PART 2

ANALYTICAL DYNAMICS

This part develops most of the important theoretical topics in classical mechanics in the general setting of symplectic manifolds. Chapter 3 sets out the basic theory of Hamiltonian and Lagrangian mechanics. This is followed by a rather extensive chapter on systems with symmetry, including current accounts of reduction by algebras of integrals, and topology of invariant manifolds. The final chapter of this part has, as a focus, the Hamilton–Jacobi theory, with numerous related topics such as action angle variables and Lagrangian submanifolds, as well as offshoots to topics like quantization and the equations of mathematical physics as Hamiltonian systems.

The use of differential forms in mechanics and its eventual formulation in terms of symplectic manifolds has been slowly evolving since Cartan [1922].

The first modern exposition of Hamiltonian systems on symplectic manifolds seems to be due to Reeb [1952e]. An early version of Lagrangian systems in this context appears in Mackey [1963]. This formulation of mechanics was widely known in mathematical circles by 1962, and was explained in a letter by Richard Palais that circulated privately at about that time. The first systematic treatise concerning mechanics on Riemannian manifolds that we know of is Synge [1926]. The reader is referred to Whittaker [1959] for additional historical details.

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Def. A symplectic manifold is a C^∞ manifold M^{2n} together with a closed 2-form Ω of rank $2n$ (i.e. $\Omega(X, Y)$ is a non-singular (alternating) bilinear form on each M_p)

Lemma. Given $p \in M$ $\exists$ a coordinate system $x_1, \dots x_{2n}$ near p s.t. $\Omega = \sum_{i=1}^{n} dx_i \wedge dx_{n+i}$

Proof. $\Omega = d\omega$ locally and by classical result on one-forms $\omega = \sum_{i=1}^{n} x_i \wedge dx_{n+i}$ locally.

Such coordinates are called canonical – this shows that all symplectic manifolds are locally equivalent.

If L is a vector field on M, Θ a p-form on M we write $i_L\Theta$ for contraction of Θ with L, i.e. $i_L\Theta$ is the ~~p-f~~ ~~...~~ the $p-1$-form given by $i_L\Theta(X_1, \dots X_{p-1}) = \Theta(L, X_1, \dots X_{p-1})$

We write $\partial_L\theta$ for the Lie derivative of θ w.r.t. L

We recall $\partial_L\theta = i_L d\theta + d i_L\theta$ i.e. $\boxed{\partial_L = i_L d + d i_L}$

If L is a vector field we write ~~$L^* = i_L\Omega$~~ $L^* = i_L\Omega$ a one form on M. The fact that Ω is non-singular $\Rightarrow$ $L \to L^*$ is a vector space isomorphism between vector fields and one-forms. We write $\theta \to \theta^*$ for the inverse isomorphism.

Remark. If $x_1, \dots x_{2n}$ are canonical coordinates and $L = \sum X_i \frac{\partial}{\partial x_i}$ then $L^* = \sum_{i=1}^{n} X_{i+n}\, dx_i - \sum_{i=1}^{n} X_i\, dx_{i+n}$

Example: M a Riemannian manifold P = bundle of covariant vectors over M, $\pi : P \to M$ usual projection. Define a one form ω on P by $\omega_\theta = \delta\pi^*(\theta)$, put $\Omega = d\omega$.

If $x_1, \dots x_n$ are coordinates in $O \subseteq M$ get coordinates $\bar{x}_1, \dots \bar{x}_n, p_1, \dots p_n$ in $\pi^{-1}(O)$ by

$$\begin{cases} \bar{x}_i(\theta) = x_i(\pi(\theta)) \\ \theta = \sum p_i(\theta)\, dx_i \end{cases}$$

i.e. $\bar{x}_i$ gives coordinates of base point relative to $x_1, \dots x_n$ and p_i gives i^{th} component of θ relative to $x_1, \dots x_n$

It is trivial to check that $\omega = \sum p_i\, d\bar{x}_i$, hence $\Omega = d\omega = \sum dp_i \wedge d\bar{x}_i$ which shows that Ω has rank $2n$

$\therefore$ P is a symplectic manifold and $p_1 \dots p_n, \bar{x}_1, \dots \bar{x}_n$ are canonical coordinates. This symplectic manifold is called the <u>phase space</u> of M.

