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A GLOBAL FORMULATION OF THE LIE THEORY OF TRANSPORTATION GROUPS

BY

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A GLOBAL FORMULATION OF THE LIE THEORY

OF TRANSFORMATION GROUPS

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Preface

The goal of this memoir is to formulate in a modern global way the theory, due in its local form to Sophus Lie, which connects Lie algebras of vector fields on a differentiable manifold with local groups and groups of transformations acting on the manifold.

Chapter I is preliminary to the main trend of the memoir and is concerned with the question of giving a natural 'quotient' differentiable structure to the set of leaves of an involutive differential system. I have decided to develop this separately, rather than in context with its application to transformation groups, since I feel that it may be of some independent interest.

In chapter II we develop the theory of infinitesimal and local transformation groups in its greatest generality. Aside from proving the basic tool theorems that will be needed in the following mor specialized chapters, we give a uniqueness theorem for a local transformation group with a given domain and given infinitesimal generator and also a global form of Lie's Second Fundamental Theorem (Hauptsatz der Gruppentheorie).

In chapter III we characterize in a number of ways the class of infinitesimal transformation groups which generate global transformation groups. In chapter IV we use the results of chapter III to develop a Lie theory connecting the Lie algebra of differentiable vector fields on a

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manifold with the group of differentiable homeomorphisms of the manifold and use this to study the automorphisms of a structure given by a manifold and a set of tensor fields on the manifold.

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LIE THEORY OF TRANSFORMATION GROUPS

Chapter I

QUOTIENT MANIFOLDS DEFINED BY FOLIATIONS

A completely integrable differential system Θ on a differentiable manifold M defines a partitioning (foliation) of M into maximal connected integral manifolds (leaves) of Θ. In this chapter we investigate under what conditions the quotient space admits a natural manifold structure, and the elementary properties of the quotient manifolds that result.

1. Differentiable Manifolds.

We will use the word 'differentiable' as a substitute for 'C^∞' or 'analytic' in contexts where both of the latter would be appropriate, in order to avoid having to give separate proofs for the C^∞ case and the analytic case of various theorems.

In order to get a smooth theory of quotient manifolds it is expedient to drop the Hausdorff separation axiom in the definition of a manifold. When this is done it is possible to modify the definition of a manifold in terms of overlapping coordinate systems in such a way that the topology of the manifold is a derived concept. Since there are several novel points in this approach we will explain it briefly and at the same time develop the notation we will need.

The reader familiar with the work of Ehresmann will recognize the debt the author owes to this pioneer in manifold theory, both in concepts and in terminology. It is a debt which we gratefully acknowledge.

We denote real Euclidian n-space by R^n and by $u_1 \ldots u_n$ we denote the natural coordinates on R^n. If M is a set, an <u>n-dimensional</u> <u>chart</u> in M is a one-to-one map φ of a subset of M onto an open subset of R^n. A real-valued function f with domain $S \subseteq M$ is said to be

<u>differentiable</u> <u>with</u> <u>respect</u> <u>to</u> φ at a point $p \varepsilon S$ if $p \varepsilon (\text{domain } \varphi)$ and there is a differentiable function g defined in a neighborhood N of $\varphi(p)$ such that $f \circ \varphi^{-1} \upharpoonright N = g \upharpoonright \varphi(S)$, i.e. $f \circ \varphi^{-1}$ and g agree where both are defined. Two n-dimensional charts in M, φ and ψ, with domains U and V respectively, are said to be <u>differentiably</u> <u>related</u> if each maps $U \cap V$ onto an open set and the mappings $\varphi \circ \psi^{-1}$ and $\psi \circ \varphi^{-1}$ are differentiable. If this is the case and f is a real-valued function in M and $p \varepsilon U \cap V$ then the differentiability of f at p with respect to φ and with respect to ψ are equivalent.

An <u>n-dimensional</u> <u>differentiable</u> <u>atlas</u> <u>for</u> M is a set of mutually differentiably related n-dimensional charts in M whose domains cover M. An n-dimensional differentiable atlas for M is called <u>complete</u> if it is not a proper subset of an n-dimensional differentiable atlas for M. An <u>n-dimensional</u> <u>differentiable</u> <u>manifold</u> is a pair (M,Ψ) where M is a set (called the point set of the manifold) and Ψ is a complete n-dimensional differentiable atlas for M (called the atlas of the manifold). If Φ is any n-dimensional differentiable atlas for a set M, then the set Ψ of ψ such that $\Phi \cup \{\psi\}$ is an n-dimensional differentiable atlas for M is the unique complete n-dimensional differentiable atlas including Φ. It is called the <u>complete</u> <u>differentiable</u> <u>atlas</u> <u>associated</u> <u>with</u> Φ and (M,Ψ) is called the <u>differentiable</u> <u>manifold</u> <u>defined</u> <u>by</u> Φ.

If (M,Φ) is an analytic manifold then Φ is a C^∞ atlas for M. If Ψ is the complete C^∞ atlas associated with Φ then (M,Ψ) is called the C^∞ <u>manifold</u> <u>associated</u> <u>with</u> (M,Φ).

If (M,Ψ) is an n-dimensional differentiable manifold then the domains of the charts in Ψ form a base for a topology $\mathfrak{J}$, called the <u>manifold</u> <u>topology</u> of (M,Ψ), and $(M,\mathfrak{J})$ is called the underlying topological space of (M,Ψ). $\mathfrak{J}$ is the weakest topology for M rendering each $\psi\varepsilon\Psi$ continuous, and with respect to $\mathfrak{J}$ each $\psi\varepsilon\Psi$ is a homeomorphism. It follows that $\mathfrak{J}$ is a T_1 topology for M; it need not be a T_2 topology but if it is we call (M,Ψ) a Hausdorff differentiable manifold. Similarly all adjectives conventionally applied to $(M,\mathfrak{J})$ will be applied to (M,Ψ), e.g. (M,Ψ) will be called a compact or a connected differentiable manifold if $\mathfrak{J}$ is a compact or connected topology for M. A real-valued function in M with domain S is called differentiable at $p\varepsilon M$ if for some $\psi\varepsilon\Psi$ (and then automatically for all $\psi'\varepsilon\Psi$ with $p\varepsilon(\text{domain } \psi')$) f is differentiable at p with respect to ψ. We call f differentiable on $S' \subseteq M$ if it is differentiable at each point of S', and differentiable in M if it is differentiable on S. In the latter case f is continuous.

A coordinate system for the n-dimensional differentiable manifold (M,Ψ) is an ordered n+1-tuple $(x_1 \ldots x_n, \mathcal{O})$ such that $\mathcal{O}$ is the domain of a chart $\psi\varepsilon\Psi$ and the x_i are real-valued functions in M such that $x_i \restriction \mathcal{O} = u_i \circ \psi$. If f is a real-valued function in M then $f\circ\psi^{-1}$ is called the expression for f in terms of the coordinate system $(x_1 \ldots x_n, \mathcal{O})$. We shall say that $(x_1 \ldots x_n, \mathcal{O})$ is a cubical coordinate system of breadth $2a$ centered at $p\varepsilon M$ if $\psi(p) = (0 \ldots 0)$ and $\psi(\mathcal{O}) = \{(t_1 \ldots t_n)\,\varepsilon R^n : |t_i| < a\}$. In this case if $|t_{m+i}| < a$ $i = 1 \ldots n\text{-}m$ then we call $\Sigma_t = \{q\varepsilon\mathcal{O} : x_{m+i}(q) = t_{m+i}\}$ the m-dimensional slice of $(x_1 \ldots x_n, \mathcal{O})$ defined by $t = (t_{m+1} \ldots t_n)$. The mapping $\varphi : p \rightarrow (x_1(p) \ldots x_m(p))$ is an m-dimensional chart in Σ_t and $\{\varphi\}$ is an m-dimensional differentiable atlas for Σ_t. We shall often refer to Σ_t as an m-dimensional differentiable manifold, meaning the manifold defined by $\{\varphi\}$.

Let (M,Ψ) be a differentiable manifold and $p\varepsilon M$. For the moment denote by $\mathcal{A}(p)$ the class of real-valued functions with domain open in M and differentiable at p. Then the notion of tangent vector at p can be defined exactly as in [1 Chapter III]. Formula (1) of [1 page 77] is proved for the C^∞ case in [2]. Except for this all properties of the tangent space etc. can be proved exactly as in [1]. The following additional elementary concepts are treated in [1] and the reader will be assumed to be familiar with them: (differentiable) vector field, bracket of two differentiable vector fields, differential of a differentiable function f (denoted by df), differentiable mapping F of one manifold into another and the differential of such a mapping (denoted by δF). F is called non-singular at p if δF maps the tangent space at p one-to-one.

Let (M,Ψ) and (N,Φ) be differentiable manifolds with $N \subseteq M$ and let i be the inclusion map of N in M. We say that (N,Φ) is a <u>differentiable submanifold</u> of (M,Ψ) if i is differentiable and everywhere non-singular. If moreover i is a homeomorphism into with respect to the respective manifold topologies then (N,Φ) is said to be <u>regularly imbedded in</u> (M,Ψ); and if further N is a closed subspace of M with respect to the manifold topology of (M,Ψ) then (N,Φ) is called a <u>closed submanifold of</u> (M,Ψ). We identify the tangent space of the submanifold (N,Φ) at a point $p \varepsilon N$ with its image under δi (a subspace of the tangent space to (M,Ψ) at p) via the linear isomorphism given by δi.

If (M,Ψ) is a differentiable manifold, $\mathcal{O}$ a subset of M open with respect to the manifold topology and if $\Psi_{\mathcal{O}} = \{\psi \varepsilon \Psi : \text{domain } \psi \subseteq \mathcal{O}\}$ then $(\mathcal{O}, \Psi_{\mathcal{O}})$ is a regularly imbedded differentiable submanifold of (M,Ψ) called the <u>open submanifold defined by</u> $\mathcal{O}$.

Let (M,Ψ) and (N,Φ) be manifolds. Following Ambrose we call a one-to-one map F of M onto N a <u>diffeomorphism</u> of (M,Ψ) onto (N,Φ) if F and F^{-1} are differentiable or, equivalently, if $\varphi \to \varphi \circ F$ is a one-to-one correspondence of Φ with Ψ. A mapping F defined near $p\epsilon M$ and into N will be called a local diffeomorphism of (M,Ψ) into (N,Φ) at p if it maps an open submanifold of (M,Ψ) containing p diffeomorphically onto an open submanifold of (N,Φ). By the implicit function theorem a necessary and sufficient condition for this is that F be differentiable at p and δF map the tangent space to (M,Ψ) at p isomorphically onto the tangent space to (N,Φ) at $F(p)$. If $F : M \to N$ is a local diffeomorphism of (M,Ψ) into (N,Φ) at each point of M we call F a <u>local diffeomorphism of</u> (M,Ψ) <u>into</u> (N,Φ).

Whenever no confusion will result (i.e. when a single complete atlas Ψ is being considered) we will use the symbol M alone to denote a manifold (M,Ψ), its underlying point set and underlying topological space.

2. Foliations.

Let M be an n-dimensional differentiable manifold. We use the term <u>m-dimensional differential system on</u> M for what Chevalley [1 page 86] calls an m-dimensional distribution on M, i.e. a mapping Θ which assigns to each $p\epsilon M$ an m-dimensional subspace Θ_p of the tangent space to M at p. A vector field L in M will be said to <u>belong to</u> Θ if for each p in the domain of L, $L_p \epsilon \Theta_p$. The differential system Θ will be called <u>differentiable</u> if for each $p\epsilon M$ there is a neighborhood $\mathcal{O}$ of p and m differentiable vector fields $L_1 \dots L_m$ defined in $\mathcal{O}$ such that $(L_1)_q \dots (L_m)_q$ is a base for Θ_q at each $q \epsilon \mathcal{O}$. Θ is called <u>involutive</u> if it is differentiable and if whenever X and Y are two differentiable vector fields in M with the same domain, both belonging to Θ, their bracket $[X,Y]$ also belongs to Θ. A submanifold N of M will be called an <u>integral manifold</u> of the differential system Θ on M if for each point $p \varepsilon N$ the tangent space to N at p is included in Θ_p .

If Θ is an m-dimensional differential system on M , a coordinate system $(x_1 \dots x_n , \mathcal{O})$ will be called <u>flat with respect to</u> Θ if for each $q \varepsilon \mathcal{O}$ $(X_1)_q \dots (X_m)_q$ is a base for Θ_q , where $X_i = \partial / \partial x_i$. If $(x_1 \dots x_n , \mathcal{O})$ is a cubical coordinate system for M then a necessary and sufficient condition that it be flat with respect to Θ is that each of its m-dimensional slices be an integral manifold of Θ .

THEOREM I. *If Θ is an m-dimensional differential system on M then a necessary and sufficient condition that Θ be involutive is that for each $p \varepsilon M$ there is a cubical coordinate system centered at p and flat with respect to Θ .*

PROOF. Since the property of being involutive is local it suffices to prove the theorem in the case that M is Hausdorff. The proof is given in [1 page 89] for the analytic case and as the proof depends only on the implicit function theorem and the existence and uniqueness theorems for differential equations (which have exact C^∞ analogues), the same proof works in the C^∞ case.

COROLLARY. *Let Θ be an m-dimensional involutive differential system in the n-dimensional differentiable manifold M . If $p \varepsilon M$ then the set of domains of cubical coordinate systems centered at p and flat with respect to Θ form a basis of neighborhoods of p with respect to the manifold topology for M .*

PROOF. Let $(x_1 \dots x_n, \mathcal{O})$ be a cubical coordinate system centered at p of breadth 2a. Then for any $b < a$ if $\mathcal{O}_b = \{q \varepsilon \mathcal{O} : |x_i(q)| < b\}$

then $(x_1 \ldots x_n, \mathcal{O}_b)$ is a cubical coordinate system centered at p and flat with respect to Θ, and the $\mathcal{O}_b$ are a basis of neighborhoods for p.

THEOREM II. *Let Θ be an m-dimensional involutive differential system on an n-dimensional differentiable manifold M. Let $(x_1 \ldots x_n, U)$ and $(y_1 \ldots y_n, V)$ be cubical coordinate systems in M flat with respect to Θ and let $p \varepsilon U \cap V$. Then there is a diffeomorphism*

$$f : (t_{m+1} \ldots t_n) \to (f_{m+1}(t_{m+1} \ldots t_n) \ldots f_n(t_{m+1} \ldots t_n))$$

of a neighborhood of $(y_{m+1}(p) \ldots y_n(p))$ in R^{n-m} onto a neighborhood of $(x_{m+1}(p) \ldots x_n(p))$ in R^{n-m} such that $x_{m+i}(q) = f_{m+i}(y_{m+1}(q) \ldots y_n(q))$ for all $q \varepsilon \mathcal{O}$ = component of p in $U \cap V$. Moreover if Σ is the m-dimensional slice of $(x_1 \ldots x_n, U)$ defined by $(x_{m+1}(p) \ldots x_n(p))$ and Σ' is the m-dimensional slice of $(y_1 \ldots y_n, V)$ defined by $(y_{m+1}(p) \ldots y_n(p))$ then

$$\varphi : \Sigma \to R^m \quad q \to (x_1(q) \ldots x_m(q)) \quad \text{and}$$

$$\psi : \Sigma' \to R^m \quad q \to (y_1(q) \ldots y_m(q))$$

are differentiably related m-dimensional charts in M.

PROOF. Let g_i be the expression for x_i in terms of the coordinate system $(y_1 \ldots y_n, V)$. Then

$$(dx_{m+i})_q = \sum_{j=1}^{n} (\partial g_{m+i} / \partial u_j)(y_1(q) \ldots y_n(q))(dy_j)_q \quad \text{for} \quad q \varepsilon U \cap V .$$

Since $(x_1 \ldots x_n, U)$ and $(y_1 \ldots y_n, V)$ are both flat with respect to Θ, $((dx_{m+1})_q \ldots (dx_n)_q)$ and $((dy_{m+1})_q \ldots (dy_n)_q)$ are both bases for the annihilator of Θ_q for $q \varepsilon U \cap V$ and hence $(\partial g_{m+i} / \partial u_j)(y_1(q) \ldots y_n(q)) = 0$ for $j \leq m$. If $\widetilde{\mathcal{O}}$ is the image of $\mathcal{O}$ under the map $q \to (y_1(q) \ldots y_n(q))$ it follows that the g_{m+i}

are independent of their first m arguments, that is if $\hat{\mathcal{O}}$ is the image of $\tilde{\mathcal{O}}$ under the map $\Pi : (t_1 \dots t_n) \to (t_{m+1} \dots t_n)$ then there are differentiable functions $f_{m+1} \dots f_n$ on $\hat{\mathcal{O}}$ such that $g_{m+i} = f_{m+i} \circ \Pi$. Then $x_{m+i}(q) = f_{m+i}(y_{m\,1}(q) \dots y_n(q))$ for $q \in \mathcal{O}$ hence $(dx_{m+i})_p = \sum_{j=1}^{n-m} (\partial f_{m+i}/\partial u_j)(y_{m+1}(p) \dots y_n(p))(dy_{m+j})_p$ and since the $(dx_{m+i})_p$ are linearly independent it follows that $\det (\partial f_{m+i}/\partial u_j)(y_{m+1}(p) \dots y_n(p)) \neq 0$. By the implicit function theorem the mapping $f : (t_{m+1} \dots t_n) \to (f_{m+1}(t_{m+1} \dots t_n) \dots f_n (t_{m+1} \dots t_n))$ is a local diffeomorphism at $(y_{m+1}(p) \dots y_n(p))$.

In $\Sigma' \cap \mathcal{O}$ we have
$x_{m+i}(q) = f_{m+i}(y_{m+i}(q) \dots y_n(q)) = f_{m+i}(y_{m+1}(p) \dots y_n(p)) = x_{m+i}(p)$ so $\Sigma' \cap \mathcal{O} \subseteq \Sigma$. Now ψ is an open mapping and $\Sigma' \cap \mathcal{O}$ is open in Σ' hence $\psi(\Sigma) \supseteq \psi(\Sigma' \cap \mathcal{O})$ is a neighborhood of $\psi(p)$. It follows that $\psi(\Sigma)$ is an open subset of R^m. Defining $\bar{g}_i(t_1 \dots t_m) = g_i(t_1 \dots t_m, y_{m+1}(p) \dots y_n(p))$ on $\psi(\Sigma' \cap \mathcal{O})$ we have for $q \in \Sigma' \cap \mathcal{O}$ $u_i \circ \varphi(q) = x_i(q) = \bar{g}_i(y_1(q) \dots y_m(q)) = \bar{g}_i(\psi(q))$ or $u_i \circ \varphi \circ \psi^{-1} = \bar{g}_i$. Since the $\bar{g}_i$ are clearly differentiable this shows that $\varphi \circ \psi^{-1}$ is a differentiable map. Similarly $\varphi(\Sigma')$ is open and $\psi \circ \varphi^{-1}$ is a differentiable map so φ and ψ are differentiably related.

DEFINITION I. Let Θ be an m-dimensional involutive differential system on an n-dimensional differentiable manifold (M, Ψ) and let $(x_1 \dots x_n, \mathcal{O})$ be a cubical coordinate system for M flat with respect to Θ. If Σ is any m-dimensional slice of $(x_1 \dots x_n, \mathcal{O})$ the mapping $q \to (x_1(q) \dots x_m(q))$ of Σ into R^m is called a <u>leaf chart for</u> M <u>with respect to</u> Θ. By theorems I and II the set of all leaf charts for M with respect to Θ form an m-dimensional differentiable atlas for M. Let (M,Φ) be the m-dimensional differentiable manifold it defines (i.e. Φ is the complete atlas containing all leaf charts). Then (M,Φ) is called the maximum integral manifold of Θ. A connected component of M with respect to the manifold topology of (M,Φ) regarded as an open submanifold of (M,Φ) is called a leaf of Θ. We call the set of leaves of Θ the foliation defined by Θ and denote it by M/Θ. We denote by Π_Θ the quotient mapping of M onto M/Θ which carries $p \in M$ onto the leaf of Θ containing p. A subset of M is called saturated (with respect to Θ) if it is the union of leaves of Θ, and if $S \subseteq M$ the saturation of S is $\Pi_\Theta^{-1}(\Pi_\Theta(S))$. The quotient topology for M/Θ is the strongest topology which makes Π_Θ continuous; equivalently its open sets are the images of saturated open sets of M under Π_Θ.

We note that it is almost immediate from the definition of (M,Φ) that (M,Φ) is an m-dimensional integral manifold of Θ and that any integral manifold of Θ is a submanifold of (M,Φ) so the name maximum integral manifold of Θ is justified. It follows that a connected m-dimensional integral manifold of Θ (and in particular an m-dimensional slice of a cubical coordinate system for (M,Ψ) flat with respect to Θ) is an open submanifold of a leaf of Θ. The fact that (M,Φ) is a submanifold of (M,Ψ) implies in particular that the manifold topology for (M,Φ) is stronger than the manifold topology for (M,Ψ), hence if (M,Ψ) is a Hausdorff manifold so is (M,Φ), however, even in this case M/Θ need not be a Hausdorff space in the quotient topology as we shall see later by example.

The task we set ourselves is, picturesquely, to 'factor' our n-dimensional manifold (M,Ψ) into an m-dimensional manifold (M,Φ) 'parallel' to Θ and an n-m dimensional quotient manifold M/Θ 'transverse' to Θ. The first part of this task, which is classical, or at least well-known [1 Chapt. III § VIII] and [3], has been accomplished above. The second part, namely putting a natural n-m dimensional differentiable manifold structure on M/Θ cannot always be accomplished and we investigate below the condition under which it can.

