Appendix C

Vector Fields as Differential Operators

Let $V = (p, v)$ be a point of $\mathbf{R}^n \times \mathbf{R}^n$. We are going to regard such a pair asymmetrically as a "vector v based at the point p", and as such we will refer to it as a tangent vector at p. If $\sigma : I \to \mathbf{R}^n$ is a C^1 curve, then for each t_0 in I, we get such a pair, $(\sigma(t_0), \sigma'(t_0))$, which we will denote by $\dot{\sigma}(t_0)$ and call the *tangent vector to* σ *at time* t_0. Let $C^\infty(\mathbf{R}^n)$ denote the algebra of smooth real-valued functions on $\mathbf{R}^n$. If $f \in C^\infty(\mathbf{R}^n)$, then the directional derivative of f at $p = \sigma(t_0)$ in the direction $v = \sigma'(t_0)$ is by definition $\left(\frac{d}{dt}\right)_{t=t_0} f(\sigma(t))$, which by the chain rule is equal to $\sum_{i=1}^n v_i \frac{\partial f}{\partial x_i}(p)$. An important consequence of the latter formula is that the directional derivative depends only on $\dot{\sigma}(t_0) = (p, v)$ and not on the choice of curve σ. (So we can for example take σ to be the straight line $\sigma(t) = p + tv$.)

This justifies using Vf to denote the directional derivative and regarding V as a (clearly linear) map $V : C^\infty(\mathbf{R}^n) \to \mathbf{R}$. Moreover, since $Vx_i = v_i$, this map determines V, and it has become customary to identify the tangent vector V with this linear map and denote V alternatively by $\sum_{i=1}^n v_i \left(\frac{\partial}{\partial x_i}\right)_p$. In particular, taking $v_i = 1$ for $i = k$ and $v_i = 0$ for $i \neq k$ gives the tangent vector at p in the direction of the x_k coordinate curve, which we denote by $\left(\frac{\partial}{\partial x_k}\right)_p$.

It is an immediate consequence of the product rule of differentiation that the mapping V satisfies the so-called Liebniz Identity:

$$V(fg) = (Vf)g(p) + f(p)(Vg).$$

Any linear map $L : C^\infty(\mathbf{R}^n) \to \mathbf{R}$ that satisfies this Leibniz Identity is called a *derivation at* p. Note that such an L vanishes on a product fg if both f and g vanish at p (and hence also on any linear combination of such products).

▷ **Exercise C–1.** Show that if L is a derivation at p, then $Lf = 0$ for any constant function. (Hint: It is enough to prove this for $f \equiv 1$ [why?], but then $f^2 = f$.)

▷ **Exercise C–2.** Show that if L is a derivation at p, then it is the directional derivative operator defined by some tangent vector at p. (Hint: Use Taylor's Theorem with Integral Remainder to write any $f \in C^\infty(\mathbf{R}^n)$ as

$$f = f(p) + \sum_{i=1}^{n} \frac{\partial f}{\partial x_i}(p)(x_i - p_i) + R,$$

where R is a linear combination of products of functions vanishing at p.)

Now let O be open in $\mathbf{R}^n$. A vector field in O is a function that assigns to each p in O a tangent vector at p, $(p, V(p))$. Usually one simplifies the notation by dropping the redundant first component, p, and identifies the vector field with the mapping $V : O \to \mathbf{R}^n$. If $f : O \to \mathbf{R}$ is a smooth function on O, then $Vf : O \to \mathbf{R}$ is the function whose value at p is $V(p)f$, the directional derivative of f at p in the direction $V(p)$. If both V and f are C^∞, then clearly so is Vf, so that we may regard V as a linear operator on the vector space $C^\infty(O)$ of smooth real-valued functions on O.

▷ **Exercise C–3.** Suppose that $V : O \to \mathbf{R}^n$ is a C^∞ vector field in O. Show that $V : C^\infty(O) \to C^\infty(O)$ is a derivation of the algebra $C^\infty(O)$, i.e., a linear map satisfying $V(fg) = (Vf)g + f(Vg)$, and show also that every derivation of $C^\infty(O)$ arises in this way.

A vector field V is often identified with (and denoted by) the differential operator $\sum_{i=1}^n V_i \frac{\partial}{\partial x_i}$.

There is an important special vector field R in $\mathbf{R}^n$ called the *radial vector field*, or the Euler vector field. As a mapping $R : \mathbf{R}^n \to \mathbf{R}^n$, it is just the identity map, while as a differential operator it is given by $R := \sum_{i=1}^{n} x_i \frac{\partial}{\partial x_i}$. Recall that a function $f : \mathbf{R}^k \to \mathbf{R}$ is said to be positively homogeneous of degree k if $f(tx) = t^k f(x)$ for all $t > 0$ and $x \neq 0$.

▷ **Exercise C–4.** Prove Euler's Formula $\sum_{i=1}^{n} x_i \frac{\partial f}{\partial x_i} = kf$ for a C^1 function $f : \mathbf{R}^n \to \mathbf{R}$ that is positively homogeneous of degree k.