2.** (Polarization Again.) Show that if Q is a quadratic form on V then the bilinear form B on V such that $Q(v) = B(v, v)$ is uniquely determined by the identity $B(v_1, v_2) = \frac{1}{2}(Q(v_1 + v_2) - Q(v_1) - Q(v_2))$.

Notation. Because of this bijective correspondence between quadratic forms and bilinear forms, it will be convenient to use the same symbol to denote them both. That is, if Q is a quadratic form then we shall also write Q for the bilinear form that determines it, so that $Q(v) = Q(v, v)$.

13.1.3 Remark. Suppose that V is an inner-product space. Then the inner product is a bilinear form on V and the quadratic form it determines is of course $Q(v) = \|v\|^2$. More generally, if $A : V \to V$ is any linear operator on V, then $B^A(v_1, v_2) = \langle Av_1, v_2 \rangle$ is a bilinear form on V and B^A is symmetric (respectively, skew-symmetric) if and only if A is self-adjoint (respectively, skew-adjoint).

▷ **13.1—Exercise 3.** Show that any bilinear form on a finite dimensional inner-product space is of the form B^A for a unique choice of self-adjoint operator A on V, and hence any quadratic form on an inner-product space is of the form $Q^A(v) = \langle Av, v \rangle$ for a unique choice of self-adjoint operator A on V.

13.1.4 Remark. If B is a bilinear form on a vector space V and if $v_1, \dots, v_n$ is a basis for V then the $b_{ij} = B(v_i, v_j)$ are called the matrix of coefficients of the form B in this basis. Clearly, if $u = \sum_i u_i v_i$ and $w = \sum_i w_i v_i$, than $B(u, w) = \sum_{ij} b_{ij} u_i w_j$. The bilinear form B is symmetric if and only if the matrix b_{ij} is symmetric, and in that case the quadratic form Q determined by B is $Q(u) = \sum_{ij} b_{ij} u_i u_j$.

13.1.5 Remark. Suppose that $T : V \to V$ is a self-adjoint operator on an inner-product space V, and that $v_1, \dots, v_n$ is a basis for V. What is the relation between the matrix $b_{ij} = \langle Tv_i, v_j \rangle$ of the symmetric bilinear form B^T defined by T, and the matrix A of T in the basis $v_1, \dots, v_n$? Your first guess may be that these two matrices are equal, however life is not quite that simple.

13.1.6 Proposition. *Let $T : V \to V$ be a self-adjoint operator on an inner-producr space V. If $b = (b_{ij})$ is the matrix of coefficients of the bilinear form B^T determined by T and A is the matrix of T, both with respect to the same basis $v_1, \dots, v_n$ for V, then $A = g^{-1}b$, where g is the matrix of inner-products $g_{ij} = \langle v_i, v_j \rangle$, i.e., the matrix of coefficients of the bilinear form given by the inner-product.*

PROOF. By the definition of A, $Tv_i = \sum_{i=1}^{n} A_{ki} v_k$, hence $b_{ij} = \langle \sum_{i=1}^{n} A_{ki} v_k, v_j \rangle = A_{ki} g_{kj}$, i.e., $b = A^t g$, where A^t is the transpose of A, Hence $A^t = bg^{-1}$, and since b and g are symmetric, $A = (bg^{-1})^t = g^{-1}b$.

13.2 Quadratic Forms on a Surface

13.2.1 Definition. If $\mathcal{F} : \mathcal{O} \to \mathbf{R}^3$ is a parametric surface in $\mathbf{R}^3$, then a *quadratic form on* $\mathcal{F}$, Q, we mean a function $p \mapsto Q_p$ that assigns to each p in $\mathcal{O}$ a quadratic form Q_p on the tangent space $T\mathcal{F}_p$ of $\mathcal{F}$ at p.

13.2.2 Remark. Making use of the bases $\mathcal{F}_u(p), \mathcal{F}_v(p)$ in the $T\mathcal{F}_p$, a quadratic form Q on $\mathcal{F}$ is described by the symmetric 2×2 matrix of real-valued functions $Q_{ij} : \mathcal{O} \to \mathbf{R}$ defined by $Q_{ij}(p) := Q(\mathcal{F}_{x_i}(p), \mathcal{F}_{x_j}(p))$, (where $x_1 = u$ and $x_2 = v$). These three functions Q_{11}, Q_{12}, and Q_{22} on $\mathcal{O}$ determine the quadratic form Q on $\mathcal{F}$ uniquely: if $w \in T\mathcal{F}_p$, then $w = \xi \mathcal{F}_u(p) + \eta \mathcal{F}_v$, and $Q_p(w) = Q_{11}(p)\,\xi^2 + 2\,Q_{12}(p)\,\xi\,\eta + Q_{22}(p)\,\eta^2$. We call the Q_{ij} the *coefficients* of the quadratic form Q, and we say that Q is of class C^k if its three coefficients are C^k. Note that we can choose *any* three functions Q_{ij} and use the above formula for $Q_p(w)$ to define a unique quadratic form Q on $\mathcal{F}$ with these Q_{ij} as coefficients. This means that **we can identify quadratic forms on a surface with ordered triples of real-valued functions on its domain.**