3. The Continuation Theorem.

THEOREM III. *Let Θ be an involutive m-dimensional differential system on an n-dimensional differentiable manifold M and let $(x_1 \ldots x_n, \mathcal{O})$ be a cubical coordinate system centered at p and flat with respect to Θ. Let q be a point of the leaf $\Sigma = \Pi_\Theta(p)$ of Θ containing p and $(y_1 \ldots y_n, U)$ a cubical coordinate system flat with respect to Θ such that q is on the m-dimensional slice defined by $(0 \ldots 0)$. Then there is an $\varepsilon > 0$ and a diffeomorphism $f : (t_{m+1} \ldots t_n) \to (f_{m+1}(t_{m+1} \ldots t_n) \ldots f_n(t_{m+1} \ldots t_n))$ of $T_\varepsilon = \{(t_{m+1} \ldots t_n) \varepsilon R^{n-m} : |t_{m+i}| < \varepsilon\}$ into R^{n-m} such that for all $t \varepsilon T_\varepsilon$ the m-dimensional slice of $(y_1 \ldots y_n, U)$ defined by t and the m-dimensional slice of $(x_1 \ldots x_n, \mathcal{O})$ defined by $f(t)$ are parts of the same leaf of Θ.*

PROOF. Let Σ' be the set of $q \varepsilon \Sigma$ for which the conclusion of the theorem holds. It follows from Theorem II that $p \varepsilon \Sigma'$ so Σ' is not empty. Since Σ is connected it will suffice to show that if $\bar{q}$ is adherent to Σ' in Σ then $\bar{q}$ is interior to Σ' with respect to Σ.

Let $(z_1 \ldots z_n, V)$ be a cubical coordinate system centered at $\bar{q}$ and flat with respect to Θ and let W be the m-dimensional slice of $(z_1 \ldots z_n, V)$ defined by $(0 \ldots 0)$. Then W is a neighborhood of $\bar{q}$ in Σ so we can find $q' \varepsilon W \cap \Sigma'$. By definition of Σ' we can find a $\delta < 0$ and functions $g_{m+1} \ldots g_n$ defined on T_δ such that $t \to g(t) = (g_{m+1}(t) \ldots g_n(t))$ is a diffeomorphism and for $t \varepsilon T_\delta$ the m-dimensional slice of $(z_1 \ldots z_n, V)$ defined by t and the m-dimensional slice of $(x_1 \ldots x_n, \mathcal{O})$ defined by $g(t)$ are parts of the same leaf of Θ. Let $q \varepsilon W$ and let $(y_1 \ldots y_n, U)$ be a cubical coordinate system flat with respect to Θ containing q in its m-dimensional slice defined by $(0 \ldots 0)$. By theorem II there are functions $h_{m+1} \ldots h_n$ defined in a neighborhood of the origin in R^{n-m} such that $z_{m+i}(r) = h_{m+i}(y_{m+1}(r) \ldots y_n(r))$ for r in an M neighborhood of q and moreover if ε is chosen sufficiently small $t \to h(t) = (h_{m+1}(t) \ldots h_n(t))$ is a diffeomorphism of T_ε into T_δ. Define f on T_ε by $f = g \circ h$. Then f being the composition of two diffeomorphisms is a diffeomorphism. Moreover if $t \varepsilon T$ then the m-dimensional slice of $(x_1 \ldots x_n, \mathcal{O})$ defined by $f(t) = g(h(t))$ is part of the same leaf of Θ as the m-dimensional slice of $(z_1 \ldots z_n, V)$ defined by $h(t)$ which in turn is a part of the same leaf of Θ as the m-dimensional slice of $(y_1 \ldots y_n, U)$ defined by t. This verifies that $q \varepsilon \Sigma'$ and hence that $W \subseteq \Sigma'$. Since W is a neighborhood of $\bar{q}$ in Σ, $\bar{q}$ is interior to Σ' with respect to Σ as was to be shown.

DEFINITION II. Let $(x_1 \ldots x_n, \mathcal{O})$ be a cubical coordinate system of breadth 2a in a differentiable manifold M which is flat with respect to an m-dimensional involutive differential system Θ. A coordinate system $(y_1 \ldots y_n, U)$ in M is said to be <u>subordinate to</u> $(x_1 \ldots x_n, \mathcal{O})$ <u>with respect to</u> Θ

if it is flat with respect to Θ, cubical of breadth $2b < 2a$, and if $|t_{m+i}| < b$ $i = 1 \ldots n-m$ implies that the m-dimensional slices of $(x_1 \ldots x_n, \mathcal{O})$ and $(y_1 \ldots y_n, U)$ defined by $(t_{m+1} \ldots t_n)$ are parts of the same leaf of Θ.

COROLLARY 1. *Let* $(x_1 \ldots x_n, \mathcal{O})$ *be a cubical coordinate system centered at* p *on the differentiable manifold* M *which is flat with respect to an involutive differential system* Θ. *If* $q \varepsilon \Pi_\Theta(p)$ *then there is a coordinate system centered at* q *and subordinate to* $(x_1 \ldots x_n, \mathcal{O})$ *with respect to* Θ.

PROOF. Let $(y_1 \ldots y_n, U)$ be any cubical coordinate system centered at q and flat with respect to Θ. Then, letting $f_{m+1} \ldots f_n$ be the functions given by the theorem, define functions $z_1 \ldots z_n$ near q by $z_i = y_i$ $i = 1 \ldots m$ $z_{m+i} = f_{m+i}(y_{m+1} \ldots y_n)$ $i = 1 \ldots n-m$. Then if W is a suitably chosen neighborhood of q $(z_1 \ldots z_n, W)$ is centered at q and is subordinate to $(x_1 \ldots x_n, \mathcal{O})$.

COROLLARY 2. *If* Θ *is an involutive differential system on a differentiable manifold* M *then* Π_Θ *is an open mapping of* M *onto* M/Θ *with respect to the quotient topology for* M/Θ. *Equivalently the saturation of an open set of* M *with respect to* Θ *is open.*

PROOF. The equivalence of the two statements is clear. Let $\mathcal{O}$ be an open set of M and let q be in the saturation $\tilde{\mathcal{O}}$ of $\mathcal{O}$. Let p be a point of $\mathcal{O}$ belonging to the same leaf of Θ as q. Let $(x_1 \ldots x_n, U)$ be a cubical coordinate system centered at p and flat with respect to Θ with $U \subseteq \mathcal{O}$ (see corollary of theorem I). By corollary I we can find a coordinate system $(y_1 \ldots y_n, V)$ centered at q and subordinate to $(x_1 \ldots x_n, U)$ with respect to Θ. If $q' \varepsilon V$ then

q' belongs to the same leaf of Θ as does the m-dimensional slice of $(x_1 \dots x_n, U)$ defined by $(y_{m+1}(q') \dots y_n(q'))$, so in particular q' is in the saturation of U and hence of $\mathcal{O}$. Thus $V \subseteq \tilde{\mathcal{O}}$ so, as V is a neighborhood of q, q is interior to $\tilde{\mathcal{O}}$. Hence $\tilde{\mathcal{O}}$ is open.

Now in general if Π is a mapping of a topological space X onto a set Y there is clearly at most one topology for Y such that Π is both continuous and open. Hence:

COROLLARY 3. *If Θ is an involutive differential system on a differentiable manifold M then the quotient topology for M/Θ is uniquely characterized by the conditions that with respect to it Π_Θ is continuous and open.*

4. Regularity.

DEFINITION III. Let Θ be an involutive m-dimensional differential system on an n-dimensional differentiable manifold M. A coordinate system $(x_1 \dots x_n, \mathcal{O})$ in M will be called *regular with respect to* Θ if it is cubical, flat with respect to Θ, and if each leaf of Θ intersects $\mathcal{O}$ in at most one m-dimensional slice of $(x_1 \dots x_n, \mathcal{O})$. A leaf of Θ will be called a *regular leaf of* Θ if it intersects the domain of a coordinate system regular with respect to Θ. We call Θ *regular* if every leaf of Θ is a regular leaf of Θ.

THEOREM IV. *If Θ is an involutive differential system on a differentiable manifold M and $(x_1 \dots x_n, \mathcal{O})$ is a coordinate system in M regular with respect to Θ then any coordinate system in M subordinate* to $(x_1 \dots x_n, \mathcal{O})$ *with respect to Θ is also regular with respect to Θ.*

PROOF. Obvious.

THEOREM V. *Let* Θ *be an involutive* m*-dimensional differential system on an* n*-dimensional differentiable manifold* M *. A necessary and sufficient condition that a leaf* Σ *of* Θ *be a regular leaf of* Θ *is that for each* $q \varepsilon \Sigma$ *there is a cubical coordinate system centered at* q *which is regular with respect to* Θ *. A necessary and sufficient condition that* Θ *be regular is that for each* $q \varepsilon M$ *there is a cubical coordinate system centered at* q *and regular with respect to* Θ *.*

PROOF. Let Σ be a leaf of Θ . If there is so much as one point $q \varepsilon \Sigma$ for which there is a coordinate system regular with respect to Θ centered at q then it is immediate from the definition that Σ is regular. Conversely suppose Σ is regular. Then there is a coordinate system $(x_1 \ldots x_n, \mathcal{O})$ regular with respect to Θ such that Σ intersects $\mathcal{O}$ in some point, say p . Define $y_i = x_i - x_i(p)$ and let U be neighborhood of p which is a cube with center at p with respect to t coordinates $(x_1 \ldots x_n, \mathcal{O})$. Then $(y_1 \ldots y_n, U)$ is clearly a coordin system regular with respect to Θ centered at p . If q is any point of Σ then by corollary 1 of theorem III there is a coordinate system $(z_1 \ldots z_n, V)$ centered at q and subordinate to $(y_1 \ldots y_n, U)$ with respect to Θ . By theorem IV $(z_1 \ldots z_n, V)$ is a regular coordinate system with respect to Θ . This proves the first statement of the theorem, and the second is an immediate consequence of the first.

COROLLARY 1. *If* Θ *is an involutive differential system on a differentiable manifold* M *then* M' *, the union of the regular leaves of* Θ *, is the union of the domains of coordinate* *systems regular with respect to Θ and hence is a saturated open set of M. If Θ' is the restriction of Θ to the open submanifold M' of M then Θ' is regular and M'/Θ' being the image of M' under Π_Θ is an open set of M/Θ with respect to the quotient topology.*

PROOF. Immediate from the theorem.

COROLLARY 2. *If Θ is a regular differential system on a differentiable manifold M and $p \varepsilon M$ then the domains of coordinate systems regular with respect to Θ and centered at p for a basis of neighborhoods at p with respect to the manifold topology of M.*

PROOF. Similar to the proof of the corollary of theorem I.

We have the following stability theorem for compact regular leaves.

THEOREM VI. *Let Θ be a m-dimensional involutive differential system on the differentiable Hausdorff manifold M. If Σ is a compact, regular leaf of Θ and W is any neighborhood of Σ there is a saturated neighborhood $\widetilde{W}$ of Σ included in W which is a union of regular, compact leaves of Θ.*

PROOF. Let $p \varepsilon \Sigma$ and let $(x_1 \ldots x_n, \mathcal{O})$ be a cubical coordinate system centered at p and regular with respect to Θ. By corollary 1 of theorem III for each $q \varepsilon \Sigma$ we can find a cubical coordinate system $(x_1^q \ldots x_n^q, U^q)$, say of breadth $4a_q$, centered at q and subordinate to $(x_1 \ldots x_n, \mathcal{O})$. By choosing U^q sufficiently small we can suppose that it is relatively compact and included in W. Let V^q be the cube of breadth $2a_q$ centered at q with respect to the coordinate system

$(x_1^q \ldots x_n^q, U^q)$. Choose $q_1 \ldots q_k$ such that $\Sigma \subseteq V = \bigcup_{j=1}^{k} V^{q_j}$ and let $U = \bigcup_{j=1}^{k} U^{q_j}$. Then $\bar{V}$ is compact and included in U and there are n-m uniquely determined differentiable functions $f_{m+1} \ldots f_n$ defined on U such that $f_{m+i} \upharpoonright U^{q_j} = x_{m+i}^{q_j}$ for $i = 1 \ldots n-m$ and $j = 1 \ldots k$. Let $F = f_{m+1}^2 + \ldots + f_n^2$. Clearly Σ is the set of points where F is zero. Since Σ is interior to V, and the frontier of V is compact, F has a positive minimum on the frontier of V. Choose r such that $0 < r <$ minimum of F on frontier of V and let $\widetilde{W} = \{q \varepsilon V : F(q) < r\}$. Then $\widetilde{W}$ is a neighborhood of Σ included in W. Let $q \varepsilon \widetilde{W}$. To complete the proof we must show that the leaf Σ_q of Θ containing q is compact and included in W. Let $\widetilde{\Sigma}_q = \{q' \varepsilon V : f_{m+i}(q') = f_{m+i}(q)\}$. Since $F(q') = F(q) < r$ for $q' \varepsilon \widetilde{\Sigma}_q$ it follows that $\widetilde{\Sigma}_q \subseteq \widetilde{W}$. Also $\widetilde{\Sigma}_q$ is clearly closed in V and as $F(q') < r <$ minimum of F on frontier of V it follows that $\widetilde{\Sigma}_q$ has no limit points on the frontier of V. Thus $\widetilde{\Sigma}_q$ is closed in $\bar{V}$ and hence is compact. On the other hand $\widetilde{\Sigma}_q$ is obviously the union of the m-dimensional slices of the coordinate system $(x_1^{q_j} \ldots x_n^{q_j}, V^{q_j})$ defined by $(f_{m+1}(q) \ldots f_n(q))$ and hence is an open submanifold of Σ_q. Since M is Hausdorff so is Σ_q hence $\widetilde{\Sigma}_q$ is open and closed in Σ_q. Since leaves are connected $\widetilde{\Sigma}_q = \Sigma_q$. Thus Σ_q is a compact leaf included in W as was to be proved.

COROLLARY 1. <u>If Θ is an involutive differential system on a Hausdorff differentiable manifold M then the set K of compact regular leaves of Θ is an open Hausdorff subspace of M/Θ with respect to the quotient topology.</u>

PROOF. Let $\Sigma \varepsilon K$ and take $W = M$ in the theorem. Then $\Pi_\Theta(\widetilde{W}) \subseteq K$ and is a neighborhood of Σ in M/Θ so K is open. If Σ' is also in K then since M is Hausdorff and Σ and Σ' are disjoint compact sets there are disjoint open sets W and W' of M including Σ and Σ' respectively. By the theorem we can find open sets $\widetilde{W}$ and $\widetilde{W}'$ in M which are unions of regular compact leaves of Θ such that $\widetilde{W} \subseteq W$ and $\widetilde{W}' \subseteq W'$ and $\Sigma \subseteq \widetilde{W}$ and $\Sigma' \subseteq \widetilde{W}'$. Then $\Pi_\Theta(\widetilde{W})$ and $\Pi_\Theta(\widetilde{W}')$ are disjoint neighborhoods of Σ and Σ' with respect to the quotient topology for M/Θ.

COROLLARY 2. *If Θ is a regular differential system on a Hausdorff differentiable manifold M and each leaf of Θ is compact then the quotient topology for M/Θ is Hausdorff and with respect to it Π_Θ is a closed mapping.*

PROOF. That the quotient topology for M/Θ is Hausdorff is immediate from corollary 1. If F is a closed set in M and $\Sigma \epsilon (M/\Theta - \Pi_\Theta(F))$ then $\Sigma \subseteq M-F$, hence there is a saturated open set $\widetilde{W} \subseteq M-F$ and including Σ. Then $\Sigma \epsilon \Pi_\Theta(\widetilde{W}) \subseteq (M/\Theta - \Pi_\Theta(F))$. Since $\Pi_\Theta(\widetilde{W})$ is open in the quotient topology it follows that $M/\Theta - \Pi_\Theta(F)$ is open in the quotient topology so $\Pi_\Theta(F)$ is closed in the quotient topology.

We recall that a submanifold Σ of a differentiable manifold M is said to be regularly imbedded in M if the inclusion map i of Σ into M is a homeomorphism (or, since i is always differentiable and hence continuous, if i^{-1} is continuous), and that Σ is called a closed submanifold of M if its point set is closed in M and it is regularly imbedded in M. It can be shown that if Σ is a regular leaf of an involutive differential system then Σ is regularly imbedded in M (the converse is false), however it is easy to construct an example where Σ is not a closed submanifold. Thus the following theorem shows that regularity of a differential system has strong consequences for the topological structure of the leaves.

THEOREM VII. *If* Θ *is a regular differential system on the differentiable manifold* M *then every leaf of* Θ *is a closed submanifold of* M .

PROOF. Let Σ be a leaf of Θ and let $\{p_k\}$ be a sequence in Σ approaching $p \varepsilon M$. By theorem V there is a regular coordinate system $(x_1 \ldots x_n, \mathcal{O})$ with respect to Θ centered at p . For k sufficiently large $p_k \varepsilon \ \mathcal{O} \cap \Sigma$ which is a single m-dimensional slice W of $(x_1 \ldots x_n, \mathcal{O})$, say the one defined by $(t_{m+1} \ldots t_n)$. Since $0 = x_{m+j}(p) = \lim_{k \to \infty} x_{m+j}(p_k) = t_{m+j}$, W is the slice of $(x_1 \ldots x_n, \mathcal{O})$ defined by $(0 \ldots 0)$. Thus $p \varepsilon W$ so $p \varepsilon \Sigma$ proving that Σ is closed in M . Moreover by definition of the manifold structure in Σ $(x_1 \ldots x_m, W)$ is a coordinate system in Σ and as $p_k \varepsilon W$ for k sufficiently large and $x_i(p_k) \to x_i(p)$ $i = 1 \ldots m$ it follows that $p_k \to p$ in the manifold topology for Σ . Since $\{p_k\}$ was any sequence in Σ approaching p in the topology of M this proves that the inclusion map of Σ in M has continuous inverse and hence that Σ is regularly imbedded.

COROLLARY. *If* M *is a compact, Hausdorff, differentiable manifold and* Θ *is a regular differential system on* M *then every leaf of* Θ *is compact. Moreover the quotient topology for* M/Θ *is compact and Hausdorff and with respect to it* Π_Θ *is a closed mapping.*

PROOF. Since M/Θ is the continuous image of M under Π_Θ it is compact. If Σ is a leaf of Θ then it is closed in M by the theorem and hence compact. The rest of the corollary follows from corollary 2 of theorem VI.

5. Quotient Manifolds.

THEOREM VIII. *Let Θ be an m-dimensional involutive differential system on an n-dimensional differentiable manifold M and let $(x_1 \ldots x_n, \mathcal{O})$ be a coordinate system in M regular with respect to Θ. Then there is a unique n-m-dimensional chart φ in M/Θ with domain $\Pi_\Theta(\mathcal{O})$ such that $\varphi \circ \Pi_\Theta(q) = (x_{m+1}(q) \ldots x_n(q))$ for all $q \varepsilon \mathcal{O}$. Two such charts in M/Θ are differentiably related and the set of all such charts for M/Θ is a differentiable atlas for M/Θ if and only if Θ is regular.*

PROOF. If $\Sigma \varepsilon \Pi_\Theta(\mathcal{O})$ then Σ intersects $\mathcal{O}$ and since $(x_1 \ldots x_n, \mathcal{O})$ is regular with respect to Θ this intersection is a single m-dimensional slice of $(x_1 \ldots x_n, \mathcal{O})$, say that defined by $(t_{m+1} \ldots t_n)$. We define $\varphi(\Sigma) = (t_{m+1} \ldots t_n)$. Clearly $\varphi \circ \Pi_\Theta(q) = (x_{m+1}(q) \ldots x_n(q))$ for $q \varepsilon \mathcal{O}$. It is obvious that φ is one-to-one and its image is the cube in R^{n-m} of the same breadth as $(x_1 \ldots x_n, \mathcal{O})$. Thus φ is an n-m dimensional chart in M/Θ. That two such charts are differentiably related is a consequence of theorem III. The last statement is a consequence of the definition of regularity of a differential system.

DEFINITION IV. Let Θ be an involutive differential system on the differentiable manifold M. A chart in M/Θ such as described in theorem VIII will be called a *natural chart* (with respect to Θ). If Θ is regular we call the set of all natural charts in M/Θ the *natural atlas* for M/Θ, and we call the manifold defined by the natural atlas for M/Θ the *quotient manifold of* M *defined by* Θ, i.e. the quotient manifold is $(M/\Theta, \Psi)$ where Ψ is the unique complete atlas for M/Θ

including the natural atlas. In general we denote it simply by M/Θ .

It is important to note that although M/Θ is defined whenever Θ is involutive, it is a manifold only when Θ is regular. Also it can happen that Θ is regular and M a Hausdorff manifold and that M/Θ is not a Hausdorff manifold. For example let $M = R^2 - \{(0,0)\}$ and for $p \varepsilon M$ let Θ_p be the subspace of the tangent space at p spanned by $(\partial / \partial u_1)_p$. Then if $\mathcal{O}$ is an open set in M cubical with respect to (u_1,u_2) and $p \varepsilon \mathcal{O}$ then $(u_1-u_1(p), u_2-u_2(p), \mathcal{O})$ is a regular coordinate system with respect to Θ . It follows that Θ is regular. The point set of M/Θ consists of the sets $\Sigma_r = \{(t,r) : t \varepsilon R\}$ with $r \neq 0$, $\Sigma_0^- = \{(t,0) : t < 0\}$, and $\Sigma_0^+ = \{(t,0) : t > 0\}$. The points Σ_0^- and Σ_0^+ cannot be separated by open sets. Thus picturesquely M/Θ is a line with two infinitely near origins.

Though the following theorem is trivial it is useful in applying other theorems and we state it for reference purposes.

THEOREM IX. Let M be a differentiable manifold and let Θ be the zero-dimensional differential system in M . Then Θ is a regular differential system on M , its leaves are the unit classes of points of M and $\Pi_\Theta : p \longrightarrow \{p\}$ is a diffeomorphism of M onto M/Θ .

Usually we identify M with M/Θ via the diffeomorphism Π_Θ .

THEOREM X. Let Θ be a regular m-dimensional differential system on the n-dimensional differentiable manifold M . Then Π_Θ is a differentiable map of M onto M/Θ . If $p \varepsilon M$ then the null space of $(\delta\Pi_\Theta)_p$ is Θ_p and the range of $(\delta\Pi_\Theta)_p$ is the entire tangent space at $\Pi_\Theta(p)$.

PROOF. Let $(x_1 \ldots x_n, \mathcal{O})$ be a regular coordinate system with respect to Θ centered at p. By definition of the manifold structure for M/Θ there is a coordinate system $(\tilde{x}_{m+1} \ldots \tilde{x}_n, \Pi_\Theta(\mathcal{O}))$ in M/Θ such that $\tilde{x}_{m+i} \circ \Pi_\Theta = x_{m+i}$. This implies the differentiability of Π_Θ at p. Moreover we clearly have $\delta\Pi_\Theta(\partial/\partial x_i)_p = 0$ if $i \leq m$ and $\delta\Pi_\Theta(\partial/\partial x_{m+i})_p = (\partial/\partial \tilde{x}_{m+i})_{\Pi_\Theta(p)}$ from which the other remarks follow.