<u>Definition</u>: A vector field L is <u>symplectic</u> if $\cancel{\partial_L \Omega = 0}$ $\partial_L \Omega = 0$ (if M is compact, or more generally if L generates a global one parameter group φ_t this is equivalent to the φ_t being symplectic transformations, i.e. preserving Ω.

<u>Theorem</u>. A vector field L is symplectic $\iff$ L^* is closed

Proof $L^* = i_L \Omega$ $\therefore$ $dL^* = d(i_L \Omega)$. Recall that $\partial_L \Omega = d\, i_L \Omega + i_L\, d\Omega$ $\cancel{\partial_L \Omega = d \dots}$ but $d\Omega = 0$

So $\partial_L \Omega =$

Lemma. $\partial_L \Omega = dL^*$

Proof. Recall that $\partial_L = \iota_L d + d\iota_L$. Since $d\Omega = 0$

$$\partial_L \Omega = \cancel{\iota_L d\Omega} + d\iota_L\Omega = dL^*$$

Theorem. L is symplectic if and only if L^* is closed, in other words the isomorphism $L \longrightarrow L^*$ between vector fields and one forms restricts to an isomorphism of symplectic vector fields and closed one-forms.

Remark. Let L be a symplectic vector field, $p_1, \ldots p_n, x_1, \ldots x_n$ a canonical coordinate system (with disc domain), then

$L^* = dH = \sum \frac{\partial H}{\partial p_i} dp_i + \frac{\partial H}{\partial x_i} dx_i$ (H is called a local hamiltonian for L and is determined up to an additive constant)

$$\therefore L = -\sum_{i=1}^{n} \frac{\partial H}{\partial x_i}\frac{\partial}{\partial p_i} + \sum_{i=1}^{n} \frac{\partial H}{\partial p_i}\frac{\partial}{\partial x_i}$$

$\therefore$ differential equation for integral curves of L are

$$\begin{cases} \dfrac{dx_i}{dt} = \dfrac{\partial H}{\partial p_i} & \text{(Hamiltonian system)} \\[2ex] \dfrac{dp_i}{dt} = -\dfrac{\partial H}{\partial x_i} \end{cases}$$

Conversely of course if d.e. corresponding to L take this form then locally $L^* = dH$ so L^* is closed and $\therefore L$ is symplectic

i.e.,

Theorem. A vector field is symplectic $\iff$ it corresponds to a Hamiltonian system of d.e. in each canonical coordinate system.

Remark. Note that a symplectic manifold has natural forms of degree $2k$ $k = 1, 2, \ldots n$, namely $\Omega \wedge \Omega \ldots \Omega$ k-times in particular it has a natural volume element Ω^n. Note that in canonical coordinates $\Omega^n = dx_1 \wedge dx_2 \ldots \wedge dx_{2n}$

Since $\partial_L(\theta \wedge \psi) = \partial_L \theta \wedge \psi + \theta \wedge \partial_L \psi$ it follows that if L is symplectic then $\partial_L \Omega^n = 0$, i.e. a symplectic vector field generates a volume preserving one parameter group, Thus

Liouville's Theorem. If a system of differential equations in phase space P are in Hamiltonian form in canonical coordinates they determine a volume preserving one parameter group of transformations of P.

Remark. In general if Φ is a form the set of vector fields L s.t. $\partial_L \Phi = 0$ is a Lie algebra, ∴ in particular the bracket $[L_1, L_2]$ of two symplectic vector fields is symplectic and hence $[L_1, L_2]^*$ is closed (Note this gives a Lie algebra structure to closed forms on a symplectic manifold). For some mysterious reason $[L_1, L_2]^*$ turns out to be exact

i.e. if S represents symplectic vector fields then the derived algebra $[S, S]$ gets mapped into exact forms by natural isomorphism. If it is onto then $S/[S, S]$ (which corresponds to abelianized group of symplectic transformations) is $\simeq H^1(M, R)$

In fact $[L_1, L_2]^* = d(\Omega(L_1, L_2))$