Notation. Because of the preceding remark, it is convenient to have a simple way of referring to the quadratic form Q on a surface having the three coefficients A, B, C. There is a classical and standard notation for this, namely:

$$Q = A(u,v)\,du^2 + 2B(u,v)\,du\,dv + C(u,v)\,dv^2.$$

▷ **13.2—Exercise 1.** To see the reason for this notation—and better understand its meaning—consider a curve in $\mathcal{O}$ given parametrically by $t \mapsto (u(t), v(t))$, and the corresponding image curve $\alpha(t) := \mathcal{F}(u(t), v(t))$ on $\mathcal{F}$. Show that

$$Q(\alpha'(t)) = A(u(t), v(t)) \left(\frac{du}{dt}\right)^2 + B(u(t), v(t)) \left(\frac{du}{dt}\right)\left(\frac{dv}{dt}\right) + C(u(t), v(t)) \left(\frac{dv}{dt}\right)^2.$$

The point is that curves on $\mathcal{F}$ are nearly always given in the form $t \mapsto \mathcal{F}(u(t), v(t))$, so a knowledge of the coefficients A, B, C as functions ot u, v is just what is needed in order to compute the values of the form on tangent vectors to such a curve from the parametric functions $u(t)$ and $v(t)$. As a first application we shall now develop a formula for the length of the curve α.

Definition of the First Fundamental Form of a Surface $\mathcal{F}$
Since $T\mathcal{F}_p$ is a linear subspace of $\mathbf{R}^3$ it becomes an inner-product space by using the restriction of the inner product on $\mathbf{R}^3$. Then the First Fundamental Form on $\mathcal{F}$, denoted by $I^{\mathcal{F}}$, is defined by $I_p^{\mathcal{F}}(w) := \|w\|^2$, and its coefficients are denoted by $E^{\mathcal{F}}, F^{\mathcal{F}}, G^{\mathcal{F}}$. When there is no chance of ambiguity we will omit the superscript from $I^{\mathcal{F}}$ and its coefficients. Thus:

$$I^{\mathcal{F}} = E^{\mathcal{F}}(u,v)\,du^2 + 2F^{\mathcal{F}}(u,v)\,du\,dv + G^{\mathcal{F}}(u,v)\,dv^2,$$

where the functions $E^{\mathcal{F}}$, $F^{\mathcal{F}}$, and $G^{\mathcal{F}}$ are defined by:

$$E^{\mathcal{F}} := \mathcal{F}_u \cdot \mathcal{F}_u = x_u^2 + y_u^2 + z_u^2,$$
$$F^{\mathcal{F}} := \mathcal{F}_u \cdot \mathcal{F}_v = x_u x_v + y_u y_v + z_u z_v,$$
$$G^{\mathcal{F}} := \mathcal{F}_v \cdot \mathcal{F}_v = x_v^2 + y_v^2 + z_v^2.$$

The Length of a Curve on a Surface

Let $t \mapsto (u(t), v(t))$ be a parametric curve in $\mathcal{O}$ with domain $[a, b]$. By the above exercise, the length, L, of the curve $\alpha : t \mapsto \mathcal{F}(u(t), v(t))$ is:

$$L = \int_a^b \sqrt{I(\alpha'(t))}\, dt$$

$$= \int_a^b \sqrt{E(u(t), v(t)) \left(\frac{du}{dt}\right)^2 + F(u(t), v(t)) \left(\frac{du}{dt}\right)\left(\frac{dv}{dt}\right) + G(u(t), v(t)) \left(\frac{dv}{dt}\right)^2}\, dt\,.$$

Alternative Notations for the First Fundamental Form.
The First Fundamental Form of a surface is so important that there are several other standard notational conventions for referring to it. One whose origin should be obvious is to denote to it by ds^2, and call $ds = \sqrt{ds^2}$ the "line element" of the surface.