COROLLARY 1. *If Θ is a regular differential system on a differentiable manifold M then Π_Θ is a continuous and open mapping with respect to the manifold topology for M/Θ.*

PROOF. Since Π_Θ is differentiable it is continuous with respect to the manifold topology for M/Θ. If $\mathcal{O}$ is the domain of a coordinate system in M regular with respect to Θ then $\Pi_\Theta(\mathcal{O})$ is the domain of a natural chart for M/Θ and hence is open with respect to the manifold topology for M/Θ. By corollary 2 of theorem V Π_Θ is open with respect to the manifold topology for M/Θ.

COROLLARY 2. *If Θ is a regular differential system on a differentiable manifold M then the manifold topology for M/Θ and the quotient topology for M/Θ are the same.*

PROOF. Corollary 1 above and corollary 3 of theorem III.

COROLLARY 3. *If Θ is a regular differential system on a Hausdorff differentiable manifold M then the set of compact leaves of Θ is an open Hausdorff submanifold of M/Θ.*

PROOF. Corollary 2 above and corollary 1 of theorem VI.

COROLLARY 4. *Let Θ be a regular differential system on a Hausdorff differentiable manifold M such that each leaf of*

Θ is compact. Then M/Θ is a Hausdorff differentiable manifold and Π_Θ is a closed mapping. If M/Θ is connected (in particular if M is connected) then the leaves of Θ are the fibers of a C^∞ fibering of M with base space M/Θ and projection Π_Θ, and in particular the leaves of Θ are all C^∞ isomorphic.

PROOF. The first conclusion follows from corollary 2 above and corollary 2 of theorem VI. By the present theorem the rank of $\delta\Pi_\Theta$ at each point of M is the dimension of M/Θ so the second conclusion follows from the proposition on page 31 of [4].

COROLLARY 5. *If Θ is a regular differential system on a compact, Hausdorff, differentiable manifold M then every leaf of Θ is compact, M/Θ is a compact, Hausdorff, differentiable manifold and Π_Θ is a closed mapping. If moreover M/Θ is connected (in particular if M is connected) then the leaves of Θ are the fibers of a C^∞ fibering of M with base space M/Θ and projection mapping Π_Θ, and in particular the leaves of Θ are all C^∞ isomorphic.*

PROOF. The corollary of theorem VIII and corollary 2 above prove all but the last remark, and that follows from corollary 4.

6. Factorization of Mappings.

A trivial but important result in the theory of topological quotien spaces is the following factorization of mappings theorem: if R and are equivalence relations on topological spaces M and N respectively and f is a continuous map of M into N which carries R-equivalent pairs of points into S-equivalent pairs of points, then there is a uniq mapping F of M/R into N/S satisfying $F \circ \Pi_R = \Pi_S \circ f$ and F

is continuous with respect to the quotient topologies [5, Chapt. 1, §9, corollary of theorem 1]. As is to be expected this has a natural 'infinitesimal' analogue when M and N are differentiable manifolds, R and S the relations of belonging to the same leaf of an involutive differential system, and f a differentiable mapping. This is expressed by (2) of the following theorem.

THEOREM XI. *Let Θ and Ψ be involutive differential systems on differentiable manifolds M and N respectively and let f be a differentiable map of M into N. Then the following three conditions are equivalent:*

(1) There is a mapping F of M/Θ into N/Ψ such that $F \circ \Pi_\Theta = \Pi_\Psi \circ f$. If such a mapping exists it is unique and is continuous relative to the respective quotient topologies, and if Θ and Ψ are regular it is a differentiable mapping relative to the respective quotient manifold structures.

(2) For all $p \varepsilon M$ $\delta f(\Theta_p) \subseteq \Psi_{f(p)}$.

(3) Each leaf of Θ is mapped by f into a single leaf of Ψ.

PROOF. Suppose a mapping F of M/Θ into N/Ψ satisfying $F \circ \Pi_\Theta = \Pi_\Psi \circ f$ exists and let $p \varepsilon M$. Let $(y_1 \dots y_s, V)$ be a cubical coordinate system centered at $f(p)$ and flat with respect to Ψ. Since f is continuous, by the corollary of theorem I we can find a cubical coordinate system $(x_1 \dots x_n, \mathcal{O})$ centered at p and flat with respect to Θ such that $f(\mathcal{O}) \subseteq V$. The relation $F \circ \Pi_\Theta = \Pi_\Psi \circ f$ implies that f maps a leaf Σ of Θ into a leaf $F(\Sigma)$ of Ψ, so in particular f maps an m-dimensional slice of $(x_1 \dots x_n, \mathcal{O})$ into an r-dimensional slice of $(y_1 \dots y_s, V)$ where m is the dimension of Θ and r is the dimension of Ψ. It follows that the $y_{r+j} \circ f$ have expressions with respect to $(x_1 \ldots x_n, \mathcal{O})$ which are independent of their first m arguments, i.e. there exist real valued functions in R^{n-m} such that $y_{r+j} \circ f(q) = g_{r+j}(x_{m+1}(q) \ldots x_n(q))$. The differentiability of f implies the differentiability of the g_{r+j}. If $i \leq m$ then $dy_{r+j}(\delta f(\partial/\partial x_i)_p) = (\partial/\partial x_i)_p(y_{r+j} \circ f) = 0$ from the above expression for $y_{r+j} \circ f$. Since the $(\partial/\partial x_i)_p$ $i \leq m$ span Θ_p and the dy_{r+j} span the annihilator of $\Psi_{f(p)}$ this implies that $\delta f(\Theta_p) \subseteq \Psi_{f(p)}$, proving that (1) implies (2). The continuity of F with respect to the quotient topologies for M/Θ and N/Ψ follows from the corollary of theorem 1 of [5, Chapt. 1, §9]. If Θ and Ψ are regular then using corollary of theorem V we can assume that $(x_1 \ldots x_n, \mathcal{O})$ is regular with respect to Θ and that $(y_1 \ldots y_s, V)$ is regular with respect to Ψ. Then by definition of the manifold structures on M/Θ and N/Ψ there is a coordinate system $(\bar{x}_{m+1} \ldots \bar{x}_n, \Pi_\Theta(\mathcal{O}))$ in M/Θ such that $x_{m+j} = \bar{x}_{m+j} \circ \Pi_\Theta$ and a coordinate system $(\bar{y}_{r+j} \ldots \bar{y}_s, \Pi_\Psi(V))$ in N/Ψ such that $y_{r+j} = \bar{y}_{r+j} \circ \Pi_\Psi$. If $\Sigma \varepsilon \Pi_\Theta(\mathcal{O})$ let $q \varepsilon \mathcal{O}$ with $\Pi_\Theta(q) = \Sigma$. Then $\bar{y}_{r+j} \circ F(\Sigma) = y_{r+j} \circ f(q) = g_{r+j}(x_{m+1}(q) \ldots x_n(q)) = g_{r+j}(\bar{x}_{m+1}(\Sigma) \ldots \bar{x}_n(\Sigma))$. Since the g_{r+j} are differentiable F is differentiable in $\Pi_\Theta(\mathcal{O})$ and in particular at p. Since p was arbitrary F is differentiable.

Now suppose that (2) holds. To prove (3) it suffices, in view of the connectivity of leaves, to prove that the set of points of a leaf Σ of Θ mapped by f into a fixed leaf of Ψ is open Σ. Using the notation introduced above, the m-dimensional slice of $(x_1 \ldots x_n, \mathcal{O})$ containing p is a neighborhood of p in the leaf of Θ containing p, hence it will suffice to show that an m-dimensional slice of $(x_1 \ldots x_n, \mathcal{O})$ is carried by f into a fixed leaf of Ψ, or better still into a fixed r-dimensional slice of $(y_1 \ldots y_s, V)$. Equivalently it will suffice to show that the expression for $y_{r+j} \circ f$ with respect to the coordinate system $(x_1 \ldots x_n, \Theta)$ is independent of its first m variables and this in turn is equivalent to showing that $(\partial/\partial x_i)(y_{r+j} \circ f) = 0$ for $i \leqslant m$. Now if $i \leqslant m$ then $(\partial/\partial x_i)$ belongs to Θ so $\delta f(\partial/\partial x_i)$ belongs to Ψ by (2) and hence, since $(dy_{r+j})_q$ annihilates Ψ_q for $q \varepsilon V$, $(\partial/\partial x_i)(y_{r+j} \circ f) = dy_{r+j}(\delta f(\partial/\partial x_i)) = 0$. This completes the proof that (2) implies (3).

Finally suppose (3) holds. Given a leaf Σ of Θ let $F(\Sigma)$ be the leaf of Ψ into which Σ is mapped by f. Then clearly $F \circ \Pi_\Theta = \Pi_\Psi \circ f$ proving (1).

COROLLARY. *Let Θ be an involutive differential system on a differentiable manifold M and let f be a differentiable map of M into a differentiable manifold N. Then the following three statements are equivalent:*

(1) *There is a mapping F of M/Θ into N such that $f = F \circ \Pi_\Theta$. If such a mapping exists it is unique and is continuous with respect to the quotient topology for M/Θ. If Θ is regular then F is differentiable with respect to the quotient manifold structure on M/Θ.*

(2) *For all $p \varepsilon M$ $\delta f(\Theta_p) = 0$.*

(3) *f is constant on leaves of Θ.*

PROOF. In the theorem take Ψ to be the zero-dimensional differential system on N and use theorem IX.

7. Projection-like Mappings.

DEFINITION V. Let M and N be differentiable manifolds. A mapping Π of M into N will be called *projection-like* if it is differentiable and for each $p \varepsilon M$ $\delta\Pi$ maps the tangent space to M at p onto the tangent space to N at $\Pi(p)$.

THEOREM XII. *If Π is a projection-like mapping of a differentiable manifold M into a differentiable manifold N then Π is open and in particular $\Pi(M)$ is an open submanifold of N.*

PROOF. See the remark page 80 of [1].

THEOREM XIII. *Let Π be a projection-like mapping of a differentiable manifold M of dimension n into a differentiable manifold N of dimension s and let Ψ be an r-dimensional regular differential system in N. For each $p \varepsilon M$ let Θ_p be the set of vectors in the tangent space to M at p mapped by $\delta\Pi$ into $\Psi_{\Pi(p)}$. Then $\Theta : p \to \Theta_p$ is an $m = r+n-s$ dimensional regular differential system in M. There is a local diffeomorphism h of M/Θ into N/Ψ with the following properties:*

(1) $\Pi_\Psi \circ \Pi = h \circ \Pi_\Theta$.

(2) *For each* $\Sigma \varepsilon M/\Theta$ $\Pi | \Sigma$ *is a projection-like map of* Σ *into* $h(\Sigma)$.

(3) *If M/Θ is compact, Hausdorff, and connected and if N/Ψ is Hausdorff and connected (in particular, by corollary 5 of theorem X, if M and N are compact, Hausdorff, and connected) then $(M/\Theta, h)$ is a covering space for N/Ψ.*

PROOF. If $p \varepsilon M$ then $\delta\Pi_p$ being onto has a null space of dimension $n-s$, hence since $\dim(\Psi_{\Pi(p)}) = r$ and $\Theta_p = \delta\Pi_p^{-1}(\Psi_{\Pi(p)})$, $\dim(\Theta_p) = r+n-s = m$. Let $(y_1 \ldots y_s, V)$ be a coordinate system in N centered at $\Pi(p)$ regular with respect to Ψ. By [1, proposition 2, page 80] we can find a coordinate system $(x_1 \ldots x_n, \mathcal{O})$ in M, such that $x_{m+i} = y_{r+i} \circ \Pi$ $i = 1 \ldots s-r$. We can assume that $(x_1 \ldots x_n, \mathcal{O})$ is cubical, centered at p and that $\Pi(\mathcal{O}) \subseteq V$. The relation $dx_{m+i} = \delta\Pi^*(dy_{r+i})$

together with the fact that the dy_{r+i} are a base for the annihilator of Ψ implies that dx_{m+i} are a base for the annihilator of $\Theta = \delta\Pi^{-1}(\Psi)$. Thus $(x_1 \ldots x_n, \mathcal{O})$ is flat with respect to Θ and hence Θ is involutive. Since $\delta\Pi(\Theta) = \Psi$ it follows from theorem XI that there is a map h of M/Θ into N/Ψ satisfying $\Pi_\Psi \circ \Pi = h \circ \Pi_\Theta$. Let Σ be a leaf of Θ and suppose Σ intersects $\mathcal{O}$ in the m-dimensional slice of $(x_1 \ldots x_n, \mathcal{O})$ defined by $(t_{m+1} \ldots t_n)$. Then if q is in this slice $y_{r+i} \circ \Pi(q) = x_{m-i}(q) = t_{m+i}$ and since $h(\Sigma) = h \circ \Pi_\Theta(q) = \Pi_\Psi \circ \Pi\,(q)$, $h(\Sigma)$ intersects V in the r-dimensional slice of $(y_1 \ldots y_s, V)$ defined by $(t_{m+1} \ldots t_n)$. Since $(y_1 \ldots y_s, V)$ is regular $h(\Sigma)$ can intersect V in but one r-dimensional slice, hence Σ can intersect $\mathcal{O}$ in but one m-dimensional slice. Thus $(x_1 \ldots x_n, \mathcal{O})$ is regular with respect to Θ so, as p is arbitrary, Θ is regular by theorem V. It follows from theorem XI that h is differentiable. Since Π is projection-like by hypothesis and Π_Ψ and Π_Θ are projection-like by theorem X, it follows from the relation $\delta h \circ \delta\Pi_\Theta = \delta\Pi_\Psi \circ \delta\Pi$ that h is also projection-like. Since $\dim(M/\Theta) = n-m = s-r = \dim(N/\Psi)$, δh maps a tangent space to M/Θ isomorphically onto a tangent space to N/Ψ, whence h is a local diffeomorphism.

Let Σ be the leaf of Θ containing p and let i be the injection mapping of Σ into M. Then as Π and i are differentiable $g = \Pi \circ i = \Pi \upharpoonright \Sigma$ is a differentiable mapping of Σ into N. Since $h(\Sigma)$ cuts V in a single slice of $(y_1 \ldots y_s, V)$ the proof of proposition 1 [1, page 95] shows that g is differentiable at p. Moreover if Φ is the tangent space to Σ at p then $\delta g(\Phi) = \delta\Pi \circ \delta i(\Phi) = \delta\Pi(\Theta_p) = \Psi_{\Pi(p)}$, and since the latter is the tangent space to $h(\Sigma)$ at $\Pi(p)$ it follows that $g = \Pi \upharpoonright \Sigma$ is a projection-like map of Σ into $h(\Sigma)$.

Finally suppose that M/Θ is compact, Hausdorff, and connected and that N/Ψ is Hausdorff and connected. Then $h(M/\Theta)$ is an open and compact and hence open and closed subset of N/Ψ so $h(M/\Theta) = N/\Psi$. If $\Sigma \varepsilon N/\Psi$ then, since h is a local diffeomorphism, $h^{-1}(\Sigma)$ is a discrete subset of M/Θ. By the proposition of page 31 of [4] it follows that $(M/\Theta, N/\Psi, h)$ is a C^∞ fiber bundle with discrete fiber, i.e. $(M/\Theta, h)$ is a covering space for N/Ψ.

Taking for Ψ the zero-dimensional differential system in N and using theorem IX we get the following.

COROLLARY. *Let Π be a projection-like mapping of a differentiable manifold M into a differentiable manifold N and for each $p \varepsilon M$ let Θ_p be the null space of $\delta\Pi_p$. Then $\Theta : p \to \Theta_p$ is a regular differential system on M and there is a local diffeomorphism h of M/Θ into N satisfying $\Pi = h \circ \Pi_\Theta$. If $q \varepsilon N$ then $h^{-1}(q)$ is the set of components of $\Pi^{-1}(q)$. If M/Θ is compact, Hausdorff, and connected (and in particular if M is compact, Hausdorff, and connected) and N is Hausdorff and N is Hausdorff and connected, then $(M/\Theta, h)$ is a covering space for N.*

8. The Uniqueness Theorem.

Suppose Θ is an involutive differential system on M. By theorem X if Θ is regular then Π_Θ is a projection-like mapping of M onto M/Θ with respect to the quotient manifold structure. Conversely we will now show that if M/Θ can be given a manifold structure with respect to which Π_Θ is projection-like then Θ is regular and the given manifold structure on M/Θ is the quotient manifold structure.

THEOREM XIV. *Let Θ be an involutive differential system on a differentiable manifold M. If M/Θ can be given a* differentiable manifold structure with respect to which Π_Θ is projection-like, then Θ is regular and the given differentiable manifold structure on M/Θ coincides with the quotient manifold structure.

PROOF. Denote by N M/Θ with a given differentiable manifold structure with respect to which Π_Θ is projection-like. For each $p \in M$ let Ψ_p be the null space of $(\delta\Pi_\Theta)_p$. By the corollary of theorem XII Ψ is regular and the leaves of Ψ are the components of inverse images of points of N under Π_Θ, i.e. leaves of Θ. Since an involutive differential system is determined by its leaves $\Theta = \Psi$ so Θ is regular. Also by the corollary of theorem XII it follows that there is a local diffeomorphism of M/Θ onto N satisfying $\Pi_\Theta = h \circ \Pi_\Theta$. Clearly h is the identity map. But to say that the identity map of M/Θ onto N is a local diffeomorphism is just to say that the given manifold structure on M/Θ coincides with the quotient manifold structure.

9. Products of Quotient Manifolds.

If (M,Φ) and (N,Ψ) are differentiable manifolds then the set of maps $\varphi \times \psi : (p,q) \longrightarrow (\varphi(p),\psi(q))$ where $\varphi \varepsilon \Phi$ and $\psi \varepsilon \Psi$ is a differentiable atlas for $M \times N$ and the manifold it defines is called the product of (M,Φ) and (N,Ψ). In general we denote the product of differentiable manifolds M and N simply by $M \times N$ and we identify the tangent space to $M \times N$ at (p,q) with the direct sum of the tangent space to M at p and the tangent space to N at q as in [1 page 82]. If Θ is a differential system on M and Φ a differential system on N we denote by $\Theta \oplus \Phi$ the differential system $(p,q) \longrightarrow \Theta_p \oplus \Phi_q$ on $M \times N$.

THEOREM XV. If Θ and Φ are regular differential systems on the differentiable manifolds M and N respectively *then* $\Theta \oplus \Phi$ *is a regular differential system on* $M \times N$ *. The mapping* $h : (\Sigma, \Sigma') \to \Sigma \times \Sigma'$ *is a diffeomorphism of* $(M/\Theta) \times (N/\Psi)$ *onto* $M \times N/\Theta \oplus \Phi$ *and if we identify* $M \times N/\Theta \oplus \Phi$ *with* $(M/\Theta) \times (N/\Psi)$ *via* h *then* $\Pi_{\Theta \oplus \Phi}$ *goes into* $\Pi_\Theta \times \Pi_\Phi : (p,q) \to (\Pi_\Theta(p), \Pi_\Phi(q))$.

PROOF. Since Π_Θ and Π_Φ are each projection-like by theorem X so is $\Pi_\Theta \times \Pi_\Phi$ and clearly the null space of $\delta(\Pi_\Theta \times \Pi_\Phi)_{(p,q)} = (\delta\Pi_\Theta)_p \oplus (\delta\Pi_\Phi)_q$ is $\Theta_p \oplus \Phi_q = (\Theta \oplus \Phi)_{(p,q)}$ by theorem X. Also the inverse image of (Σ, Σ') under $\Pi_\Theta \times \Pi_\Phi$ is $\Sigma \times \Sigma'$. The corollary of theorem XIII completes the proof.

Chapter II

LOCAL AND INFINITESIMAL TRANSFORMATION GROUPS

If a Lie group G acts locally (and differentiably) on a manifold M, then one can define in a natural way a homomorphism of the Lie algebra of G into the Lie algebra of differentiable vector fields on M. Such a homomorphism is called an infinitesimal G-transformation group acting on M, and if it arises from a local action of G on M, then it is said to be the infinitesimal generator of this local action. Lie's 'Hauptsatz der Gruppentheorie' or 'Second Fundamental Theorem' [6, page 390] can be interpreted as saying that an infinitesimal G-transformation group acting on M generates a local action of G on some neighborhood of each point of M.

Our goals in this chapter are two-fold. First, in sections 1-5, we develop the basic general theory of infinitesimal transformation groups that will be used in the following two sections and in later chapters. The key idea here is the detailed study of the topological structure and imbedding of the leaves of a certain involutive differential system on $G \times M$ associated with an infinitesimal G-transformation group acting on M and called its infinitesimal graph. The theorems look rather technical in nature but their value is amply indicated later on.

Secondly, in sections 6 and 7, we prove a uniqueness theorem (which is easy) and an existence theorem (which is not so easy) for local G-transformation groups with a given infinitesimal generator. The existence theorem is a globalization of Lie's Second Fundamental Theorem (as stated above) with respect to M: it states that an infinitesimal G-transformation group acting on M generates a local action of G on <u>all</u> of M.

The question of globalizing Lie's Second Fundamental Theorem also with respect to G will be taken up in chapter III, and in chapter IV we apply the results to develop a Lie theory for the group of all diffeomorphisms of a manifold onto itself.

1. Notation.

Besides the definitions, notations, and terminology of chapter I, we introduce the following standing notation. G will denote a connected r-dimensional Lie group, e its identity, and $\mathfrak{g}$ its Lie algebra of right invariant vector fields. M will denote an n-dimensional differentiable (i.e. C^∞ or analytic) manifold. We denote by Π_G and Π_M the projections of $G \times M$ on G and M respectively. If X is a tangent vector to G at g and Y a tangent vector to M at p then we denote by (X,Y) the vector Z tangent to $G \times M$ at (g,p) such that $\delta\Pi_G(Z) = X$ and $\delta\Pi_M(Z) = Y$. If X and Y are vector fields on G and M respectively, then (X,Y) is the vector field on $G \times M$ defined by $(X,Y)_{(g,p)} = (X_g, Y_p)$. For each $g \varepsilon G$ right translation by g^{-1} will be denoted by R_g, i.e. for $h \varepsilon G$ $R_g(h) = hg^{-1}$. We denote by $\bar{R}_g$ the map $(h,p) \rightarrow (hg^{-1},p)$ of $G \times M$ onto itself. Thus $\bar{R}_g = R_g \times I$ where I is the identity map of M, and $R_g \circ \Pi_G = \Pi_G \circ \bar{R}_g$. Finally, we denote by $\bar{R}$ the mapping $(g,h,p) \rightarrow (hg^{-1},p)$ of $G \times G \times M$ into $G \times M$.

