Appendix G

Canonical Form for Linear Operators

G.1. The Spectral Theorem

If V is an orthogonal vector space, then each element v of V defines a linear functional $f_v : V \to R$, namely $u \mapsto \langle u, v \rangle$, and since $f_v(u) = \langle u, v \rangle$ is clearly linear in v as well as u, we have a linear map $v \mapsto f_v$ of V into its dual space V^*. Moreover the kernel of this map is clearly 0 (since, if v is in the kernel, then $\|v\|^2 = \langle v, v \rangle = f_v(v) = 0$), and since V^* has the same dimension as V, it follows by basic linear algebra that this map is in fact a linear isomorphism of V with V^*. We say that v is *dual* to f_v and vice versa.

Now let $A : V \to V$ be a linear map, and for each v in V let A^*v in V be the element dual to the linear functional $u \mapsto \langle Au, v \rangle$; that is, A^*v is defined by the identity $\langle Au, v \rangle = \langle u, A^*v \rangle$. It is clear that $v \mapsto A^*v$ is linear, and we call this linear map $A^* : V \to V$ the *adjoint* of A. If $A^* = A$, then we say that A is *self-adjoint*.

▷ **Exercise G–1.** Let $L(V, V)$ denote the space of linear operators on V. Show that $A \mapsto A^*$ is a linear map of $L(V, V)$ to itself and that it is its own inverse (i.e., $A^{**} = A$). Show also that $(AB)^* = B^*A^*$.

G.1.1. Proposition. *Let A be a self-adjoint linear operator on V and let W be a linear subspace of V. If W is invariant under A, then so is $W^\perp$.*

Proof. If $u \in W^{\perp}$, we must show that Au is also in $W^{\perp}$, i.e., that $\langle w, Au\rangle = 0$ for any $w \in W$. Since $Aw \in W$ by assumption, $\langle w, Au\rangle = \langle Aw, u\rangle = 0$ follows from $u \in W^{\perp}$. ∎

In what follows, A will denote a self-adjoint linear operator on V. If λ is any scalar, than we denote by $E_\lambda(A)$ the set of v in V such that $Av = \lambda v$. It is clear that $E_\lambda(A)$ is a linear subspace of V, called the λ-*eigenspace of* A. If $E_\lambda(A)$ is not the 0 subspace of V, then we call λ an *eigenvalue* of A, and every nonzero element of $E_\lambda(A)$ is called an *eigenvector* corresponding to the eigenvalue λ.

G.1.2. Proposition. *If $\lambda \neq \mu$, then $E_\lambda(A)$ and $E_\mu(A)$ are orthogonal subspaces of V.*

▷ **Exercise G–2.** Prove this. (Hint: Let $u \in E_\lambda(A)$ and $v \in E_\mu(A)$. You must show $\langle u, v\rangle = 0$. But $\lambda\langle u, v\rangle = \langle Au, v\rangle = \langle u, Av\rangle = \mu\langle u, v\rangle$.)

Note that it follows that a self-adjoint operator on an N-dimensional orthogonal vector space can have at most N distinct eigenvalues.

G.1.3. Spectral Theorem for Self-Adjoint Operators. *If A is a self-adjoint operator on an orthogonal vector space V, then V is the orthogonal direct sum of the eigenspaces $E_\lambda(A)$ corresponding to the eigenvalues λ of A. Equivalently, we can find an orthonormal basis for V consisting of eigenvectors of A.*

▷ **Exercise G–3.** Prove the equivalence of the two formulations.

We will base the proof of the Spectral Theorem on the following lemma.

G.1.4. Spectral Lemma. *A self-adjoint operator $A : V \to V$ always has at least one eigenvalue unless $V = 0$.*

Here is the proof of the Spectral Theorem. Let W be the direct sum of the eigenspaces $E_\lambda(A)$ corresponding to the eigenvalues λ of A. We must show that $W = V$, or equivalently that $W^{\perp} = 0$. Now W is clearly invariant under A, so by the first proposition of this section, so is $W^{\perp}$. Since the restriction of a self-adjoint operator to an invariant subspace is clearly still self-adjoint, by the Spectral Lemma, if $W^\perp \neq 0$, then there would be an eigenvector of A in $W^\perp$, contradicting the fact that all eigenvectors of A are in W.

The proof of the Spectral Lemma involves a rather pretty geometric idea. Recall that we have seen that A is derivable from the potential function $U(v) = \frac{1}{2}\langle Av, v\rangle$, i.e., $Av = (\nabla U)_v$ for all v in V. So what we must do is find a unit vector v where $(\nabla U)_v$ is proportional to v provided $V \neq 0$, i.e., provided the unit sphere in V is not empty. In fact, something more general is true.

G.1.5. Lagrange Multiplier Theorem (Special Case). *Let V be an orthogonal vector space and $f : V \to R$ a smooth real-valued function on V. Let v denote a unit vector in V where f assumes its maximum value on the unit sphere S of V. Then $(\nabla f)_v$ is a scalar multiple of v.*

Proof. The scalar multiples of v are exactly the vectors normal to S at v, i.e., orthogonal to all vectors tangent to S at v. So we have to show that if u is tangent to S at v, then $(\nabla f)_v$ is orthogonal to u, i.e., that $\langle u, (\nabla f)_v\rangle = df_v(u) = 0$. Choose a smooth curve $\sigma(t)$ on S with $\sigma(0) = v$ and $\sigma'(0) = u$ (for example, normalize $v + tu$). Then since $f(\sigma(t))$ has a maximum at $t = 0$, it follows that $(d/dt)_{t=0} f(\sigma(t)) = 0$. But by definition of df, $(d/dt)_{t=0} f(\sigma(t)) = df_v(u)$. ∎

G.1.6. Definition. An operator A on an orthogonal vector space V is *positive* if it is self-adjoint and if $\langle Av, v\rangle > 0$ for all $v \neq 0$ in V.

▷ **Exercise G–4.** Show that a self-adjoint operator is positive if and only if all of its eigenvalues are positive.

▷ **Exercise G–5.** Verify the intuitive fact that a unit vector v is orthogonal to all vectors tangent to the unit sphere at v. (Hint: Choose σ as above and differentiate the identity $\langle \sigma(t), \sigma(t)\rangle = 1$.)

▷ **Exercise G–6.** Show that another equivalent formulation of the Spectral Theorem is that a linear operator on an orthogonal vector space is self-adjoint if and only if it has a diagonal matrix in some orthonormal basis.