13.3 The Shape Operator and Second Fundamental Form

We next consider the differential $D\nu_p$ of the Gauss map $\nu : \mathcal{O} \to \mathbf{S}^2$ at a point p of $\mathcal{O}$. Strictly speaking it is a linear map of $\mathbf{R}^2 \to \mathbf{R}^3$, but as we shall now see it has a natural interpretation as a map of $T_p\mathcal{F}$ to itself. As such it plays a central role in the study of the extrinsic properties of $\mathcal{F}$ and is called the *shape operator* of $\mathcal{F}$ at p. Moreover, we shall also establish the important fact that the shape operator is a self-adjoint operator on $T_p\mathcal{F}$ and so defines a quadratic form on $\mathcal{F}$, the Second Fundamental Form of the surface.

In fact, since $D\mathcal{F}_p$ is by definition an isomorphism of $\mathbf{R}^2$ onto $T_p\mathcal{F}$, given $w \in T_p\mathcal{F}$, we can define $D\nu_p(w) := (\frac{d}{dt})_{t=0} \nu(\alpha(t))$, where α is any curve of the form $\alpha(t) := \mathcal{F}(\gamma(t))$ with $\gamma(t)$ a curve in $\mathcal{O}$ with $\gamma(0) = p$ such that $D\mathcal{F}_p(\gamma'(0)) = w$. Then since $\nu(\alpha(t)) \in \mathbf{S}^2$ for all t, it follows that $D\nu_p(w) \in T_{\nu(p)}\mathbf{S}^2 = \nu_p^\perp = T\mathcal{F}_p$, completing the proof that $D\nu_p$ maps $T\mathcal{F}_p$ to itself.

13.3.1 Definition. The linear map $-D\nu_p : T_p\mathcal{F} \to T_p\mathcal{F}$ is called the *shape operator* of the surface $\mathcal{F}$ at p.

13.3.2 Remark. The reason for the minus sign will appear later. (It gives the curvatures of the standard surfaces their correct sign.)

13.3.3 Theorem. *The shape operator is self-adjoint.*

▷ **13.3—Exercise 1.** Prove this. Hint—you must show that for all $w_1, w_2 \in T_p\mathcal{F}$, $\langle D\nu_p w_1, w_2 \rangle = \langle w_1, D\nu_p w_2 \rangle$. However it suffices to to prove this for w_1 and w_2 taken from some basis for $T_p\mathcal{F}$. (Why?) In particular you only need to show this when the w_i are taken from $\{\mathcal{F}_u, \mathcal{F}_v\}$. For this, take the partial derivatives of the identities $\langle \mathcal{F}_u, \nu \rangle = 0$ and $\langle \mathcal{F}_v, \nu \rangle = 0$ with respect to u and v (remembering that $\nu_u = D\nu(\mathcal{F}_u)$), so that for example $\langle D\nu(\mathcal{F}_u)), \mathcal{F}_v \rangle = \langle \nu_u, \mathcal{F}_v \rangle = (\langle \nu, \mathcal{F}_v \rangle)_u - \langle \nu, \mathcal{F}_{vu} \rangle = -\langle \nu, \mathcal{F}_{vu} \rangle$, etc.

Definition of the Second Fundamental Form of a Surface $\mathcal{F}$

We define the *Second Fundamental Form* of a surface $\mathcal{F}$ to be the quadratic form defined by the shape operator. It is denoted by $II^{\mathcal{F}}$, so for $w \in T_p\mathcal{F}$,

$$II_p^{\mathcal{F}}(w) = -\langle D\nu_p(w), w \rangle.$$

We will denote the components of the Second Fundamental Form by $L^{\mathcal{F}}, M^{\mathcal{F}}, N^{\mathcal{F}}$, so that

$$II^{\mathcal{F}} = L^{\mathcal{F}}(u,v)\, du^2 + 2M^{\mathcal{F}}(u,v)\, du\, dv + N^{\mathcal{F}}(u,v)\, dv^2,$$

where the functions $L^{\mathcal{F}}$, $M^{\mathcal{F}}$, and $N^{\mathcal{F}}$ are defined by:

$$L^{\mathcal{F}} := -D\nu(\mathcal{F}_u) \cdot \mathcal{F}_u = \nu \cdot \mathcal{F}_{uu},$$
$$M^{\mathcal{F}} := -D\nu(\mathcal{F}_u) \cdot \mathcal{F}_v = \nu \cdot \mathcal{F}_{uv},$$
$$N^{\mathcal{F}} := -D\nu(\mathcal{F}_v) \cdot \mathcal{F}_v = \nu \cdot \mathcal{F}_{vv}.$$

As with the First Fundamental Form, we will usually omit the superscript $\mathcal{F}$ from $II^{\mathcal{F}}$ and its components when it is otherwise clear from the context.