2. Elementary Definitions.

DEFINITION I. *A local transformation group domain in* $G \times M$ is an open subset D of $G \times M$ such that for each $p \varepsilon M$ the set $\{ g \varepsilon G : (g,p) \varepsilon D \}$ is a connected neighborhood of e.

THEOREM I. *Let* O *be an open set in* $G \times M$ *which includes*

$\{e\} \times M$, *and for each* $p \in M$ *let* D_p *be the component of* e *in* $\{g \in G : (g,p) \in O\}$. *Then* $D = \bigcup_{p \in M} (D_p \times \{p\})$ *is the largest local transformation group domain in* $G \times M$ *which is included in* O.

PROOF. The only non-obvious fact is that D is open. Suppose that $(g,p) \in D$ so that $g \in D_p$. Since D_p is arcwise connected we can find a continuous map σ of $[0,1]$ into D_p such that $\sigma(0) = e$ and $\sigma(1) = g$. Denoting by $|\sigma|$ the range of σ, $|\sigma| \times \{p\} \subseteq O$ and, since O is open, for each $h \in |\sigma|$ we can find a neighborhood V_h of h and a neighborhood U_h of p such that $V_h \times U_h \subseteq O$. Since $|\sigma|$ is compact we can choose $h_1 \ldots h_k$ such that $|\sigma| \subseteq \bigcup_{i=1}^{k} V_{h_i}$. Then putting $U = \bigcap_{i=1}^{k} U_{h_i}$ we see that U is a neighborhood of p and that $|\sigma| \times U \subseteq O$. Let V be an arcwise connected neighborhood of g and W a neighborhood of p included in U such that $V \times W \subseteq O$. Given $(h,q) \in V \times W$ we can find an arc in V jointing g to h. Then σ followed by this arc is an arc in $\{k \in G : (k,q) \in O\}$ joining e to h. It follows that $h \in D_q$ so $(h,q) \in D$ and thus $V \times W \subseteq D$. Since $V \times W$ is a neighborhood of (g,p), D is open.

COROLLARY. *The set of local transformation group domains in* $G \times M$ *is a lattice under inclusion. If* U *and* V *are local transformation group domains in* $G \times M$ *then their least upper bound is* $U \cup V$ *and their greatest lower bound is* $\bigcup_{p \in M} (W_p \times \{p\})$ *where* W_p *is the component of* e *in* $\{g \in G : (g,p) \in U \cap V\}$.

DEFINITION II. A *local* G-*transformation group acting on* M is a differentiable mapping $\varphi : D_\varphi \longrightarrow M$ where D_φ is a local G transformation group domain in $G \times M$ and

(1) for each $p \varepsilon M$ $\varphi(e,p) = p$

(2) if $(h,p)\varepsilon D_\varphi$, $(g,\varphi(h,p))\varepsilon D_\varphi$, and $(gh,p)\varepsilon D_\varphi$, then $\varphi(gh,p) = \varphi(g,\varphi(h,p))$.

If $D_\varphi = G \times M$ then φ is called a <u>global</u> G-<u>transformation group acting on</u> M .

NOTATION. We shall always denote the domain of a local G-transformation group φ acting on M by D_φ . Moreover for each $p\varepsilon M$ we put $D_{\varphi p} = \{ g\varepsilon G : (g,p)\varepsilon D_\varphi \}$ and for each $g\varepsilon G$ we put $D_{\varphi_g} = \{ p\varepsilon M : (g,p)\varepsilon D_\varphi \}$. We define $\varphi^p : D_{\varphi p} \longrightarrow M$ by $\varphi^p(g) = \varphi(g,p)$ and $\varphi_g : D_{\varphi_g} \longrightarrow M$ by $\varphi_g(p) = \varphi(g,p)$.

We note that if φ is a local G-transformation group acting on M then for each $p\varepsilon M$ $D_{\varphi p}$ is a connected open neighborhood of e and φ^p maps it differentiably into M . Similarly for each $g\varepsilon G$ D_{φ_g} is an open subset of M mapped differentiably into M by φ_g . Property (1) of definition II says that φ_e is the identity map of M . If φ is global then property (2) of definition II says that for all $g,h\varepsilon G$ we have $\varphi_{gh} = \varphi_g \circ \varphi_h$. In particular it follows that $\varphi_{g^{-1}} = \varphi_g^{-1}$ so each φ_g is a diffeomorphism of M onto itself when φ is global. We shall use these facts freely.

DEFINITION III. If φ is a local G-transformation group acting on M then we define a mapping with domain $\mathfrak{g}$ called the <u>infinitesimal generator of</u> φ and denoted by φ^+ as follows: for each $L\varepsilon\mathfrak{g}$ $\varphi^+(L)$ is the vector field on M defined by $\varphi^+(L)_p = \delta\varphi^p(L_e)$ for all $p\varepsilon M$.

DEFINITION IV. An <u>infinitesimal</u> G-<u>transformation group acting on</u> M is a homomorphism of $\mathfrak{g}$ into the Lie algebra of differentiable vector fields on M.

THEOREM II. *If φ is a local G-transformation group acting on M then its infinitesimal generator φ^+ is an infinitesimal G-transformation group acting on M. Moreover for each $(g,p)\varepsilon D_\varphi$ $\delta\varphi^p(L_g) = \varphi^+(L)_{\varphi(g,p)}$.*

PROOF. Let $L\varepsilon \mathfrak{g}$ and let f be a differentiable function at p in M. Let $F = f\circ\varphi$ so that F is a differentiable function at (e,p) in $G\times M$. The vector field $X : (h,q)\longrightarrow (L_h,0)$ is differentiable on $G\times M$ and hence $H = XF$ is a differentiable function at (e,p) in $G\times M$. It follows that $\bar{H} : q\rightarrow H(e,q)$ is a differentiable function at p in M. Now $\bar{H}(q) = L_e(f\circ\varphi^q) = \delta\varphi^q(L_e)f = (\varphi^+(L)f)(q)$, i.e. $\bar{H} = \varphi^+(L)f$. Thus $\varphi^+(L)f$ is a differentiable at p in M. Since f was an arbitrary differentiable function at p this shows that $\varphi^+(L)$ is a differentiable vector field at p and as p was an arbitrary point of M it follows that $\varphi^+(L)$ is a differentiable vector field on M.

Now let $(g,p)\varepsilon D_\varphi$ and let $q = \varphi(g,p)$. Then $D_{\varphi q}\cap D_{\varphi p}g^{-1}$ is a neighborhood of e and for all h in the latter set (g,p), (hg,p), and $(h,\varphi(g,p))$ are in D_φ so that $\varphi(hg,p) = \varphi(h,\varphi(g,p))$, i.e. $\varphi^p\circ R_{g^{-1}}(h) = \varphi^q(h)$. Since this holds for all h in a neighborhood of e it follows that φ^q and $\varphi^p\circ R_{g^{-1}}$ have the same differential at e. Now an element L of $\mathfrak{g}$ is right invariant and hence satisfies $\delta R_{g^{-1}}(L_e) = L_g$ and so $\varphi^+(L)_{\varphi^p(g)} = \varphi^+(L)_q = \delta\varphi^q(L_e) = \delta\varphi^p\circ\delta R_{g^{-1}}(L_e) = \delta\varphi^p(L_g)$. Thus L and $\varphi^+(L)$ are φ^p related [1, page 84]. If $L'\varepsilon\mathfrak{g}$ then of course L' and $\varphi^+(L')$ are also φ^p related so by [1, page 85] $[L,L']$ and $[\varphi^+(L), \varphi^+(L')]$ are φ^p related, i.e. $\varphi^+([L,L'])_p = \delta\varphi^p([L,L']_e) = [\varphi^+(L),\varphi^+(L')]_p$. Since φ^+ is manifestly linear, this proves that it is a Lie algebra homomorphism.

DEFINITION V. Let Θ be an infinitesimal G-transformation group acting on M. We shall say that Θ <u>generates</u> any local G-transformation group acting on M for which it is the infinitesimal generator and that Θ is <u>generating</u> if it generates at least one local G-transformation group acting on M.

Suppose now that D is any local transformation group domain in $G \times M$. If we define $\varphi : D \to M$ by $\varphi(g,p) = p$, then φ is a local G-transformation group acting on M. Thus every local transformation group domain in $G \times M$ is the domain of at least one local G-transformation group acting on M. Aside from this trivial remark there seems to be little of interest that can be said about the set of local transformation groups acting on M with a given domain.

Two more interesting questions, which will be the major concern of this chapter, are the following. Given an infinitesimal G-transformation group Θ acting on M, what are the local G-transformation group domains D in $G \times M$ for which there exists a local G-transformation group φ with domain D and infinitesimal generator Θ, and to what extent is φ determined by Θ and D, supposing it does exist? The answer that we shall give to the second question is quite straightforward: a local G-transformation group is uniquely determined by its domain and its infinitesimal generator. As to the first question, perhaps the most interesting point is whether or not there exist any local G-transformation groups acting on M generated by Θ, i.e. whether or not Θ is generating. Lie's Second Fundamental Theorem gives an affirmative answer to this question in the following local sense: for each $p \in M$ there exists an open submanifold O of M containing p such that the infinitesimal G-transformation group acting on O defined by $L \to \Theta(L) \restriction O$ is generating. We shall show (corollary of theorem XI) that if M is Hausdorff, and even more generally, we can take $O = M$, that is, that Θ itself is generating.

3. 'Factoring' a Transformation Group.

In chapter I we have defined what it means for an involutive differential system Θ to be regular, and given a regular differential system Θ on a differentiable manifold M we have defined a manifold structure on the set M/Θ of leaves of Θ with respect to which the quotient mapping $\Pi_\Theta : M \longrightarrow M/\Theta$ is differentiable. We will now show that, under suitable conditions, a global G-transformation group acting on M induces one acting on M/Θ, and that the two 'commute' with Π_Θ.

Let Θ be an involutive differential system on M and let φ be a global G-transformation group acting on M. If $g \varepsilon G$ and Σ is a leaf of Θ, then since φ_g maps M diffeomorphically onto itself it maps Σ diffeomorphically onto a submanifold Σ' of M. If $p \varepsilon \Sigma$ then the tangent space to Σ at p is Θ_p, so the tangent space to Σ' at $\varphi_g(p)$ is $\delta\varphi_g(\Theta_p)$. Thus a necessary and sufficient condition that Σ' be an integral manifold of Θ is that $\delta\varphi_g(\Theta_p) = \Theta_{\varphi(g,p)}$ for each $p \varepsilon \Sigma$. It follows that a necessary and sufficient condition for $\varphi_g(\Sigma)$ to be a leaf of Θ, whenever Σ is and $g \varepsilon G$, is that $\delta\varphi_g(\Theta_p) = \Theta_{\varphi(g,p)}$ for each $(g,p) \varepsilon G \times M$.

DEFINITION VI. Let Θ be an involutive differential system on M, and let φ be a global G-transformation group acting on M. We say that φ is _compatible with_ Θ if either and hence both of the following two equivalent conditions hold:

(1) For all $(g,p) \varepsilon G \times M$ $\delta\varphi_g(\Theta_p) = \Theta_{\varphi(g,p)}$.

(2) For each $g \varepsilon G$ and each leaf Σ of Θ, φ_g maps Σ diffeomorphically onto a leaf of Θ.

THEOREM III. _Let_ Θ _be a regular differential system on_ M

<u>and let</u> φ <u>be a global G-transformation group acting on</u> M. <u>Then if</u> φ <u>is compatible with</u> Θ, <u>there is a uniquely determined global G-transformation group</u> $\bar{\varphi}$ <u>acting on</u> M/Θ <u>such that</u> $\Pi_\Theta \circ \varphi_g = \bar{\varphi}_g \circ \Pi_\Theta$ <u>for all</u> $g \varepsilon G$. <u>Moreover</u> $\bar{\varphi}^+ = \delta\Pi_\Theta \circ \varphi^+$.

PROOF. Let Φ be the zero-dimensional differential system on G. By theorems IX and XV of chapter I, we can identify $G \times M/\Phi \oplus \Theta$ with $G \times (M/\Theta)$ in such a way that $\Pi_{\Phi \oplus \Theta} = I \times \Pi_\Theta : (g,p) \longrightarrow (g, \Pi_\Theta(p))$. Now if X is a vector tangent to M at p then $\delta\varphi(0,X) = \delta\varphi^p(0) + \delta\varphi_g(X) = \delta\varphi_g(X)$ so $\delta\varphi((\Phi \oplus \Theta)_{(g,p)}) = \delta\varphi_g(\Theta_p) = \Theta_{\varphi(g,p)}$. By theorem XI of chapter I there is a uniquely determined differentiable map $\bar{\varphi}$ of $G \times (M/\Theta)$ into M/Θ such that $\bar{\varphi} \circ (I \times \Pi_\Theta) = \Pi_\Theta \circ \varphi$, or equivalently such that $\bar{\varphi}_g \circ \Pi_\Theta = \Pi_\Theta \circ \varphi_g$ for all $g \varepsilon G$. It is obvious that $\bar{\varphi}$ is a global G-transformation group acting on M/Θ. If $p \varepsilon M$ and $\Sigma = \Pi_\Theta(p)$ then by the above commutativity relation $\Pi_\Theta \circ \varphi^p = \bar{\varphi}^\Sigma$, hence if $L \varepsilon \mathfrak{g}$ then $\bar{\varphi}^+(L)_{\Pi_\Theta(p)} = \delta\bar{\varphi}^\Sigma(L_e) = \delta\Pi_\Theta \circ \delta\varphi^p(L_e) = \delta\Pi_\Theta \circ \varphi^+(L)_p$.

4. The Infinitesimal Graph.

DEFINITION VII. Let Θ be an infinitesimal G-transformation group acting on M. We define a mapping Θ^* on $G \times M$ called the <u>infinitesimal graph</u> of Θ by:

$$\Theta^*_{(g,p)} = \{ (L_g, \Theta(L)_p) : L \varepsilon \mathfrak{g} \}.$$

If φ is a local G-transformation group acting on M, then from a strictly set theoretical point of view φ^p <u>is</u> the set of ordered pairs $\{ (g, \varphi^p(g)) : g \varepsilon D_{\varphi^p} \}$. However, we shall use the term 'graph of φ^p' for this geometrical object and reserve the symbol φ^p for situations when, intuitively speaking, the mapping properties of φ^p are being considered. We shall show presently that Θ^* is an involutive r-dimensional differential system on $G \times M$, and that if Θ generates φ then for each $p \varepsilon M$ the graph of φ^p is an open submanifold of the leaf Σ_p of Θ^* containing (e,p), in fact, it is the component of (e,p) in $\Sigma_p \cap \Pi_G^{-1}(D_\varphi p)$, with respect to the manifold topology of Σ_p. This is the reason for the appelation 'infinitesimal graph'.

THEOREM IV. _If_ Θ _is an infinitesimal_ G-_transformation group acting on_ M _then_ Θ^* _is an involutive_ r-_dimensional differential system on_ $G \times M$. _If_ Σ _is a leaf of_ Θ^* _then the restriction of_ Π_G _to_ Σ _is a local diffeomorphism of_ Σ _into_ G. _If_ $(x_1 \ldots x_{r+n}, 0)$ _is a cubical coordinate system centered at_ $(g,p) \varepsilon G \times M$ _and flat with respect to_ Θ^*, _then the functions_ $w_1 \ldots w_n$ _defined near_ p _in_ M _by_ $w_i(q) = x_{r+i}(g,q)$ _form a coordinate system in_ M _about_ p.

PROOF. Given $(g,p) \varepsilon G \times M$, $L \longrightarrow (L_g, \Theta(L)_p)$ is clearly a linear map of $\mathfrak{g}$ into the tangent space to $G \times M$ at (g,p). If $(L_g, \Theta(L)_p) = 0$ then $L_g = 0$ and hence, as L is right invariant, $L = 0$. Thus the above mapping is non-singular and so its range, which is $\Theta^*_{(g,p)}$, is an r-dimensional subspace of the tangent space to $G \times M$ at (g,p). Choosing a basis $L_1 \ldots L_r$ for $\mathfrak{g}$ it is clear that $(L_1, \Theta(L_1)) \ldots (L_r, \Theta(L_r))$ form a basis of differentiable vector fields for Θ^* on all of $G \times M$; moreover by the homomorphism property of Θ we have $[(L_i, \Theta(L_i)), (L_j, \Theta(L_j))] = ([L_i, L_j], [\Theta(L_i), \Theta(L_j)]) = ([L_i, L_j], \Theta([L_i, L_j]))$, and so by [1, proposition 1, page 88] Θ^* is an involutive r-dimensional differential system on $G \times M$.

If Σ is a leaf of Θ^* and $(g,p) \varepsilon \Sigma$, then the tangent space to Σ at (g,p) is $\Theta^*_{(g,p)}$. Since $\delta\Pi_G(L_g, \Theta(L)_p) = L_g$, $\delta\Pi_G$ maps $\Theta^*_{(g,p)}$ is isomorphically onto the tangent space to G at g, and hence $\Pi_G \restriction \Sigma$ is a local diffeomorphism into G at (g,p).

Finally suppose that $(x_1 \ldots x_{r+n}, 0)$ is a cubical coordinate

coordinate system at (g,p) in $G \times M$ which is flat with respect Θ^*. Define $\sigma : M \to G \times M$ by $\sigma(q) = (g,q)$. Then σ is a differential map and we wish to show that $w_i = x_{r+i} \circ \sigma$, $i = 1 \ldots n$ is a coordinate system in a neighborhood of p. It suffices to show that the $(dw_i)_p$ are linearly independent, or since $dw_i = \delta\sigma^*(dx_{r+i})$, that the null space of $\delta\sigma_p^*$ is disjoint from the linear span of the $(dx_{r+i})_{(g,p)}$. Now as $(x_1 \ldots x_{r+n}, 0)$ is flat with respect to Θ^*, the $(dx_{r+i})_{(g,p)}$ form a base for the annihilator of $\Theta^*_{(g,p)}$. On the other hand, the null space of $\delta\sigma^*$ is the annihilator of the range of $\delta\sigma$ and, as the range of $\delta\sigma$ (= the set of vectors of the form $(0,Y)$ where 0 is the zero vector at g and Y any vector at p) is clearly supplementary to $\Theta^*_{(g,p)}$, its annihilator is in fact disjoint from the annihilator of $\Theta^*_{(g,p)}$.

COROLLARY. *If Θ is an infinitesimal G-transformation group acting on M then every leaf Σ of Θ^* satisfies the second axiom of countability, and any differentiable map of a differentiable manifold into $G \times M$ with its range in Σ is a differentiable map into Σ.*

PROOF. G is a connected Lie group and hence satisfies the second axiom of countability. Since Σ is connected and Π_G maps Σ locally diffeomorphically into G, the proof of [1, lemma 3, page 97] (with G playing the role of R^d) shows that Σ satisfies the second axiom of countability. The second statement of the corollary follows from [1, proposition 1, page 95].

The following rather technical result plays a central role in the further development of the theory.

THEOREM V. *Let Θ be an infinitesimal G-transformation group acting on M, and let Σ be a leaf of Θ^* which is Hausdorff in its manifold topology. If Π_G maps an open subset O of Σ one-to-one onto a connected subset V of G, then O is a component of $\Sigma \cap \Pi_G^{-1}(V)$ in the manifold topology of Σ.*

PROOF. Since O is an open submanifold of Σ it follows from theorem IV that Π_G maps O locally diffeomorphically onto V, and being one-to-one on O it actually maps O diffeomorphically onto V. It follows that O is connected. Hence, since O is open, it will suffice to show that O is closed in $\Sigma \cap \Pi_G^{-1}(V)$. Let (g,p) be any point of $\Sigma \cap \Pi_G^{-1}(V)$ not belonging to O; we shall show that (g,p) is not adherent to O in the manifold topology of Σ. Now there is a unique point of O whose first component is g, say (g,q). Since Σ is Hausdorff we can find Σ neighborhoods U and W of (g,p) and (g,q) respectively which are disjoint. Once again making use of the local diffeomorphism property of Π_G restricted to Σ, we can assume that Π_G maps each of U and W diffeomorphically onto the same neighborhood of g. Moreover since O is open in Σ, we can assume that $W \subseteq O$. It is then immediate from the fact that Π_G is one-to-one on O that $U \cap O \subseteq W$. Since $U \cap W$ is empty it follows that U does not meet O, and since U is a Σ neighborhood of (g,p), that (g,p) is not adherent to O in the topology of Σ.

Note that the topology of Σ induced by $G \times M$ is possibly *strictly* weaker than the manifold topology of Σ (as a leaf of Θ^*) and that it is necessary to distinguish these two possible topologies for Σ. However, using the fact that Σ satisfies the second axiom of countability (corollary of theorem IV), it can be shown that if O is open in the manifold topology of Σ, then the components of O are the same in both topologies. Thus, though we shall never need the fact, in the last line of theorem V we could replace 'manifold topology of Σ' by 'topology of $G \times M'$.

THEOREM VI. *The mapping* $\bar{R} : (g,h,p) \longrightarrow (hg^{-1},p)$ *of* $G \times G \times M$ *into* $G \times M$ *is a global G-transformation group acting on* $G \times M$ *which is compatible* (definition VI) *with the infinitesimal graph of every infinitesimal G-transformation group acting on* M. *If* $L \varepsilon \mathfrak{g}$ *then* $\bar{R}^+ (L) = (-\bar{L},0)$ *where* $\bar{L}$ *is* the left invariant vector field on G such that $\bar{L}_e = L_e$.

PROOF. That $\bar{R}$ is a global G-transformation group acting on $G \times M$ is obvious. If Θ is an infinitesimal G-transformation group acting on M and $L \varepsilon \mathfrak{g}$, then since $\bar{R}_g = R_g \times I$ and L is right invariant, it follows that $\delta\bar{R}_g(L_h,\Theta(L)_p) = (\delta R_g(L_h),\delta I(\Theta(L)_p)) = (L_{hg^{-1}}, \Theta(L)_p)$ and so $\delta\bar{R}_g\Theta^*_{(h,p)} = \Theta^*_{\bar{R}(g,h,p)}$ for all $(g,h,p) \varepsilon G \times G \times M$; so $\bar{R}$ is compatible with Θ^*.

Let J be the map $h \rightarrow h^{-1}$ of G onto itself. Given $g \varepsilon G$ and $p \varepsilon M$ let T be left translation on G by g and let f be the map $h \rightarrow p$ of G into M. Then $\bar{R}^{(g,p)} = (T \circ J) \times f$ so $\delta\bar{R}^{(g,p)} = \delta T \circ \delta J \oplus \delta f$. Now clearly $\delta f = 0$, and if $L \varepsilon \mathfrak{g}$ then the relation $J(\exp(tL)) = \exp(-tL)$ together with the fact that the tangent vector to the one parameter subgroup $t \rightarrow \exp(tL)$ at e is L_e implies that $\delta J(L_e) = -L_e$. Hence $\bar{R}^+(L)_{(g,p)} = \delta\bar{R}^{(g,p)}(L_e) = (\delta T \circ \delta J(L_e),\delta f(L_e)) = (\delta T(-L_e), 0) = (-\bar{L}_g, 0)$.

We shall henceforth be continually referring to the following corollary.

COROLLARY. *If* Θ *is an infinitesimal* G-*transformation group acting on* M , Σ *a leaf of* Θ^* , *and* $g\varepsilon G$, *then* $\Sigma' = \bar{R}_g(\Sigma)$ *is a leaf of* Θ^* *and* $\bar{R}_g$ *maps* Σ *diffeomorphically onto* Σ' . *If* V *is any open set in* G , *then* $\bar{R}_g$ *maps each component of* $\Sigma \cap \Pi_G^{-1}(V)$ *(relative to* Σ *) diffeomorphically onto a component of* $\Sigma' \cap \Pi_G^{-1}(Vg^{-1})$ *(relative to* Σ'*).*

PROOF. The first conclusion is an immediate consequence of the theorem and definition VI. Since $\bar{R}_g$ maps $G \times M$ diffeomorphically onto itself and Σ diffeomorphically onto Σ' it maps $\Sigma \cap \Pi_G^{-1}(V)$ diffeomorphically onto $\bar{R}_g(\Sigma) \cap \bar{R}_g \Pi_G^{-1}(V) = \Sigma' \cap \Pi_G^{-1}(Vg^{-1})$, and hence each component of $\Sigma \cap \Pi_G^{-1}(V)$ diffeomorphically onto a component of $\Sigma' \cap \Pi_G^{-1}(Vg^{-1})$.

THEOREM VII. *If* φ *is a local* G-*transformation group acting on* M , *then for each* $p\varepsilon M$ *the graph of* φ^p *is an open, connected submanifold of the leaf of* φ^{+*} *containing* (e,p) . *It is regularly imbedded in* $G \times M$ *and* Π_G *maps it diffeomorphically onto* $D_\varphi p$.

PROOF. Given $p\varepsilon M$, $D_\varphi p$ is an open, connected submanifold of G and $\Phi^p : g \rightarrow (g,\varphi(g,p))$ maps it differentiably into $G \times M$ and onto the graph of φ^p . Putting $q = \Phi^p(g)$ we have, by theorem II, that for each $L\varepsilon \mathfrak{G}$ $\delta\Phi^p(L_g) = (L_g, \delta\varphi^p(L_g)) = (L_g, \varphi^+(L)_{\varphi(g,p)}) = (L,\varphi^+(L))_q$. Hence the tangent space to $D_\varphi p$ is mapped isomorphically by $\delta\Phi^p$ onto φ_q^{+*} , and it follows that if we carry the manifold structure of $D_\varphi p$ over to the graph of φ^p via Φ^p then the graph of φ^p becomes a connected r-dimensional integral manifold of φ^{+*} , and hence an open submanifold of a leaf of φ^{+*} . Since $\Phi^p(e) = (e,p)$, this leaf is the one containing (e,p) . The inverse of Φ^p is Π_G restricted to the graph of Φ^p . Since Π_G is continuous on $G \times M$ it follows that Φ^p is a homeomorphism into $G \times M$, and hence that the graph of φ^p is regularly imbedded in $G \times M$. The last conclusion is also a consequence of the fact that Π_G on the graph of φ^p is inverse to Φ^p .

LEMMA. Let φ be a local G-transformation group acting on M , $p \in M$, and Σ the leaf of φ^{+*} containing (e,p) . If (e,q) is adherent to the graph of φ^p in the topology of Σ , then $p = q$.

PROOF. Let O be a neighborhood of q and V a symmetric neighborhood of e such that $V \times O \subseteq D_\varphi$. Let U be a neighborhood of (e,q) in the graph of φ^q such that $U \subseteq V \times O$. By theorem VII U is a neighborhood of (e,q) in Σ . Since (e,q) is adherent to the graph of φ^p in the topology of Σ , there exists a point $(g,\varphi(g,p)) \in U \cap$ graph of φ^p . Then since $(g,\varphi(g,p)) \in U \subseteq$ graph of φ^q it follows that $\varphi(g,p) = \varphi(g,q)$. Now $g^{-1} \in \Pi_G(U)^{-1} \subseteq V^{-1} = V$ and $\varphi(g,p) \in \Pi_M(U) \subseteq O$ so $(g^{-1},\varphi(g,p)) = (g^{-1},\varphi(g,q)) \in V \times O \subseteq D_\varphi$. Since (e,p) and (e,q) are also in D_φ , it follows from definition II that $p = \varphi(e,p) = \varphi(g^{-1},\varphi(g,p)) = \varphi(g^{-1},\varphi(g,q)) = \varphi(e,q) = q$.

THEOREM VIII. If Θ is a generating infinitesimal G-transformation group acting on M , then every leaf of Θ^* is a Hausdorff manifold.

PROOF. Let φ be a local G-transformation group generated by Θ and let Σ be a leaf of $\Theta^* = \varphi^{+*}$. Let (g,p) and (g',q) be points of Σ that cannot be separated by open sets of Σ . Since Σ is a sub-manifold of $G \times M$, its manifold topology is stronger than the topology induced from $G \times M$; hence (g,p) and (g',q) cannot be separated by open sets of $G \times M$. Since G is Hausdorff it follows that $g = g'$. By the corollary of theorem VI $\bar{R}_g$ maps Σ diffeomorphically onto the leaf Σ' of φ^* containing (e,p), hence (e,p) and (e,q) are points of Σ' that cannot be separated by open sets of Σ'. By theorem VII the graph of φ^p is a neighborhood of (e,p) in Σ' and hence meets each Σ' neighborhood of (e,q). Thus (e,q) is adherent to the graph of φ^p in the topology of Σ' and hence, by the lemma, $p = q$. Thus $(g,p) = (g',q)$, which proves that Σ is Hausdorff.

COROLLARY I. *Let Θ be an infinitesimal G-transformation group acting on M, $p \varepsilon M$, Σ the leaf of Θ^* containing (e,p), and φ any local G-transformation group generated by Θ. Then the graph of φ^p is the component of (e,p) in $\Sigma \cap \Pi_G^{-1}(D_\varphi p)$ with respect to the manifold topology of Σ.*

PROOF. By the theorem Σ is a Hausdorff manifold. By theorem VII the graph of φ^p is an open submanifold of Σ which Π_G maps one-to-one onto $D_\varphi p$. Since $D_\varphi p$ is connected theorem V completes the proof.

COROLLARY II. *There exists a non-generating infinitesimal G-transformation group.*

PROOF. Choose $e' \notin G$ and let $M = G \cup \{e'\}$. We make M into an analytic manifold by decreeing that G and $M - \{e\}$ shall be open submanifolds and that the map $g \to g$ $(g \neq e)$, $e \to e'$ shall be a diffeomorphism of G onto $M - \{e\}$. Given $L \varepsilon \mathfrak{g}$ there is a unique analytic vector field $\Theta(L)$ on M such that $\Theta(L) \restriction G = L$. Clearly Θ is an infinitesimal G-transformation group acting on M. The leaf of Θ^* containing (e,e) consists of the points (g,g) with $g \varepsilon G$ and the point (e,e'), and Π_M maps it diffeomorphically onto M. Clearly (e,e) and (e,e') cannot be separated by open sets of this leaf so it is not

Hausdorff. By the theorem Θ is not generating.

According to the above theorem, a necessary condition that an infinitesimal G-transformation group acting on M be generating is that each leaf of Θ^* is a Hausdorff manifold. Theorem XI states that this condition is also sufficient, so that if M is Hausdorff then every infinitesimal G-transformation group acting on M is generating.

5. The Local Existence Theorem.

LEMMA. *Let* Θ *be an infinitesimal* G-*transformation group acting on* M , $p \varepsilon M$, *and* $(x_1 \dots x_{r+n}, Q)$ *a coordinate system in* $G \times M$ *centered at* (e,p) *and flat with respect to* Θ^* . *If* $(z_1 \dots z_r, V)$ *is a coordinate system in* G *centered at* e , *and if we define* $\bar{z}_i$ $(i = 1 \dots r)$ *in* $\Pi_G^{-1}(V)$ *by* $\bar{z}_i = z_i \circ \Pi_G$, *then there is a neighborhood* O *of* (e,p) *in* $G \times M$ *such that* $(\bar{z}_1 \dots \bar{z}_r, x_{r+1} \dots x_{r+n}, O)$ *is a coordinate system in* $G \times M$.

PROOF. Let S be the null space of $(\delta\Pi_G)_{(e,p)}$, i.e. the set of all vectors tangent to $G \times M$ at (e,p) which are of the form $(0,Y)$. Then $(d\bar{z}_1)_{(e,p)} \dots (d\bar{z}_r)_{(e,p)}$ is a basis for the annihilator of S . On the other hand, since $(x_1 \dots x_{r+n}, Q)$ is flat with respect to Θ^* , $(dx_{r+1})_{(e,p)} \dots (dx_{r+n})_{(e,p)}$ is a basis for the annihilator of $\Theta^*_{(e,p)}$ Then as S and $\Theta^*_{(e,p)}$ are clearly supplementary subspaces of the tangent space to G at (e,p) , it follows that $(d\bar{z}_1)_{(e,p)} \dots (d\bar{z}_n)_{(e,p)}, (dx_{r+1})_{(e,p)} \dots (dx_{r+n})_{(e,p)}$ is a basis for the space of differentials at (e,p) , and the lemma is an immediate consequence of this.

THEOREM IX. *Let Θ be an infinitesimal G-transformation group acting on M such that each leaf of Θ^* is a Hausdorff manifold, and for each $q \in M$ denote by Σ_q the leaf of Θ^* which contains (e,q). Given $p \in M$ there exists an open neighborhood U of p such that if V is any sufficiently small open, connected neighborhood of e, then there is a unique diffeomorphism ψ of $V \times U$ into $G \times M$ satisfying the two conditions*

(1) $\Pi_G \circ \psi = \Pi_G \upharpoonright V \times U$

(2) *for each* $q \in U$ *the map* $g \to \psi(g,q)$ *is a diffeomorphism of* V *onto the connected component of* (e,q) *in* $\Sigma_q \cap \Pi_G^{-1}(V)$ (*relative to* Σ_q).

PROOF. Let $(z_1 \ldots z_r, V)$ be a coordinate system in G centered at e, and define $\bar{z}_i$ $(i = 1 \ldots r)$ in $\Pi_G^{-1}(V)$ by $\bar{z}_i = z_i \circ \Pi_G$. Let $(x_1 \ldots x_{r+n}, Q)$ be a flat coordinate system with respect to Θ^* centered at (e,p). If we define $w_1 \ldots w_n$ by $w_i(q) = x_{r+i}(e,q)$, then by theorem IV there is a neighborhood U of p such that $(w_1 \ldots w_n, U)$ is a coordinate system in M. Define $\bar{w}_i$ $(i = 1 \ldots n)$ on $\Pi_M^{-1}(U)$ by $\bar{w}_i = w_i \circ \Pi_M$. By definition of the manifold structure on $G \times M$, $(\bar{z}_1 \ldots \bar{z}_r, \bar{w}_1 \ldots \bar{w}_n, V \times U)$ is a coordinate system in $G \times M$. By the lemma we can find a neighborhood O of (e,p) such that $(\bar{z}_1 \ldots \bar{z}_r, x_{r+1} \ldots x_{r+n}, O)$ is also a coordinate system in $G \times M$. By reducing the sets O, U, and V we can suppose that both of the above coordinate systems in $G \times M$ are cubical and of breadth 2a for some $a > 0$. Then there is a uniquely determined diffeomorphism ψ of $V \times U$ onto O such that

(1') $\bar{z}_i \circ \psi = \bar{z}_i$,

(2') $x_{r+i} \circ \psi = \bar{w}_i$.

Since $z_i \circ \Pi_G = \bar{z}_i = \bar{z}_i \circ \psi = z_i \circ \Pi_G \circ \psi$, and the z_i are a coordinate system in V it follows that $\Pi_G \circ \psi = \Pi_G \restriction V \times U$ which

If $q \varepsilon U$ and $g \varepsilon V$, then by (2') $x_{r+i}(\psi(g,q)) = \bar{w}_i(g,q) = w_i$ $x_{r+i}(e,q)$. Since ψ is onto O it follows that $\psi(V,q)$ is the r-dimensional slice of $(x_1 \dots x_{r+n}, O)$ defined by $(x_{r+1}(e,q) \dots$ $x_{r+n}(e,q))$. Then as $(x_1 \dots x_{r+n}, O)$ is flat with respect to follows that $\psi(V,q)$ is an open submanifold of Σ_q . Now by (1) $\Pi_G(\psi(g,q)) = g$, so Π_G maps $\psi(V,q)$ one-to-one onto V . Since connected and Σ_q is by hypothesis a Hausdorff manifold, it foll theorem V that $\psi(V,q)$ is the connected component of (e,q) in $\Sigma_q \cap \Pi_G^{-1}(V)$. By theorem IV Π_G maps $\psi(V,q)$ diffeomorphically and hence the inverse of $\Pi_G \restriction \psi(V,q)$, which is $g \rightarrow \psi(g,q)$, map diffeomorphically onto $\psi(V,q)$ = the component of (e,q) in Σ_q This proves (2).

Clearly if V' is any open, connected neighborhood of e in V then $\psi \restriction V' \times U$ also has the desired properties.

COROLLARY. *If* Θ *is an infinitesimal* G-*transformation group acting on* M *such that every leaf of* Θ^* *is a Hausd manifold, then given any* $p \varepsilon M$ *there exists a neighborhood of* p *and an open, connected neighborhood* V *of* e *such for each* $q \varepsilon U$ Π_G *maps the component of* (e,q) *in* $\Sigma_q \cap \Pi$ (*with respect to the manifold topology of* Σ_q) *diffeomorph onto* V .

Lie's Second Fundamental Theorem, in essentially its classi form, follows easily from theorem IX. Define φ on $V \times U$ by and let $O = \{ (g,p) \varepsilon V \times U : \varphi(g,p) \varepsilon U \}$. Then O is open in and $\{e\} \times U \subseteq O$. If D is the largest local transformation domain in O (theorem I), then $\varphi \upharpoonright D$ is a local G-transformation group acting on U and $(\varphi \upharpoonright D)^{+}(L) = \Theta(L) \upharpoonright U$ for all $L \varepsilon \mathfrak{g}$. However, we omit details since we shall show in section 7 that we can actually paste together a lot of φ's in such a way as to get a local G-transformation group acting on <u>all</u> of M which is generated by Θ.

The corollary of theorem IX is the last result of this chapter that will be needed in later chapters. The results of the next two sections, while of some interest in themselves, are rather complicated, and since they do not seem to lead anywhere in particular the reader may well prefer to skip to chapter III where the theory becomes considerably more interesting and elegant.

6. The Uniqueness Theorem.

We are now able to derive some important information concerning the order properties of the set of local G-transformation groups acting on M under the partial ordering relation 'φ is a restriction of ψ'. As a consequence we get a uniqueness theorem for local G-transformation groups acting on M with a given domain and a given infinitesimal generator.

THEOREM X. *Let the set of local G-transformation groups acting on M be partially ordered by restriction, i.e. we say 'φ is less than ψ' if $\varphi = \psi \upharpoonright D_{\varphi}$. Then a necessary and sufficient condition that two local G-transformation groups, φ and ψ, have a lower bound is that they have the same infinitesimal generator. If this is the case then φ and ψ actually have a greatest lower bound, σ, whose domain D is the largest local transformation group domain included in $D_{\varphi} \cap D_{\psi}$, (see the corollary of theorem I).*

PROOF. Suppose that φ and ψ have a lower bound θ . Then for any $p \varepsilon M$ $\theta^p = \varphi^p \restriction D_\theta p$, and since $D_\theta p$ is a neighborhood of e it follows that for any $L \varepsilon \mathfrak{G}$ we have $\varphi^+(L)_p = \delta\varphi^p(L_e) = \delta\theta^p(L_e) = \theta^+(L)_p$ so $\varphi^+ = \theta^+$. Similarly $\psi^+ = \theta^+$ so $\varphi^+ = \psi^+$, i.e. φ and ψ have the same infinitesimal generator.

Conversely suppose that φ and ψ have the same infinitesimal generator, Θ , and for each $p \varepsilon M$ denote by Σ_p the leaf of Θ^* containing (e,p) . Let D_p be the component of (e,p) in $D_\varphi p \cap D_\psi p$, so that by the corollary of theorem I $D = \bigcup_{p \varepsilon M} (D_p \times \{p\})$. By corollary 1 of theorem VII, given $p \varepsilon M$ the graph of φ^p is the component of (e,p) in $\Sigma_p \cap \Pi_G^{-1}(D_\varphi p)$. Since Π_G maps the graph of φ^p diffeomorphically onto $D_\varphi p$ (theorem VII), it follows that the graph of $\varphi^p \restriction D_p$ is the component of (e,p) in $\Sigma_p \cap \Pi_G^{-1}(D_p)$. Similarly the graph of $\psi^p \restriction D_p$ is also the component of (e,p) in $\Sigma_p \cap \Pi_G^{-1}(D_p)$. Hence $\varphi^p \restriction D_p = \psi^p \restriction D_p$, and since p is arbitrary, $\varphi \restriction D = \psi \restriction D$. Putting σ for the common restriction of φ and ψ to D , it is clear from the fact that D is the largest local transformation group domain included in $D_\varphi \cap D_\psi$ that σ is a greatest lower bound for φ and ψ .

COROLLARY 1. *If φ and ψ are two local G-transformation groups acting on M with the same infinitesimal generator and $D_\varphi \subseteq D_\psi$, then $\varphi = \psi \restriction D_\varphi$.*

COROLLARY 2. *A local G-transformation group acting on M is uniquely determined by its domain and its infinitesimal generator.*

COROLLARY 3. *If Θ is an infinitesimal G-transformation group acting on M , and φ a local G-transformation group generated by Θ , then there is a local G-transformation group*

ψ *acting on* M , *maximal under the ordering by restriction, such that* $\varphi = \psi \restriction D_\varphi$. *Any such* ψ *has* Θ *as its generator.*

PROOF. The existence of ψ follows by an elementary application of Zorn's lemma, and that $\psi^+ = \Theta$ follows from the theorem.

COROLLARY 4. *If* Θ *is an infinitesimal* G-*transformation group acting on* M , *then the following three properties are* equivalent:

(1) Θ *is generating and under the partial ordering by restriction any two local* G-*transformation groups generated by* Θ *have an upper bound.*

(2) *The set of local* G-*transformation groups generated by* Θ *have a maximum element under the ordering by restriction.*

(3) *The set of local* G-*transformation groups generated by* Θ *form a non-empty lattice under the ordering by restriction.*

PROOF. Suppose that (1) holds. Since Θ is generating, it follows from corollary 3 that there is a maximal G-transformation group, σ , generated by Θ . If φ is any local G-transformation group generated by Θ , then φ and σ have an upper bound which, by the maximality of σ , must be σ itself. This shows that σ is actually a maximum element in the set of all local G-transformation groups generated by Θ . Thus 1) implies (2).

Next suppose that (2) holds and let σ be the maximum element of the et of local G-transformation groups generated by Θ . Then, if φ and ψ re two local G-transformation groups generated by Θ , it is clear from rollary of theorem I that $\sigma \restriction D_\varphi \cup D_\psi$ is a least upper bound for φ d ψ , so (2) implies (3).

That (3) implies (1) is obvious.

An infinitesimal G-transformation group, Θ, acting on M is called <u>univalent</u> if for each $p \varepsilon M$ the leaf Σ_p of Θ^* containing (e,p) is mapped one-to-one (and hence by theorem IV diffeomorphically) into G by Π_G. The theory of univalent infinitesimal groups is very rich and elegant, and we will devote chapter III to it. For the present we will content ourselves with a few elementary remarks. Suppose then that Θ is univalent. It follows from theorem XI of the next section that Θ is generating. If φ and Ψ are two local G-transformation groups generated by Θ and $(g,p) \varepsilon D_\varphi \cap D_\psi$, then by theorem IV it follows that $(g,\varphi(g,p))$ and $(g,\psi(g,p))$ both belong to Σ_p. Since Π_G maps Σ_p one-to-one it follows that $\varphi(g,p) = \psi(g,p)$ and hence φ and ψ agree on their common domain. Then if we define θ on $D_\varphi \cup D_\psi$ by $\theta \upharpoonright D_\varphi = \varphi$ and $\theta \upharpoonright D_\psi = \psi$, θ is an upper bound for φ and ψ. Thus univalent infinitesimal groups satisfy the conditions of corollary 4 (we do not know if, conversely, every infinitesimal group satisfying the properties of corollary 4 is univalent, but we suspect so). The maximum local G-transformation group generated by a univalent infinitesimal group can be constructed quite explicitly, without any appeal to Zorn's lemma: in fact, as we shall see in chapter III, it is uniquely characterized by the property that for each $p \varepsilon M$ the graph of φ^p is the entire leaf Σ_p.