Matrix Notation for First and Second Fundamental Form Components

It is convenient when making computations involving the two fundamental forms to have a more uniform matrix style notation for their components relative to the standard basis $\mathcal{F}_u, \mathcal{F}_v$ for $T_p\mathcal{F}$. In such situations we will put $t_1 = u$ and $t_2 = v$ and write $I = \sum_{i,j} g_{ij}\, dt_i\, dt_j$ and $II = \sum_{i,j} \ell_{ij}\, dt_i\, dt_j$. Thus $g_{11} = E$, $g_{12} = g_{21} = F$, $g_{22} = G$, and $\ell_{11} = L$, $\ell_{12} = \ell_{21} = M$, $\ell_{22} = N$. The formulas giving the g_{ij} and ℓ_{ij} in terms of partial derivatives of $\mathcal{F}$ are more uniform with this notation (and hence easier to compute with): $g_{ij} = \mathcal{F}_{t_i} \cdot \mathcal{F}_{t_j}$, and $\ell_{ij} = -\nu_{t_i} \cdot \mathcal{F}_{t_j} = \nu \cdot \mathcal{F}_{t_i t_j}$.

We will refer to the 2×2 matrix g_{ij} as g and its inverse matrix by g^{-1}, and we will denote the matrix elements of the inverse matrix by g^{ij}. By Cramer's Rule:

$$g^{-1} = \frac{1}{\det(g)} \begin{pmatrix} g_{22} & -g_{12} \\ -g_{12} & g_{11} \end{pmatrix},$$

i.e., $g^{11} = g_{22}/\det(g)$, $g^{22} = g_{11}/\det(g)$, and $g^{12} = g^{21} = -g_{12}/\det(g)$.

13.3.4 Remark. By Proposition 13.1.6, the matrix of the Shape operator in the basis $\mathcal{F}_{t_1}, \mathcal{F}_{t_2}$ is $g^{-1}\ell$.

▷ **13.3—Exercise 2.** Show that $\|\mathcal{F}_u \times \mathcal{F}_v\|^2 = \det(g) = EG - F^2$, so that the unit normal to $\mathcal{F}$ is $\nu = \frac{\mathcal{F}_u \times \mathcal{F}_v}{\sqrt{EG - F^2}}$. Hint—recall the formula $(u \times v) \cdot (x \times y) = \det \begin{pmatrix} u \cdot x & v \cdot x \\ u \cdot y & v \cdot y \end{pmatrix}$ from our quick review of the vector product.

Geometric Interpretation of the Second Fundamental Form

Definition. Let $\alpha(s) = \mathcal{F}(u(s), v(s))$ be a regular curve on $\mathcal{F}$ and let $n(s)$ denote its unit normal, so $\alpha''(s) = k(s)\,n(s)$, where $k(s)$ is the curvature of α. We define the *normal curvature* to α, denoted by $k_n(s)$, to be the component of $\alpha''(s)$ in the direction normal to $\mathcal{F}$, i.e., the dot product of $\alpha''(s) = k(s)\,n(s)$ with $\nu(\alpha(s))$, so that $k_n(s) = k(s)\cos(\theta(s))$, where $\theta(s)$ is the angle between the normal $n(s)$ to α and the normal $\nu(\alpha(s))$ to $\mathcal{F}$.

13.3.5 Meusnier's Theorem. *If $\alpha(s)$ is a regular curve on a surface $\mathcal{F}$, then its normal curvature is given by the formula $k_n(s) = II^{\mathcal{F}}(\alpha'(s))$. In particular, if two regular curves on $\mathcal{F}$ pass through the same point p and have the same tangent at p, then they have the same normal curvature at p.*

PROOF. Since α is a curve on $\mathcal{F}$, $\alpha'(s)$ is tangent to $\mathcal{F}$ at $\alpha(s)$, so $\alpha'(s) \cdot \nu(\alpha(s))$ is identically zero. Differentiating gives $\alpha''(s) \cdot \nu(\alpha(s)) + \alpha'(s) \cdot D\nu(\alpha'(s)) = 0$, so $k_n(s) := \alpha''(s) \cdot \nu(\alpha(s)) = -\alpha'(s) \cdot D\nu(\alpha'(s)) = II(\alpha'(s))$. ∎

13.3.6 Remark. Recall that the curvature of a curve measures its second order properties, so the remarkable thing about Meusnier's Theorem is that it says, for a curve α that lies on a surface $\mathcal{F}$, k_n, the normal component of the curvature of α depends only on its first order properties (α') and the second order properties of $\mathcal{F}$ $(D\nu)$. The obvious conclusion is that k_n measures the curvature of α that is a consequence of its being constrained to lie in the surface.