7. The Existence Theorem.

In this section Θ will be an infinitesimal G-transformation group acting on M <u>such that each leaf of</u> Θ^* <u>is Hausdorff</u>. The latter hypothesis will be necessary in order to apply theorem IX. Our goal is to show that Θ is generating, and we shall give a hammer and tongs construction of a local G-transformation group acting on M which is generated by Θ. As usual for each $p \varepsilon M$ we will denote by Σ_p the leaf of Θ^* containing (e,p). Whenever we speak of a component of a subset of Σ_q we shall mean with respect to the manifold topology of Σ_q as a leaf of Θ^* and not with respect to the possibly weaker topology induced from $G \times M$.

Let $(z_1 \dots z_r, W)$ be a fixed canonical coordinate system in G. By <u>a cubical neighborhood of</u> e we shall mean a subset of G which is an open cube centered at e with respect to $(z_1 \dots z_r, W)$. We note that the cubical neighborhoods of e form a linearly ordered (by inclusion) basis of neighborhoods of e each of which is open, connected, and symmetric. These are the only relevant properties for what follows.

DEFINITION VIII. Let V be a cubical neighborhood of e, U an open set in M, and ψ a diffeomorphism of $V \times U$ into $G \times M$. We shall call ψ an <u>auxiliary map</u> if the following three conditions are satisfied.

(1) $\Pi_G \circ \psi = \Pi_G \upharpoonright V \times U$

(2) For each $q \varepsilon U$, $g \to \psi(g,q)$ maps V diffeomorphically onto the component of (e,q) in $\Sigma_q \cap \Pi_G^{-1}(V)$.

(3) For each $q \varepsilon U$, Π_G maps the component of (e,q) in $\Sigma_q \cap \Pi_G^{-1}(V^2)$ one-to-one onto V^2.

We note in passing that if we were to replace V^2 by V in condition (3), then we would get a statement which is an immediate consequence of conditions (1) and (2).

LEMMA a. <u>For each</u> $p \varepsilon M$ <u>there is a neighborhood</u> U <u>of</u> p <u>such that if</u> W <u>is any sufficiently small cubical neighborhood of</u> e, <u>then there is a unique auxiliary map of</u> $W \times U$ <u>into</u> $G \times M$.

PROOF. Choose $\psi : V \times U \to G \times M$ as in theorem IX, and let O be a neighborhood of e such that $O^2 \subseteq V$. Then if W is any cubical neighborhood of e which is included in O, $\psi \upharpoonright W \times U$ is the unique auxiliary map of $W \times U$ into $G \times M$.

LEMMA b. Let $\psi : V \times U \to G \times M$ be an auxiliary map, and let $p \varepsilon U$. Then $\psi(e,p) = (e,p)$ and if $\psi^p : V \to G \times M$ is defined by $\psi^p(g) = \psi(g,p)$, then for any $L \varepsilon \mathfrak{g}$ $\Theta(L)_p = \delta\Pi_M \circ \delta\psi^p(L_e)$.

PROOF. $\psi(e,p) \varepsilon$(component of (e,p) in $\Sigma_p \cap \Pi_G^{-1}(V)$). But of course $(e,p) \varepsilon$(component of (e,p) in Σ_p $\Pi_G^{-1}(V)$) also. Then since Π_G maps the component of (e,p) in $\Sigma_p \cap \Pi_G^{-1}(V)$ one-to-one and $\Pi_G \psi(e,p) = e = \Pi_G(e,p)$, it follows that $\psi(e,p) = (e,p)$.

By (2) of definition VIII, ψ^p maps V diffeomorphically onto the component of (e,p) in $\Sigma_p \cap \Pi_G^{-1}(V)$. Now the latter is an open submanifold of Σ_p and hence its tangent space at (e,p) is $\Theta^*_{(e,p)}$, so by the first part of the lemma, if $L \varepsilon \mathfrak{g}$ then $\delta\psi^p(L_e) \varepsilon \Theta^*_{(e,p)}$. On the other hand, by (1) of definition VIII $\Pi_G \circ \psi^p$ maps V identically, so $\delta\Pi_G \circ \delta\psi^p(L_e) = L_e$. Now $(L_e, \Theta(L)_p)$ is the only element of $\Theta^*_{(e,p)}$ which is mapped onto L_e by $\delta\Pi_G$, hence $\delta\Pi_M \circ \delta\psi^p(L_e) = \delta\Pi_M(L_e, \Theta(L)_p) = \Theta(L)_p$.

LEMMA c. Two auxiliary maps agree in the common part of their domains.

PROOF. Let $\psi : V \times U \to G \times M$ and $\psi' : V' \times U' \to G \times M$ be two auxiliary maps. As both V and V' are cubical neighborhoods of e, either $V \subseteq V'$ or $V' \subseteq V$, and for definiteness we assume the latter. Let $(g,p) \varepsilon (V \times U) \cap (V' \times U')$. Then $\psi'(g,p) \varepsilon$(component of (e,p) in $\Sigma_p \cap \Pi_G^{-1}(V')) \subseteq$ (component of (e,p) in $\Sigma_p \cap \Pi_G^{-1}(V)$). Also $\psi(g,p) \varepsilon$(component of (e,p) in $\Sigma_p \cap \Pi_G^{-1}(V)$). Then since $\Pi_G(\psi(g,p)) = g = \Pi_G(\psi'(g,p))$, and Π_G maps the component of (e,p) in $\Sigma_p \cap \Pi_G^{-1}(V)$ one-to-one, it follows that $\psi(g,p) = \psi'(g,p)$.

LEMMA d. *Let* $\psi : V \times U \to G \times M$ *be an auxiliary map and let* $\varphi = \Pi_M \circ \psi$. *If* $(h,p) \varepsilon V \times U$, *then* $\bar{R}_h$ *maps* Σ_p *diffeomorphically onto* $\Sigma_{\varphi(h,p)}$, *and if* $W \subseteq G$ *then* $\bar{R}_h$ *maps any component of* $\Sigma_p \cap \Pi_G^{-1}(W)$ *diffeomorphically onto a component of* $\Sigma_{\varphi(h,p)} \cap \Pi_G^{-1}(Wh^{-1})$.

PROOF. Since $(h,\varphi(h,p)) = \psi(h,p) \varepsilon$(component of (e,p) in $\Sigma_p \cap \Pi_G^{-1}(V)$), in particular $(h,\varphi(h,p)) \varepsilon \Sigma_p$ so $(e,\varphi(h,p)) \varepsilon \bar{R}_h(\Sigma_p)$. The lemma now follows from the corollary of theorem VI.

LEMMA e. *Let* $\psi : V \times U \to G \times M$ *be an auxiliary map, and let* $\varphi = \Pi_M \circ \psi$. *If* (h,p) , $(g,\varphi(h,p))$, *and* (gh,p) *all are contained in* $V \times U$, *then* $\varphi(gh,p) = \varphi(g,\varphi(h,p))$.

PROOF. Since $V = Vhh^{-1} \subseteq V^2h^{-1}$, $(g,\varphi(g,\varphi(h,p))) = \psi(g,\varphi(h,p)) \varepsilon$ (component of $(e,\varphi(h,p))$ in $\Sigma_{\varphi(h,p)} \cap \Pi_G^{-1}(V)) \subseteq$ (component of $(e,\varphi(h,p))$ in $\Sigma_{\varphi(h,p)} \cap \Pi_G^{-1}(V^2h^{-1})$. It follows that $(gh,\varphi(g,\varphi(h,p))) \varepsilon$ (component of $(h,\varphi(h,p))$ in $\Sigma_p \cap \Pi_G^{-1}(V^2)$) , from lemma d. Now $(h,\varphi(h,p)) = \psi(h,p) \varepsilon$(component of (e,p) in $\Sigma_p \cap \Pi_G^{-1}(V)) \subseteq$ (component of (e,p) in $\Sigma_p \cap \Pi_G^{-1}(V^2)$) , hence $(gh,\varphi(g,\varphi(h,p))) \varepsilon$(component of (e,p) in $\Sigma_p \cap \Pi_G^{-1}(V^2)$) . On the other hand, $(gh,\varphi(gh,p)) = \psi(gh,p) \varepsilon$ (component of (e,p) in $\Sigma_p \cap \Pi_G^{-1}(V)) \subseteq$ (component of (e,p) in $\Sigma_p \cap \Pi_G^{-1}(V^2)$. Since Π_G maps the component of (e,p) in $\Sigma_p \cap \Pi_G^{-1}(V^2)$ one-to-one, and $\Pi_G(gh,\varphi(gh,p)) = gh = \Pi_G(gh,\varphi(g,\varphi(h,p)))$, it follows that $\varphi(gh,p) = \varphi(g,\varphi(h,p))$.

LEMMA f. *Let* $\psi : V \times U \to G \times M$ *and* $\psi' : V' \times U' \to G \times M$ *be two auxiliary maps,* $\varphi = \Pi_M \circ \psi$, *and* $\varphi' = \Pi_M \circ \psi'$. *If* (h,p) *and* (gh,p) *belong to* $V \times U$ *and* $(g,\varphi(h,p))$ *is in* $V' \times U'$, *then* $\varphi(gh,p) = \varphi'(g,\varphi(h,p))$.

PROOF. Case I : $V \subseteq V'$. Both $(h,\varphi(h,p)) = \psi(h,p)$ and $(gh,\varphi(gh,p)) = \psi(gh,p)$ belong to the component of (e,p) in $\Sigma_p \cap \Pi_G^{-1}(V)$, so $(gh,\varphi(gh,p)) \varepsilon$(component of $(h,\varphi(h,p))$ in $\Sigma_p \cap \Pi_G^{-1}(V)$) . It follows from lemma d that $(g,\varphi(gh,p)) \varepsilon$(component of $(e,\varphi(h,p))$ in $\Sigma_{\varphi(h,p)} \cap \Pi_G^{-1}(Vh^{-1})$. Now $Vh^{-1} \subseteq VV^{-1} = V^2 \subseteq V'^2$ so *a fortiori* $(g,\varphi(gh,p)) \varepsilon$(component of $(e,\varphi(h,p))$ in $\Sigma_{\varphi(h,p)} \cap \Pi_G^{-1}(V'^2)$). On the other hand, $(g,\varphi'(g,\varphi(h,p))) = \psi'(g,\varphi(h,p)) \varepsilon$(component of $(e,\varphi(h,p))$ in $\Sigma_{\varphi(h,p)} \cap \Pi_G^{-1}(V')) \subseteq$ (component of $(e,\varphi(h,p))$ in $\Sigma_{\varphi(h,p)} \cap \Pi_G^{-1}(V'^2))$. Since Π_G maps the component of $(e,\varphi(h,p))$ in $\Sigma_{\varphi(h,p)} \cap \Pi_G^{-1}(V'^2)$ one-to-one, and $\Pi_G(g,\varphi(gh,p)) = g = \Pi_G(g,\varphi'(g,\varphi(h,p)))$ it follows that $\varphi(gh,p) = \varphi'(g,\varphi(h,p))$.

Case II: $V' \subseteq V$. Since $(g,\varphi'(g,\varphi(h,p))) = \psi'(g,\varphi(h,p)) \varepsilon$ (component of $(e,\varphi(h,p))$ in $\Sigma_{\varphi(h,p)} \cap \Pi_G^{-1}(V'))$, it follows from lemma d that $(gh,\varphi'(g,\varphi(h,p))) \varepsilon$(component of $(h,\varphi(h,p))$ in $\Sigma_p \cap \Pi_G^{-1}(V'h))$. Since $V'h \subseteq V'V \subseteq V^2$, *a fortiori* $(gh,\varphi'(g,\varphi(h,p))) \varepsilon$(component of $(h,\varphi(h,p))$ in $\Sigma_p \cap \Pi_G^{-1}(V^2))$. Now $(h,\varphi(h,p)) = \psi(h,p) \varepsilon$(component of (e,p) in $\Sigma_p \cap \Pi_G^{-1}(V)) \subseteq$ (component of (e,p) in $\Sigma_p \cap \Pi_G^{-1}(V^2))$, hence $(gh,\varphi'(g,\varphi(h,p))) \varepsilon$(component of (e,p) in $\Sigma_p \cap \Pi_G^{-1}(V^2))$. But also $(gh,\varphi(gh,p)) = \psi(gh,p) \varepsilon$(component of (e,p) in $\Sigma_p \cap \Pi_G^{-1}(V)) \subseteq$ (component of (e,p) in $\Sigma_p \cap \Pi_G^{-1}(V^2))$. Since Π_G is one-to-one on the component of (e,p) in $\Sigma_p \cap \Pi_G^{-1}(V^2)$, and $\Pi_G(gh,\varphi'(g,\varphi(h,p))) = \Pi_G(gh,\varphi(gh,p))$, $\varphi(gh,p) = \varphi'(g,\varphi(h,p))$.

LEMMA g. *Let* $\psi : V \times U \to G \times M$ *and* $\psi' : V' \times U' \to G \times M$ *be two auxiliary maps, and let* $D = (V \times U) \cup (V' \times U')$. *Then there is a uniquely determined map* φ *of* D *into* M *such that* $\varphi \restriction V \times U = \Pi_M \circ \psi$ *and* $\varphi \restriction V' \times U' = \Pi_M \circ \psi'$. *Moreover*

(1) *if* $p \varepsilon U \cup U'$ *then* $\varphi(e,p) = p$,

(2) *if* (h,p), $(g,\varphi(h,p))$, *and* (gh,p) *are all in* D , *then*

$\varphi(gh,p) = \varphi(g,\varphi(h,p))$.

PROOF. The existence and uniqueness of φ follows directly from lemma c, and that φ satisfies (1) is an immediate consequence of lemma b. In proving (2) we can by symmetry assume that $V' \subseteq V$. If $p\varepsilon U$ then $(h,p)\varepsilon V \times U$ and $(gh,p)\varepsilon V \times U$, so the relation $\varphi(gh,p) = \varphi(g,\varphi(h,p))$ follows from lemma e if $(g,\varphi(h,p))\varepsilon V \times U$ and from lemma f if $(g,\varphi(h,p))\varepsilon V' \times U'$. If $p \not\varepsilon U$, then $(h,p)\varepsilon V' \times U'$ and $(gh,p)\varepsilon V' \times U'$, so the relation $\varphi(gh,p) = \varphi(g,\varphi(h,p))$ follows from lemma e if $(g,\varphi(h,p))\varepsilon V' \times U'$ and from lemma f if $(g,\varphi(h,p))\varepsilon V \times U$.

LEMMA h. Let $\{\psi_\alpha : V_\alpha \times U_\alpha \longrightarrow G \times M\}_{\alpha \varepsilon A}$ be the set of all auxiliary maps, and let $D = \bigcup_{\alpha \varepsilon A} (V_\alpha \times U_\alpha)$. Then D is a local transformation group domain in $G \times M$, and there is a unique differentiable map, φ , of D into M such that for all $\alpha \varepsilon A$ $\varphi \restriction V_\alpha \times U_\alpha = \Pi_M \circ \psi_\alpha$. This map φ is a local G-transformation group acting on M which is generated by Θ .

PROOF. Let $A_p = \{\alpha \varepsilon A : p\varepsilon U_\alpha\}$ for each $p\varepsilon M$, and let $D_p = \bigcup_{\alpha \varepsilon A_p} V_\alpha$. By lemma a A_p is not empty, and since each V is a cubical neighborhood of e , so is D_p , so in particular each D_p is a connected neighborhood of e . Now as each V_α and U_α is open, so is D , and since D_p is clearly $\{g\varepsilon G : (g,p)\varepsilon D\}$ it follows from definition I that D is a local transformation group domain in $G \times M$. The existence and uniqueness of φ is immediate from lemma g, and that φ is a local G-transformation group follows from lemma g and the fact that D is a local transformation group domain in $G \times M$.

Let $p\varepsilon M$ and let $\psi : V \times U \longrightarrow G \times M$ be an auxiliary map with $p\varepsilon U$.

If we define $\psi^p : V \to G \times M$ by $\psi^p(g) = \psi(g,p)$ then clearly $\Pi_M \circ \psi^p = \varphi^p \circ V$. Since V is a neighborhood of e it follows that $(\delta\varphi^p)_e = \delta\Pi_M \circ (\delta\psi^p)_e$. Thus if $L \varepsilon \mathfrak{g}$, then by lemma b, $\varphi^+(L)_p = \delta\varphi^p(L_e) = \delta\Pi_M \circ \delta\psi^p(L_e) = \Theta(L)_p$; hence $\varphi^+ = \Theta$, so φ is generated by Θ.

THEOREM XI. *A necessary and sufficient condition that an infinitesimal G-transformation group, Θ, acting on M be generating is that each leaf of Θ^* be a Hausdorff manifold.*

PROOF. Necessity follows from theorem VIII, and sufficiency from lemma h above.

COROLLARY. *If M is a Hausdorff manifold, then every infinitesimal G-transformation group acting on M is generating.*

In view of the above corollary the reader may feel that the author would have been well advised to disallow non-Hausdorff manifolds in the first place. However, non-Hausdorff manifolds occur naturally, and as it were of their own accord, in chapter III. It is to prepare for this, and not out of a misdirected desire for generality, that we have allowed them.

Chapter III

GLOBALIZABLE INFINITESIMAL TRANSFORMATION GROUPS

In this chapter we will be concerned with the question of when an infinitesimal G-transformation group acting on a manifold M generates a global transformation group. If an infinitesimal G-transformation group, Θ, generates a global G-transformation group, φ, acting on M, and F is a closed subset of M which is not invariant under all the transformations φ_g, then the restriction of Θ to the open submanifold $M - F$ of M will no longer generate a global transformation group. However, it is clear that such a restricted infinitesimal transformation group is in no way inherently pathological: there are just not enough points around. For this reason, we introduce the notion of a 'globalizable' infinitesimal transformation group, one that generates a global transformation group acting on a manifold which includes the given manifold as an open submanifold. It is to the rich and elegant theory of such infinitesimal transformation groups that we now turn.

1. Globalizations.

DEFINITION I. Let Θ be an infinitesimal G-transformation group acting on M, and let O be an open submanifold of M. The <u>restriction of Θ to O</u>, denoted by $\Theta \restriction O$, is the infinitesimal G-transformation group acting on O defined by $\Theta \restriction O(L) = \Theta(L) \restriction O$ for all $L \in \mathfrak{g}$.

DEFINITION II. Let Θ be an infinitesimal G-transformation group acting on M. A <u>globalization</u> of Θ is a pair (M^*, φ) such that:

(1) M^* is a differentiable manifold an M is an open submanifold of M^*.

(2) φ is a global G-transformation group acting on M^*.

(3) $\Theta = \varphi^+ \restriction M$.

(4) Given $q \varepsilon M^*$ there exists $(g,p) \varepsilon G \times M$ such that $\varphi(g,p) = q$.

A globalization (M^*, φ) of Θ will be called <u>proper</u> if $M^* = M$.

We shall call Θ <u>globalizable</u> if it admits a globalization and <u>proper</u> if it admits a proper globalization.

We note that if (1), (2), and (3) of the above definition hold, and M^{**} is the set of $q \varepsilon M^*$ for which (4) holds, then M^{**} is an open submanifold of M^* and $(M^{**}, \varphi \restriction G \times M^{**})$ is a globalization of Θ.

DEFINITION III. Let (M^*, φ) and (M', ψ) be two globalizations of the same infinitesimal G-transformation group acting on M. A <u>homomorphism of</u> (M^*, φ) <u>into</u> (M', ψ) is a mapping f (not assumed to be continuous) of M^* into M' such that:

(1) $f \restriction M$ is the identity map of M.

(2) $f \circ \varphi_g = \psi_g \circ f$ for all $g \varepsilon G$.

If (M^*, φ) admits a homomorphism into (M', ψ), then we say that (M', ψ) is homomorphic to (M^*, φ).

THEOREM I. <u>If</u> (M^*, φ) <u>and</u> (M', ψ) <u>are two globalizations of the same infinitesimal</u> G-<u>transformation group acting on</u> M, <u>and if</u> (M', ψ) <u>is homomorphic to</u> (M^*, φ), <u>then there is a unique homomorphism</u>, f, <u>of</u> (M^*, φ) <u>into</u> (M', ψ). <u>Moreover</u> f <u>is a local diffeomorphism of</u> M^* <u>onto</u> M'.

PROOF. Let f be any homomorphism of (M^*, φ) into (M', ψ).

Given $q \varepsilon M^*$ choose $(g,p) \varepsilon G \times M$ such that $q = \varphi(g,p)$. Then $f(q) = f(\varphi_g(p)) = \psi_g(f(p)) = \psi_g(p)$. This shows that f is uniquely determined. Moreover we have $f \circ \varphi_g = \psi_g \circ f$, or $f = \psi_g \circ f \circ \varphi_g^{-1}$. Now as φ and ψ are global G-transformation groups, ψ_g and φ_g^{-1} are diffeomorphisms, and as f is a local diffeomorphism (in fact locally the identity map) at $p = \varphi_g(q)$ it follows that f is a local diffeomorphism at q.

Finally, given $p' \varepsilon M'$ we can choose $(h,p) \varepsilon G \times M$ such that $p' = \psi(h,p)$. Then since $f(p) = p$, we have $p' = \psi_h(f(p)) = f(\varphi_h(p))$, proving that f maps onto M'.

COROLLARY. _If_ Θ _is an infinitesimal G-transformation group acting on_ M, _and_ (M^*,φ) _is a globalization of_ Θ, _then the identity map of_ M^* _is the only homomorphism of_ (M^*,φ) _into itself_.

DEFINITION IV. A homomorphism of one globalization of an infinitesimal G-transformation group into another is an _isomorphism_ if and only if it is one-to-one.

We note that it follows from theorem I that a homomorphism of (M^*,φ) into (M',ψ) is an isomorphism if and only if it is a diffeomorphism of M^* onto M', and that in this case its inverse is an isomorphism of (M',ψ) into (M^*,φ).

DEFINITION V. Let Θ be an infinitesimal G-transformation group acting on M. A globalization of Θ will be called _universal_ if every globalization of Θ is homomorphic to it.

THEOREM II. _Two universal globalizations of the same infinitesimal G-transformation group are isomorphic_.

PROOF. We show that more generally if two globalizations of the same infinitesimal G-transformation group are each homomorphic to the other, then they are isomorphic. In fact if f is a homomorphism of (M^*, φ) into (M', ψ), and g is a homomorphism of (M', ψ) into (M^*, φ), then $g \circ f$ is a homomorphism of (M^*, φ) into itself, and hence by the corollary of theorem I $g \circ f$ is the identity map of M^*. Thus f has a left inverse and therefore is one-to-one, and hence an isomorphism.

It follows from the above proof that the isomorphism classes of the globalizations of a globalizable infinitesimal group are partially ordered by the relation 'is homomorphic to'. We shall see that there is a maximum element, i.e. every globalizable infinitesimal transformation group admits a universal globalization (theorem X). It would be of interest to know if there is also always a minimum element, and how to construct in an effective way a canonical set of representatives from a universal globalization. But these are problems we shall not consider in this memoir.

2. Univalent Infinitesimal Transformation Groups.

We now consider a condition which, though at first glance quite unrelated, turns out to be equivalent to globalizability.

DEFINITION VI. An infinitesimal G-transformation group, Θ, acting on M will be called *univalent* if for each $p \in M$ Π_G maps the leaf of Θ^* containing (e,p) one-to-one.

THEOREM III. *If* Θ *is a univalent infinitesimal G-transformation group acting on* M, *then* Π_G *maps each leaf of* Θ^* *diffeomorphically into* G.

PROOF. Let Σ be any leaf of Θ^* and let (g,p) be some point of Σ. Then $(e,p) = \bar{R}_g(g,p) \varepsilon \bar{R}_g(\Sigma)$, so by the corollary of theorem VI, chapter II $\bar{R}_g(\Sigma)$ is the leaf of Θ^* containing (e,p). Then Π_G maps $\bar{R}_g(\Sigma)$ one-to-one, and in fact by theorem IV of chapter II diffeomorphically into G. Now R_g and $\bar{R}_g$ are diffeomorphisms, and $\Pi_G = R_g^{-1} \circ \Pi_G \circ \bar{R}_g$, so Π_G maps Σ diffeomorphically into G.

THEOREM IV. <u>If</u> Θ <u>is a univalent infinitesimal G-transformation group acting on</u> M, <u>then</u> Θ^* <u>is a regular differential system</u> (<u>definition</u> III, <u>chapter</u> I) <u>in</u> $G \times M$, <u>and the mapping</u> $F : p \to \Pi_{\Theta^*}(e,p)$ <u>maps</u> M <u>diffeomorphically onto an open submanifold</u> of $G \times M/\Theta^*$.

PROOF. Let $(x_1 \ldots x_{r+n}, O)$ be a cubical coordinate system in $G \times M$ centered at (g,p) and flat with respect to Θ^*. By theorem IV of chapter II, we can find a coordinate system $(w_1 \ldots w_n, U)$ in M centered at p such that if $q \varepsilon M$ then $(g,q) \varepsilon O$ and $w_i(q) = x_{r+i}(g,q)$. By reducing U we can suppose that $(w_1 \ldots w_n, U)$ is cubical, say of breadth 2a. Let V be the cube of breadth 2a and centered at (g,p) with respect to the coordinate system $(x_1 \ldots x_{r+n}, O)$, and suppose that a leaf Σ of Θ^* intersects V in the r-dimensional slices of $(x_1 \ldots x_{r+n}, V)$ defined by $s = (s_{r+1} \ldots s_{r+n})$ and $t = (t_{r+1} \ldots t_{r+n})$. Then as $|s_i| < a$ and $|t_i| < a$, we can find q_s and q_t in U with $w_i(q_s) = s_{r+i}$ and $w_i(q_t) = t_{r+i}$. Then $x_{r+i}(g,q_s) = w_i(q_s) = s_{r+i}$ and $x_{r+i}(g,q_t) = w_i(q_t) = t_{r+i}$, so (g,q_s) and (g,q_t) belong to the r-dimensional slices of $(x_1 \ldots x_{r+n}, V)$ defined by s and t respectively and so belong to Σ. Now according to theorem III Π_G maps Σ one-to-one, so as $\Pi_G(g,q_s) = g = \Pi_G(g,q_t)$ it follows that

$q_s = q_t$, and so $s = t$. Thus Σ can intersect V in at most one r-dimensional slice of $(x_1 \ldots x_{r+n}, V)$. By definition III of chapter I, $(x_1 \ldots x_{r+n}, V)$ is a regular coordinate system with respect to Θ^* , and since (g,p) was an arbitrary point of $G \times M$, it follows that Θ^* is a regular differential system from theorem V of chapter I.

If $\Pi_{\Theta^*}(e,p) = \Pi_{\Theta^*}(e,q)$, then (e,p) and (e,q) belong to the same leaf of Θ^* ; since Π_G is one-to-one on leaves of Θ^* it follows that $p = q$. Thus $F : p \rightarrow \Pi_{\Theta^*}(e,p)$ is one-to-one on M . Given $p \varepsilon M$ let $(x_1 \ldots x_{r+n}, O)$ be a regular coordinate system with respect to Θ^* centered at (e,p) . By definition of the manifold structure on $G \times M/\Theta^*$, there is a coordinate system $(\bar{x}_{r+1} \ldots \bar{x}_{r+n}, \Pi_{\Theta^*}(O))$ in $G \times M/\Theta^*$ such that $x_{r+i} = \bar{x}_{r+i} \circ \Pi_{\Theta^*}$. On the other hand, if we put $\sigma(q) = (e,q)$, then we know by theorem IV of chapter II that there is a coordinate system $(w_1 \ldots w_n, U)$ in M centered at p such that $w_i = x_{r+i} \circ \sigma$. Then $w_i = \bar{x}_{r+i} \circ \Pi_{\Theta^*} \circ \sigma = \bar{x}_{r+i} \circ F$, which proves that F is a local diffeomorphism at p . Since p was an arbitrary point of M , F is a local diffeomorphism, and hence $F(M)$ is an open submanifold of $G \times M/\Theta^*$. Since F is one-to-one, it maps M diffeomorphically onto $F(M)$.

COROLLARY. *If Θ is a univalent infinitesimal G-transformation group acting on M , then each leaf of Θ^* is a closed submanifold of $G \times M$.*

PROOF. Theorem VII of chapter I.

THEOREM V. *Let Θ be a univalent infinitesimal G-transformation group acting on M . There is a unique global G-transformation group, φ , acting on $G \times M/\Theta^*$ such that $\Pi_{\Theta^*} \circ \bar{R}_g = \varphi_g \circ \Pi_{\Theta^*}$ for all $g \varepsilon G$. Moreover $(G \times M/\Theta^*, \varphi)$ is* *a globalization of* $\delta F \circ \Theta$ *, where* $F : p \to \Pi_{\Theta^*}(e,p)$ *is the diffeomorphism of* M *into* $G \times M/\Theta^*$ *considered in theorem* IV.

PROOF. By the preceding theorem Θ^* is regular, and by theorem VI of chapter II, $\bar{R}$ is compatible with Θ^*, so the existence and uniqueness of φ follows from theorem III of chapter II. It also follows from the latter theorem that $\varphi^+ = \delta\Pi_{\Theta^*} \circ \bar{R}^+$, so if $L \varepsilon \mathfrak{g}$ and $p \varepsilon M$, then by theorem VI of chapter II, $\varphi^+(L)_{F(p)} = \delta\Pi_{\Theta^*} \circ \bar{R}^+(L)_{(e,p)} = \delta\Pi_{\Theta^*}(-L_e, 0)$.

To prove that $(G \times M/\Theta^*, \varphi)$ is a globalization of $\delta F \circ \Theta$ it remains to verify (3) and (4) of definition II, which in the present case are equivalent to

(3') $\delta F(\Theta(L)_p) = \varphi^+(L)_{F(p)}$ for all $p \varepsilon M$ and $L \varepsilon \mathfrak{g}$, and

(4') given $\Sigma \varepsilon G \times M/\Theta^*$, there exists $(g,p) \varepsilon G \times M$ such that $\Sigma = \varphi_g(F(p))$.

Define $\sigma : M \to G \times M$ by $\sigma(p) = (e,p)$. Then $F = \Pi_{\Theta^*} \circ \sigma$, so $\delta F = \delta\Pi_{\Theta^*} \circ \delta\sigma$, and hence in view of the above expression for $\varphi^+(L)_{F(p)}$ (3') is equivalent to $\delta\Pi_{\Theta^*}((\delta\sigma\Theta(L)_p) + (L_e, 0)) = 0$, or, since $\delta\sigma(\Theta(L)_p) = (0, \Theta(L)_p)$, to $\delta\Pi_{\Theta^*}(L_e, \Theta(L)_p) = 0$. But as $(L_e, \Theta(L)_p) \varepsilon \Theta^*_{(e,p)}$ the latter equality is a consequence of theorem X of chapter I.

If $\Sigma \varepsilon G \times M/\Theta^*$, then choosing $(h,p) \varepsilon \Sigma$ and putting $g = h^{-1}$, we have $\varphi_g(F(p)) = \varphi_g(\Pi_{\Theta^*}(e,p)) = \Pi_{\Theta^*} \circ \bar{R}_g(e,p) = \Pi_{\Theta^*}(h,p) = \Sigma$, which proves (4').

3. Maximum Local Transformation Groups.

It may have occurred to the reader that in some ways it would have been more natural if, in the definition of a local G-transformation group acting on M (definition II of chapter II), we had replaced (2) by the stronger condition

(2') If (h,p) and $(g, \varphi(h,p))$ belong to D_φ then so does (gh,p) and $\varphi(gh,p) = \varphi(g, \varphi(h,p))$.

Local G-transformation groups satisfying this stronger property we shall call maximum, for it is clear that they do not admit a proper extension.

It turns out (theorem X) that a necessary and sufficient condition for an infinitesimal G-transformation group to generate a maximum local G-transformation group is that it be univalent (or globalizable which, again by theorem X, is equivalent to being univalent) and then it generates a unique one. Thus if we had used (2') in the definition of a local G-transformation group, then theorem XI of chapter II would not have been valid. This is why we chose the weaker condition and hence more general concept.

In this section we shall develop some of the principal properties of maximum local transformation groups. Since there are a number of maximality properties equivalent to (2') above, and there seems little reason to prefer any one above the others, we formulate the definition in the following alternative forms.

DEFINITION VII. By theorem VI below the following four conditions on a local G-transformation group, φ , acting on M are equivalent. If φ satisfies any one and hence all of these conditions, it will be called a maximum local G-transformation group acting on M .

(1) If (h,p) and $(g,\varphi(h,p))$ belong to D_φ , then so does (gh,p) ; equivalently, if $h \varepsilon D_\varphi p$ then $D_\varphi q \subseteq R_h(D_\varphi p)$, where $q = \varphi(h,p)$.

(2) For each $p \varepsilon M$ the graph of φ^p is the entire leaf of φ^{+*} containing (e,p) .

(3) If $(h,p) \varepsilon D_\varphi$ then $(g,\varphi(h,p)) \varepsilon D_\varphi$ if and only if $(gh,p) \varepsilon D_\varphi$; equivalently if $h \varepsilon D_\varphi p$ then $D_\varphi q = R_h(D_\varphi p)$, where $q = \varphi(h,p)$.

(4) If $p \varepsilon M$ and $\{g_n\}$ is a sequence in $D_\varphi p$ approaching a

point g on the frontier of $D_\varphi p$, then $\varphi(g_n,p)\to\infty$ (in the usual sense; namely that for each compact subset, K, of M $\varphi(g_n,p)\notin K$ for all sufficiently large n).

THEOREM VI. _If_ φ _is any local G-transformation group acting on_ M, _then the four conditions of definition_ VII _are equivalent_.

PROOF. We show first that (1) implies (2). Let $p\varepsilon M$ and let Σ be the leaf of φ^{+*} containing (e,p). Let (g,q) be any point of Σ which is adherent to the graph of φ^p in the topology of Σ. Choose a neighborhood O of q and a neighborhood V of e such that $V\times O\subseteq D_\varphi$. Let U be a Σ-neighborhood of (g,q) such that $\Pi_G(U)\subseteq V^{-1}g$ and $\Pi_M(U)\subseteq O$. Since (g,q) is adherent to the graph of φ^p we can find $(h,\varphi(h,p))\varepsilon U$. Then $(gh^{-1},\varphi(h,p))\varepsilon V\times O\subseteq D_\varphi$, so by (1), $(gh^{-1}h,p)\varepsilon D_\varphi$, i.e. $g\varepsilon D_\varphi p$, so $(g,q)\varepsilon\Sigma\cap\Pi_G^{-1}(D_\varphi p)$. But by corollary I of theorem VIII, chapter II, the graph of φ^p is closed in $\Sigma\cap\Pi_G^{-1}(D_\varphi p)$, and it follows that (g,q) is in the graph of φ^p. Thus the graph of φ^p is closed in Σ. Since it is also open in Σ (theorem VII of chapter II) and Σ is connected, it follows that the graph of φ^p is all of Σ. Thus we have derived (2) from (1).

We next show that (2) implies (3). Let $(h,p)\varepsilon D_\varphi$, and put $q=\varphi(h,p)$. Then $(h,q)\varepsilon$graph of φ^p = leaf of φ^{+*} containing (e,p). It follows from the corollary of theorem VI, chapter II that $\bar{R}_h$(graph of φ^p) = leaf of φ^{+*} containing (e,q) = graph of φ^q. Thus $R_h(D_\varphi p)=$ $R_h\circ\Pi_G$(graph of φ^p) $=\Pi_G\circ\bar{R}_h$(graph of φ^p) $=\Pi_G$(graph of φ^q) $=D_\varphi q$, which is one of the (trivially equivalent) forms of (3). Thus we have derived (3) from (2).

It is trivial that (3) implies (1). To complete the proof we show that (2) is equivalent to (4).

Suppose first that (2) does not hold. Then for some $p \varepsilon M$ the graph of φ^p is not the whole of the leaf, Σ, of φ^{+*} containing (e,p), so by theorem VII of chapter II it is a proper open submanifold of Σ. Since Σ is connected the graph of φ^p has a frontier point (g,q) in Σ. Let $\{(g_n, \varphi(g_n,p))\}$ be a sequence from the graph of φ^p approaching (g,q) in the topology of Σ. By the corollary of theorem VII, chapter II, Π_G maps the graph of φ^p diffeomorphically onto $D_\varphi p$, and by theorem IV of chapter II Π_G is a local diffeomorphism of Σ into G at (g,q). It follows that g is a frontier point of $D_\varphi p$. Now as Σ is a submanifold of $G \times M$, its manifold topology is stronger than the topology induced from $G \times M$, so $\{(g_n, \varphi(g_n,p))\}$ approaches (g,q) in the topology of $G \times M$, i.e. $g_n \to g$ and $\varphi(g_n,p) \to q$. This is inconsistent with (4), and so by contraposition (2) implies (4).

Finally suppose that (2) does hold. Let $p \varepsilon M$ and let $\{g_n\}$ be a sequence in $D_\varphi p$ approaching $g \varepsilon G$, and suppose that $\varphi(g_n,p) \not\to \infty$. Then to prove (4) we must show that $g \not\varepsilon$ frontier of $D_\varphi p$. Since $\varphi(g_n,p) \not\to \infty$, there is a compact set K such that $\varphi(g_n,p) \varepsilon K$ for arbitrarily large n. By passing to a subsequence of $\{g_n\}$ we can suppose that $\varphi(g_n,p) \to q \varepsilon K$. Then $(g_n, \varphi(g_n,p)) \to (g,q)$ in the topology of $G \times M$. Now (2) clearly implies that φ^+ is univalent (for the graph of a function with domain in G and range in M is mapped one-to-one by Π_G) so by the corollary of theorem IV, the graph of φ^p = the leaf of φ^{-*} containing (e,p) is closed in $G \times M$. It follows that $(g,q) \varepsilon$ graph of φ^p, and hence $g = \Pi_G(g,q) \varepsilon D_\varphi p$. Since $D_\varphi p$ is open in G, $g \not\varepsilon$ frontier of $D_\varphi p$.

In the course of the very last part of the proof we proved the following corollary.

COROLLARY. *The infinitesimal generator of a maximum local G-transformation group acting on M is univalent.*

THEOREM VII. *If φ is a maximum local G-transformation group acting on M, and ψ is any local G-transformation group acting on M with the same infinitesimal generator as φ, then $\psi = \varphi \restriction D_\psi$.*

PROOF. If $p \varepsilon M$ then by (2) of definition VII the graph of φ^p is the leaf of φ^{+*} containing (e,p). Then since $\psi^{+*} = \varphi^{+*}$, it follows from theorem VII of chapter II that the graph of ψ^p is included in the graph of φ^p, i.e. $\psi^p = \varphi^p \restriction D_{\psi^p}$. Since this holds for all $p \varepsilon M$, $\psi = \varphi \restriction D_\psi$.

COROLLARY. *Two maximum local G-transformation groups acting on M with the same infinitesimal generator are identical.*

THEOREM VIII. *Let φ be a global G-transformation group acting on M^*, and let M be an open submanifold of M^*. For each $p \varepsilon M$ let D_p be the component of e in $\{g \varepsilon G : \varphi(g,p) \varepsilon M\}$, and let $D = \bigcup_{p \varepsilon M} (D_p \times \{p\})$. Then $\varphi \restriction D$ can be characterized as the unique maximum local G-transformation group acting on M which is a restriction of φ, and also as the unique maximum local G-transformation group acting on M generated by $\varphi^+ \restriction M$.*

PROOF. Let $O = \{(g,p) \varepsilon G \times M : \varphi(g,p) \varepsilon M\}$. It is clear that O is open in $G \times M$ and that $\{e\} \times M \subseteq O$; hence by theorem I of chapter II, D is a local transformation group domain in $G \times M$. It is then obvious that $\varphi \restriction D$ is a local G-transformation group acting on M.

For each $q \in M$ let $O_q = \{ g \in G : (g,q) \in O \}$ so that D_q is the component of e in O_q. Thus if $h \in D_p$, then D_p is the component of h in O_p, and since R_h maps O_p diffeomorphically onto $R_h(O_p)$, $R_h(D_p)$ is the component of e in $R_h(O_p)$. Now if we put $q = \varphi(h,p)$, then for any $g \in G$ we have the relation $\varphi(gh,p) = \varphi(g,q)$, which implies that $O_q = R_h(O_p)$, and hence that D_q = component of e in $R_h(O_p) = R_h(D_p)$. Thus $\varphi \restriction D$ satisfies condition (3) of definition VII, and so is a maximum local G-transformation group acting on M.

Now suppose that σ is any maximum local G-transformation group acting on M such that $\sigma = \varphi \restriction D_\sigma$. Then for any $p \in M$ $\sigma^p = \varphi^p \restriction D_{\sigma}p$, and since $D_\sigma p$ is a neighborhood of e it follows that $(\delta\sigma^p)_e = (\delta\varphi^p)_e$ Then if $L \in \mathfrak{g}$ we have $\sigma^+(L)_p = \delta\sigma^p(L_e) = \delta\varphi^p(L_e) = \varphi^+(L)_p$, so $\sigma^+ = \varphi^+ \restriction M$. It is now an immediate consequence of the corollary of theorem VII that $\varphi \restriction D$ is the unique maximum local G-transformation group acting on M which is a restriction of φ, or whose infinitesimal generator is $\varphi^+ \restriction M$.

COROLLARY. *Let Θ be an infinitesimal G-transformation group acting on M, and let (M^*,φ) be a globalization of Θ. Then there is a unique maximum local G-transformation group acting on M, ψ, which is generated by Θ, and ψ can also be characterized as the unique maximum local G-transformation group acting on M which is a restriction of φ. For each $p \in M$ the leaf of Θ^* containing (e,p) is included in the graph of φ^p.*

PROOF. The first conclusion is a restatement of the theorem in slightly different terms and needs no extra proof. For each $p \in M$ the leaf of Θ^* containing (e,p) is, by (2) of definition VII, the graph of ψ^p , which, since $\psi^p = \varphi^p \restriction D_{\psi^p}$ is included in the graph of φ^p .

LEMMA. _Let_ Θ _be an infinitesimal_ G-_transformation group acting on_ M , _and let_ (M^*,φ) _be a globalization of_ Θ . _If_ Σ _is any point of_ $G \times M/\Theta^*$ (i.e. _any leaf of_ Θ^*) _then there is a unique point_ $f(\Sigma)$ _in_ M^* _such that_ $\Sigma \subseteq$ _graph of_ $\varphi^{f(\Sigma)}$.

PROOF. If $(h,p)\varepsilon D_\varphi$, and $q = \varphi(h,p)$ then the relation $\varphi(gh^{-1},q) = \varphi(g,p)$, valid for all $g\varepsilon G$, implies that the graph of $\varphi^q = \bar{R}_h(\text{graph of } \varphi^p)$. Now given $\Sigma\varepsilon G \times M/\Theta^*$ choose $(g^{-1},p)\varepsilon\Sigma$. Since $(e,p) = \bar{R}_{g^{-1}}(g^{-1},p)\varepsilon\bar{R}_{g^{-1}}(\Sigma)$, it follows from the corollary of theorem VI, chapter II, that $\bar{R}_{g^{-1}}(\Sigma)$ is the leaf of Θ^* containing (e,p) and hence by the corollary of theorem VIII $\bar{R}_{g^{-1}}(\Sigma) \subseteq$ graph of φ^p It follows from the remark at the beginning of the proof, that if we put $f(\Sigma) = \varphi(g,p)$ then $\Sigma \subseteq \bar{R}_g(\text{graph of } \varphi^p) = \text{graph of } \varphi^{f(\Sigma)}$. If q is any point of M^* such that $\Sigma \subseteq$ graph of φ^q , then as $(g^{-1},p)\varepsilon\Sigma$, it follows that $\varphi(g^{-1},q) = p$ so that $q = \varphi(g,p) = f(\Sigma)$.

THEOREM IX. _A univalent infinitesimal_ G-_transformation group acting on_ M _admits a universal globalization_ (_which, by theorem_ II, _is unique to within isomorphism_).

PROOF. Let F be the mapping $p \to \Pi_{\Theta^*}(e,p)$ of M into $G \times M/\Theta^*$. By theorem IV F is a diffeomorphism of M onto the open submanifold F(M) of $G \times M/\Theta^*$. Let us identify points of M with their corresponding points in F(M) under F , so that $\delta F \circ \Theta$ is identified with Θ . Thus, with this identification, there is by theorem V a globalization $(G \times M/\Theta^*,\varphi)$ of Θ such that $\Pi_{\Theta^*} \circ \bar{R}_g = \varphi_g \circ \Pi_{\Theta^*}$ for all $g\varepsilon G$. We shall show that this globalization is universal.