13.3.7 Remark. If w is a unit tangent vector to $\mathcal{F}$ at p, then w and $\nu(p)$ determine a plane Π through p that cuts $\mathcal{F}$ in a curve $\alpha(s)$ lying on $\mathcal{F}$ with $\alpha(0) = p$ and $\alpha'(0) = w$, This curve α is called the *normal section* of $\mathcal{F}$ in the direction by w. Since α lies in the plane Π, $\alpha''(0)$ is tangent to Π, and since it is of course orthogonal to $w = \alpha'(0)$, it follows that $\alpha''(0)$ must be parallel to $\nu(p)$—i.e., the angle θ that $\alpha''(0)$ makes with $\nu(p)$ is zero, and hence by the definition of the normal curvature, $k_n = k\cos(\theta) = k$, i.e., for a normal section, the normal curvature is just the curvature, so we could equivalently define the Second Fundamental Form of $\mathcal{F}$ by saying that for a unit vector $w \in T_p\mathcal{F}$, $II(w)$ is the curvature of the normal section at p in the direction w. (This is how I always think of II.)

▷ **13.3—Exercise 3.** Show that the First and Second Fundamental Forms of a Surface are invariant under congruence. That is, if g is an element of the Euclidean group $\boldsymbol{Euc}(\mathbf{R}^3)$, then $g \circ \mathcal{F}$ has the same First and Second Fundamental Forms as $\mathcal{F}$.

The Principal Directions and Principal Curvatures

Since the Shape operator, $-D\nu_p$, is a self-adjoint operator on $T_p\mathcal{F}$, by the Spectral Theorem there an orthonormal basis e_1, e_2 for $T_p\mathcal{F}$ consisting of eigenvectors of the Shape operator. The corresponding eigenvalues λ_1, λ_2 are called the *principal curvatures* at p, and e_1 and e_2 are called *principal directions* at p. Recall that in general, if $T : V \to V$ is a self-adjoint operator, then a point on the unit sphere of V where the corresponding quadratic form $\langle Tv, v \rangle$ assumes a minimum or maximum value is an eigenvector of T. Since $T_p\mathcal{F}$ is two-dimensional, we can define λ_1 and λ_2 as respectively the minimum and maximum values

of $II_p(w)$ on the unit sphere (a circle!) in $T_p\mathcal{F}$, and e_1 and e_2 as unit vectors where these minimum and maximum values are assumed. We define the *Gaussian Curvature K* and the *Mean Curvature H* at p to be respectively the determinant and trace of the Shape operator $-D\nu_p$, so $K = \lambda_1\lambda_2$ and $H = \lambda_1 + \lambda_2$.

13.3.8 Remark. It is important to have a good formulas for K, H, and the principal curvatures in terms of the coefficients of the first and second fundamental forms (which themselves can easily be computed from the parametric equations for the surface). Recalling from 13.3.4 that the matrix of the Shape operator in the usual basis $\mathcal{F}_u, \mathcal{F}_v$ is $g^{-1}\ell$, it follows that:

$$K = \frac{\det(\ell)}{\det(g)} = \frac{\ell_{11}\ell_{22} - \ell_{12}^2}{g_{11}g_{22} - g_{12}^2}.$$

▷ **13.3—Exercise 4.** Show that the Mean Curvature is given in terms of the coefficients of the first and second fundamental forms by the formula:

$$H = \frac{g_{22}\ell_{11} - 2g_{12}\ell_{12} + g_{11}\ell_{22}}{g_{11}g_{22} - g_{12}^2}.$$

(Hint: The trace of an operator is the sum of the diagonal elements of its matrix with respect to **any** basis.)

13.3.9 Remark. Now that we have formulas for H and K in terms of the g_{ij} and ℓ_{ij}, it is easy to get formulas for the principal curvatures λ_1, λ_2 in terms of H and K (and so in terms of g_{ij} and ℓ_{ij}). Recall that the so-called characteristic polynomial of the Shape operator is $\chi(\lambda) := \det(-D\nu - \lambda I) = (\lambda - \lambda_1)(\lambda - \lambda_2) = \lambda^2 - H\lambda + K$, so that its roots, which are the principal curvatures λ_1, λ_2 are given by $\lambda_1 = \frac{H - \sqrt{H^2 - 4K}}{2}$ and $\lambda_2 = \frac{H + \sqrt{H^2 - 4K}}{2}$.

13.3.10 Remark. There is a special case one should keep in mind, and that is when $\lambda_1 = \lambda_2$, i.e., when II_p is constant on the unit sphere of $T_p\mathcal{F}$. Such a point p is called an *umbillic point* of $\mathcal{F}$. While at a non-umbillic point the principal directions e_1 and e_2 are uniquely determined up to sign, at an umbilic point every direction is a principal direction and we can take for e_1, e_2 any orthonormal basis for the tangent space at p.

Parallel Surfaces

We define a one-parameter family of surfaces $\mathcal{F}(t) : \mathcal{O} \to \mathbf{R}^3$ associated to the surface $\mathcal{F}$ by $\mathcal{F}(t)(u,v) = \mathcal{F}(u,v) - \nu(u,v)$. Clearly $\mathcal{F}(0) = \mathcal{F}$ and $\|\mathcal{F}(t)(u,v) - \mathcal{F}(u,v)\| = t$. Also, $D\mathcal{F}(t)_p = D\mathcal{F}_p + tD\nu_p$, and since $D\nu_p$ maps $T_p\mathcal{F}$ to itself, it follows that $T_p\mathcal{F}(t) = T_p\mathcal{F}$ (at least for t sufficiently small). So, for obvious reasons, we call $\mathcal{F}(t)$ the parallel surface to $\mathcal{F}$ at distance t.

▷ **13.3—Exercise 5.** Since $T_p\mathcal{F}(t) = T_p\mathcal{F}$, it follows that the First Fundamental Forms $I^{\mathcal{F}(t)}$ of the parallel surfaces can be regarded as a one-parameter family of quadratic forms on $\mathcal{F}$. Show that $II^{\mathcal{F}} = \left(\frac{d}{dt}\right)_{t=0} I^{\mathcal{F}(t)}$.

13.3—Example 1. A plane is a surface $\mathcal{F} : \mathbf{R}^2 \to \mathbf{R}^3$ given by a map of the form $p \mapsto x_0 + T(p)$ where $T : \mathbf{R}^2 \to \mathbf{R}^3$ is a linear map of rank two. If we call Π the image of P (a two-dimensional linear subspace of $\mathbf{R}^3$), then clearly the image of $\mathcal{F}$ is $x_0 + \Pi$, the tangent space to $\mathcal{F}$ at every point is Π, and the normal vector ν_p is the same at every point (one of the two unit vectors orthogonal to Π). Here are three ways to see that the Second Fundamental Form of such a surface is zero:

a) The normal sections are all straight lines, so their curvatures vanish.

b) Since ν is constant, the parallel surfaces $\mathcal{F}(t)$ are obtained from $\mathcal{F}$ by translating it by $t\nu$, a Euclidean motion, so all of the First Fundamental Forms $I^{\mathcal{F}(t)}$ are the same, and by the preceding exercise $II^{\mathcal{F}} = 0$.

c) Since the Gauss Map $\nu : \mathbf{R}^2 \to \mathbf{S}^2$ is a constant, the Shape operator $-D\nu$ is zero.

13.3—Example 2. The sphere of radius r. We have already seen how to parametrize this using longitude and co-latitude as the parameters. Also, any hemisphere can be parametrized in the usual way as a graph. However we will not need any parmetrization to compute the Second Fundamental Form. We use two approaches.

a) The normal sections are all great circles, so in particular they are circles of radius r, and so have curvature $\frac{1}{r}$. Thus the Shape operator is $\frac{1}{r}$ times the identity.

b) If $\mathcal{F}(t)$ is the parallel surface at distance t, then clearly $\mathcal{F}(t) = \frac{r+t}{r}\mathcal{F} = (1 + \frac{t}{r})\mathcal{F}$, so $I^{\mathcal{F}(t)} = (1 + \frac{t}{r})I^{\mathcal{F}}$, and this time the exercise gives $II^{\mathcal{F}} = \frac{1}{r}I^{\mathcal{F}}$.

It follows that the Gauss Curvature of the sphere is $K = \frac{1}{r^2}$, and its mean curvature is $H = \frac{2}{r}$.