In fact let (M',ψ) be any globalization of Θ . By the lemma there is a map f of $G \times M/\Theta^*$ into M' which is uniquely characterized by the condition that $\Sigma \subseteq$ graph of $\psi^{f(\Sigma)}$. If $p \varepsilon M$ then by the corollary of theorem VIII, $F(p) = \Pi_{\Theta^*}(e,p) =$ (leaf of Θ^* containing $(e,p)) \subseteq$ (graph of φ^p), so $f(F(p)) = p$. Since we are identifying p and $F(p)$, this gives $f \upharpoonright M$ = the identity map of M.

If $\Sigma \varepsilon G \times M/\Theta^*$, and $g \varepsilon G$, then the relation $\Pi_{\Theta^*} \circ \bar{R}_g = \varphi_g \circ \Pi_{\Theta^*}$ implies that φ_g applied to the point Σ of $G \times M/\Theta^*$ is $\bar{R}_g$ applied to the subset Σ of $G \times M$. Thus as $\Sigma \subseteq$ graph of $\psi^{f(\Sigma)}$ it follows from the remark at the beginning of the proof of the lemma that $\varphi_g(\Sigma) \subseteq \bar{R}_g$(graph of $\psi^{f(\Sigma)}$) = graph of $\psi^{\psi(g,f(\Sigma))}$. This gives $f(\varphi_g(\Sigma)) = \psi_g(f(\Sigma))$. Since Σ was any point of $G \times M/\Theta^*$ it follows that $f \circ \varphi_g = \psi_g \circ f$.

The last two paragraphs show that f is a homomorphism of $(G \times M/\Theta^*, \varphi)$ into (M', ψ) (definition III). Thus every globalization of Θ is homomorphic to $(G \times M/\Theta^*, \varphi)$, so the latter is a universal globalization of Θ.

4. The Principal Theorem.

We summarize our previous results in theorem X.

THEOREM X. *Let Θ be an infinitesimal G-transformation group acting on M. The following four statements are equivalent.*

(1) *Θ admits a universal globalization (which is unique to within isomorphism).*

(2) *Θ is globalizable.*

(3) *Θ generates a maximum local G-transformation group acting on M (this maximum local G-transformation group is then uniquely determined, and any local G-transformation group acting on M generated by Θ is a restriction of it).*

(4) *Θ is univalent.*

PROOF. That (1) implies (2) is trivial. That (2) implies (3) follows from the corollary of theorem VIII. That (3) implies (4) is the statement of the corollary of theorem VI. And finally that (4) implies (1) is the statement of theorem IX.

COROLLARY. *The mapping* $\varphi \rightarrow \varphi^+$ *is a one-to-one correspondence between maximum local* G-*transformation groups acting on* M *and univalent infinitesimal* G-*transformation groups acting on* M. *Hence given a univalent infinitesimal* G-*transformation group acting on* M, *it makes sense to speak of* THE *maximum local* G-*transformation group acting on* M *it generates.*

5. Proper Infinitesimal Transformation Groups.

THEOREM XI. *A proper infinitesimal* G-*transformation group acting on* M *is univalent.*

PROOF. By definition II a proper infinitesimal G-transformation group is in particular globalizable, so theorem XI is an immediate consequence of theorem X.

We now ask what distinguishes proper infinitesimal G-transformation groups among univalent infinitesimal G-transformation groups.

THEOREM XII. *Let* Θ *be a univalent infinitesimal* G-*transformation group acting on* M, *and let* φ *be the maximum local* G-*transformation group it generates. Then the following five statements are equivalent.*

(1) *For each* $p \in M$ φ^p *maps subsets of* $D_\varphi p$ *which are relatively compact in* G *into relatively compact subsets of* M.

(2) *For each* $p \in M$ Π_G *maps the leaf of* Θ^* *containing* (e,p) *onto* G.

(3) φ <u>is global</u>.

(4) (M,φ) <u>is the unique globalization of</u> Θ .

(5) Θ is proper.

PROOF. Suppose that (1) holds, and suppose that for some $p \in M$ $D_\varphi p = \Pi_G(\text{graph of } \varphi^p) = \Pi_G(\text{leaf of } \Theta^* \text{ containing } (e,p))$ is not all of G . Then since G is connected there exists a g in the frontier of $D_\varphi p$. Choose $\{g_n\}$ a sequence in $D_\varphi p$ approaching g . Then the set K of g_n is relatively compact in G , while since $\varphi^p(g_n) \to \infty$ by (4) of definition VII, $\varphi^p(K)$ is not relatively compact in M . Since this is contrary to (1), no such p can exist, i.e. (2) holds. Thus (1) implies (2).

If (2) holds then for each $p \in M$ $D_\varphi p = \Pi_G(\text{graph of } \varphi^p) = \Pi_G(\text{leaf of } \Theta^* \text{ containing } (e,p)) = G$. Then $D_\varphi = \bigcup_{p \in M} (D_\varphi p \times \{p\}) = G \times M$, so φ is global. Thus (2) implies (3).

Next suppose that (3) holds and let (M^*,ψ) be any globalization of Θ . By the corollary of theorem VIII, $\varphi = \psi \restriction D_\varphi = \psi \restriction G \times M$. By definition II, given $q \in M$ we can find $(g,p) \in G \times M$ such that $q = \psi(g,p)$. But $\psi(g,p) = \varphi(g,p) \in M$, so $M^* = M$, and hence $\psi = \psi \restriction G \times M = \varphi$. Thus $(M^*,\psi) = (M,\varphi)$, so we have deduced (4) from (5).

That (4) implies (5) is trivial, so we show finally that (5) implies (1). In fact if (5) holds then we can find a proper globalization of Θ , (M,ψ) . Then ψ is a global and <u>a fortiori</u> maximal G-transformation group acting on M , so $\varphi = \psi$. Then if $p \in M$, then $D_\varphi p = G$, hence if K is a subset of $D_\varphi p$ relatively compact in G , then $\bar{K}$ is a compact subset of $D_\varphi p$, so $\varphi^p(\bar{K})$ is compact. Since $\varphi^p(K)$ is included in $\varphi^p(\bar{K})$, this proves (1).

COROLLARY 1. <u>The mapping</u> $\varphi \to \varphi^+$ <u>is a one-to-one corre-</u>

spondence between global G-transformation groups acting on M and proper infinitesimal G-transformation groups acting on M. Hence given a proper infinitesimal G-transformation group acting on M, it makes sense to speak of THE global G-transformation group it generates.

COROLLARY 2. *A univalent infinitesimal G-transformation group acting on a compact differentiable manifold is proper. Equivalently, a maximum local G-transformation group acting on a compact differentiable manifold is global.*

PROOF. If M is compact then condition (1) of the theorem is automatically satisfied.

In the next section we shall prove what is for all practical purposes a much stronger result; namely, (corollary 2 of theorem XVIII) that if M is compact and Hausdorff, and G simply connected, then every infinitesimal G-transformation group acting on M is automatically proper, and hence generates a unique global G-transformation group acting on M.

COROLLARY 3. *A necessary and sufficient condition that an infinitesimal G-transformation group acting on M be proper is that for each* $p \varepsilon M$ Π_G *maps the leaf of* Θ^* *containing* (e,p) *one-to-one onto* G.

PROOF. If Θ is proper and $p \varepsilon M$, then letting φ be the global G-transformation group generated by Θ, Π_G maps the leaf of Θ^* containing (e,p) = the graph of φ^p one-to-one onto $D_{\varphi^p} = G$. If conversely Θ satisfies the condition, then it is univalent by definition VI, and hence proper since it satisfies condition (2) of the theorem.

6. Uniform Infinitesimal Transformation Groups.

The 'healthiest' infinitesimal G-transformation groups are the proper ones, i.e. those generating global G-transformation groups. Then perhaps the mildest form of pathology in infinitesimal G-transformation groups is a failure to be proper occasioned solely by the lack of simple connectivity in G, for this can be cured by replacing (in an obvious sense) G by its universal covering group. The symptom of this mild sort of pathology is that for each leaf Σ of the infinitesimal graph, the pair $(\Sigma, \Pi_G \upharpoonright \Sigma)$ is a covering space of G. It is important to be able to recognize when an infinitesimal G-transformation group is no more pathological than this, and in this section, we consider a useful, necessary and sufficient condition which we call uniformity. This condition amounts, in essence, to the existence of a neighborhood V of e such that the restriction of the infinitesimal graph to $\Pi_G^{-1}(V)$ has leaves which are each mapped one-to-one onto V by Π_G. Thus, while global in the M direction, this condition is of a local nature with respect to G, and herein lies its importance.

Perhaps the most striking result of this section is the very general sufficient condition for proper-ness given in corollary 2 of theorem XVIII. However, a no less important application of the results of this section is made in proving one of the keystones of the next chapter (theorem III).

DEFINITION VIII. Let Θ be an infinitesimal G-transformation group acting on M, and for each $p \in M$ denote by Σ_p the leaf of Θ^* containing (e,p). If $S \subseteq M$, then an open, connected neighborhood, V, of e will be called a <u>uniform neighborhood for</u> S <u>with respect to</u> Θ if for each $p \in S$ the connected component of (e,p) in $\Sigma_p \cap \Pi_G^{-1}(V)$ is mapped one-to-one onto

V by Π_G. If such a V exists then we say that S *is uniform with respect to* Θ. If M itself is uniform with respect to Θ, then we say that Θ *is uniform*.

The following theorem summarizes the trivial consequences of definition VIII.

THEOREM XIII. *Let Θ be an infinitesimal G-transformation group acting on M, S a subset of M, and V a uniform neighborhood for S with respect to Θ. Then*

(1) *if V' is an open, connected neighborhood of e included in V, then V' is also a uniform neighborhood for S with respect to Θ,*

(2) *V is a uniform neighborhood with respect to Θ for any subset of S; hence a subset of a set uniform with respect to Θ is itself uniform with respect to Θ, and*

(3) *if $S' \subseteq M$ and V' is a uniform neighborhood for S' with respect to Θ, then the component of e in $V \cap V'$ is a uniform neighborhood for $S \cup S'$ with respect to Θ; hence the union of any two (and so of any finite number of) subsets of M uniform with respect to Θ is uniform with respect to Θ.*

DEFINITION IX. Let Θ be an infinitesimal G-transformation group acting on M. The *support of* Θ is the set of $p \varepsilon M$ for which $\Theta(L)_p \neq 0$ for some $L \varepsilon \mathfrak{g}$.

THEOREM XIV. *Let Θ be an infinitesimal G-transformation group acting on M. If p is in the complement of the support of Θ then the leaf of Θ^* containing (e,p) is* $\Sigma_p' = \{ (g,p) : g \varepsilon G \}$ *with the manifold structure defined by* *the condition that* $\Pi_G \upharpoonright \Sigma_p'$ *is a diffeomorphism.*

PROOF. Clearly the tangent space to Σ_p' (considered as a submanifold of $G \times M$) at (g,p) is $\{(L_g,0) : L \varepsilon \mathfrak{g}\}$ which is just $\Theta^*_{(g,p)}$ since p is not in the support of Θ. Since Σ_p' is clearly connected, it is an open submanifold of the leaf Σ_p of Θ^* containing (e,p). Now Σ_p' is closed in $G \times M$ and hence *a fortiori* it is closed in Σ_p, a submanifold of $G \times M$. Thus Σ_p' is both open and closed in Σ_p and since Σ_p is connected, it follows that $\Sigma_p' = \Sigma_p$.

COROLLARY. *If* Θ *is an infinitesimal G-transformation group acting on* M *then any connected, open neighborhood of* e *is a uniform neighborhood, with respect to* Θ, *for the complement of the support of* Θ.

THEOREM XV. *If* Θ *is an infinitesimal G-transformation group acting on* M *such that every leaf of* Θ^* *is a Hausdorff manifold, then every relatively compact subset of* M *is uniform with respect to* Θ.

PROOF. By (2) of theorem XIII, it suffices to prove that compact subsets of M are uniform with respect to Θ and for this it suffices by (3) of theorem XIII to prove that every $p \varepsilon M$ has a neighborhood which is uniform with respect to Θ. But the latter is an immediate consequence of the corollary of theorem IX of chapter II.

THEOREM XVI. *Let* Θ *be an infinitesimal G-transformation group acting on* M. *If the support of* Θ *is relatively compact in* M *and every leaf of* Θ^* *is a Hausdorff manifold then* Θ *is uniform.*

PROOF. An immediate consequence of theorem XV, the corollary of theorem XIV, and (3) of theorem XIII.

COROLLARY 1. *If M is a Hausdorff differentiable manifold then every infinitesimal G-transformation group acting on M with support relatively compact in M is uniform.*

COROLLARY 2. *If M is a compact Hausdorff differentiable manifold then every infinitesimal G-transformation group acting on M is uniform.*

LEMMA. *If Θ is a uniform infinitesimal G-transformation group acting on M and Σ is a leaf of Θ^* then*

(1) *Σ is a Hausdorff manifold*

(2) $\Pi_G(\Sigma) = G$.

PROOF. Let us denote by Σ_p the leaf of Θ^* containing (e,p). Suppose (g,p) and (h,q) are points of Σ that cannot be separated by open sets of Σ. Then as Σ is a submanifold of $G \times M$, its manifold topology is stronger than the topology induced from $G \times M$, so (g,p) and (h,q) cannot be separated by open sets of $G \times M$. Since G is Hausdorff it follows that $g = h$, so (g,p) and (g,q) are points of Σ that cannot be separated by open sets. By the corollary to theorem VI of chapter II, $\bar{R}_g$ maps Σ diffeomorphically onto $\Sigma_p = \Sigma_q$, so (e,p) and (e,q) are points of $\Sigma_p = \Sigma_q$ that cannot be separated by open sets. If V is a uniform neighborhood for M with respect to Θ then as the component of (e,p) in $\Sigma_p \cap \Pi_G^{-1}(V)$ and the component of (e,q) in $\Sigma_q \cap \Pi_G^{-1}(V)$ are neighborhoods of (e,p) and (e,q) respectively in $\Sigma_p = \Sigma_q$ they are not disjoint and hence are equal. Since Π_G is one-to-one on the component of (e,p) in $\Sigma_p \cap \Pi_G^{-1}(V)$ it follows that $p = q$. Thus $(g,p) = (h,q)$ which proves (1).

We next prove that $\Pi_G(\Sigma_p) = G$ for any $p\varepsilon M$. Again let V be a uniform neighborhood for M with respect to Θ. We will show by induction that for every positive integer n $V^n \subseteq \Pi_G(\Sigma_p)$. For $n = 1$ this is immediate for in fact Π_G maps the component of (e,p) in $\Sigma_p \cap \Pi_G^{-1}(V)$ onto V. Suppose $V^{n-1} \subseteq \Pi_G(\Sigma_p)$ and let $g\varepsilon V^{n-1}$. Then there exists $q\varepsilon M$ such that $(g,q)\varepsilon\Sigma_p$. By the corollary to theorem VI of chapter II, $\bar{R}_g(\Sigma_p) = \Sigma_q$. Since, as we have just seen, $V \subseteq \Pi_G(\Sigma_q)$, $V \subseteq \Pi_G \circ \bar{R}_g(\Sigma_p) = R_g \circ \Pi_G(\Sigma_p)$ hence $Vg = R_{g^{-1}} V \subseteq \Pi_G(\Sigma_p)$. Now g was any point of V^{n-1} so $V^n = VV^{n-1} \subseteq \Pi_G(\Sigma_p)$. This completes the induction. Since G is connected V generates G, hence $\Pi_G(\Sigma_p) = G$ as claimed.

Now choose $(g,p)\varepsilon\Sigma$, then $(e,p)\varepsilon\bar{R}_g(\Sigma)$ so by the corollary of theorem VI, chapter II $\bar{R}_g(\Sigma) = \Sigma_p$. Then $\Pi_G(\Sigma) = \Pi_G \circ \bar{R}_{g^{-1}}(\Sigma_p) = R_{g^{-1}} \circ \Pi_G(\Sigma_p) = R_{g^{-1}}(G) = G$.

THEOREM XVII. <u>A necessary and sufficient condition that an infinitesimal</u> G-<u>transformation group acting on</u> M <u>be uniform is that for each leaf</u> Σ <u>of</u> Θ^* <u>the pair</u> $(\Sigma, \Pi_G \restriction \Sigma)$ <u>is a covering space for</u> G.

PROOF. If this condition is satisfied, then clearly any simply connected open neighborhood of e in G is a uniform neighborhood for M with respect to Θ so Θ is uniform.

Conversely, suppose Θ is uniform and let Σ be a leaf of Θ^*. To show that $(\Sigma, \Pi_G \restriction \Sigma)$ is a covering space for G, we must show

(a) Σ is a Hausdorff space

(b) $\Pi_G(\Sigma) = G$

(c) For each $g\varepsilon G$ there is a neighborhood W of g such that Π_G maps each component of $\Sigma \cap \Pi_G^{-1}(W)$ diffeomorphically onto W.

Now (a) and (b) have already been proved in the lemma, so it remains to prove (c). Let V be a uniform neighborhood for M with respect to Θ and let U be an open neighborhood of e such that $UU^{-1} \subseteq V$. Then if $k \varepsilon U^{-1}$ Uk is an open, connected neighborhood of e included in V so, by (1) of theorem XIII, Uk is a uniform neighborhood for M with respect to Θ. We shall show that we can take $W = Ug$ in (c). In fact let K be any component of $\Sigma \cap \Pi_G^{-1}(Ug)$ and let (h,q) be any point of K. Then $h \varepsilon Ug$ so $gh^{-1} \varepsilon U^{-1}$ and therefore by the above remark Ugh^{-1} is a uniform neighborhood for M with respect to Θ. Now $(e,q) = \bar{R}_h(h,q) \varepsilon \bar{R}_h(\Sigma)$ so by the corollary of theorem VI of chapter II, $\bar{R}_h(\Sigma) = \Sigma_q$, and $\bar{R}_h(K)$ is the component of (e,q) in $\Sigma_q \cap \Pi_G^{-1}(Ugh^{-1})$. Hence Π_G maps $\bar{R}_h(K)$ diffeomorphically onto Ugh^{-1}. Since $\bar{R}_h{-1}$ maps $\bar{R}_h(K)$ diffeomorphically onto K and $\Pi_G \circ \bar{R}_h{-1} = R_h{-1} \circ \Pi_G$ it follows that Π_G maps K diffeomorphically onto $R_h{-1}(Ugh^{-1}) = Ug$.

COROLLARY. *If G is simply connected then an infinitesimal G-transformation group acting on M is proper if and only if it is uniform.*

PROOF. An immediate consequence of the theorem and corollary 3 of theorem XII, remembering that if (X,f) is a covering space for G, then by definition of simple connectivity f maps X homeomorphically onto G.

THEOREM XVIII. *Let G be simply connected and let Θ be an infinitesimal G-transformation group acting on M with support relatively compact in M such that each leaf of Θ^* is a Hausdorff manifold. Then Θ is proper and hence, by corollary 1 of theorem XII, Θ generates a unique global G-transformation group acting on M.*

PROOF. An immediate consequence of theorem XVI and the corollary of theorem XVII.

COROLLARY 1. *If G is simply connected and M is a Hausdorff differentiable manifold then every infinitesimal G-transformation group Θ acting on M which has relatively compact support is proper and hence generates a unique global G-transformation group acting on M.*

COROLLARY 2. *If G is simply connected and M is a compact Hausdorff manifold then every infinitesimal G-transformation group acting on M is proper and hence generates a unique global G-transformation group acting on M. In other words, the mapping $\varphi \to \varphi^{+}$ is a one-to-one correspondence between global G-transformation groups acting on M and infinitesimal G-transformation groups acting on M.*

Although corollary 2 above seems not to have been published previously, it is apparently known to a number of people who are interested in such questions (for the case $G = R$ this corollary can even be considered 'well-known').

7. R-Transformation Groups.

We denote by R the connected Lie group of real numbers under addition, by x the identity map of R onto itself and by D the vector field $\partial / \partial x$ on R. Then D is a basis for the one-dimensional Lie algebra of invariant vector fields on R, hence if M is any differentiable manifold and X a vector field on M then the map $\Theta : tD \to tX$ is clearly an infinitesimal R-transformation group acting on M which is uniquely characterized by the condition $\Theta(D) = X$. We thus have the following result.

THEOREM XIX. *The mapping $\Theta \to \Theta(D)$ is a one-to-one correspondence between infinitesimal R-transformation groups acting on M and differentiable vector fields on M.*

DEFINITION X. If X is a differentiable vector field on M then the *infinitesimal R-transformation group associated with* X is that infinitesimal R-transformation group Θ acting on M such that $\Theta(D) = X$.

DEFINITION XI. A differentiable vector field X on M will be called *generating*, *univalent*, or *proper* respectively, according as its associated infinitesimal R-transformation group Θ has these properties. A local R-transformation group generated by Θ will be said to be *generated by* X, i.e. a local R-transformation group φ acting on M is generated by X if $\varphi^{+}(D) = X$.

The following result is classical.

LEMMA a. *If M is a connected, Hausdorff, differentiable manifold and p and q are distinct points of M then there exists a continuous one-to-one map, σ, of the interval $[0,1]$ into M such that $\sigma(0) = p$ and $\sigma(1) = q$.*

LEMMA b. *If f is a local diffeomorphism of a connected, Hausdorff, differentiable manifold into R then f is one-to-one and hence a diffeomorphism.*

PROOF. Suppose on the contrary there were distinct points p and q in M with $f(p) = f(q)$ and let σ be a continuous one-to-one map of $[0,1]$ into M with $\sigma(0) = p$ and $\sigma(1) = q$. If $g = f \circ \sigma$ then, as σ is one-to-one and f locally one-to-one, g is locally one-to-one.

In particular g is not constant and hence assumes values other than $g(0) = g(1)$. For definiteness assume g takes on values greater than $g(0)$. Then as $[0,1]$ is compact g assumes a maximum value at a point t with $0 < t < 1$. But as g is one-to-one and continuous in an open interval I containing t, by a classical theorem it is strictly monotone on I and hence cannot assume its maximum at t. This contradiction completes the proof.

THEOREM XX. *Let X be a differentiable vector field on M. Then the following four conditions are equivalent.*

(1) X is univalent.

(2) There exists a (necessarily unique) maximum local R-transformation group acting on M generated by X.

(3) X is generating.

(4) If Θ is the infinitesimal R-transformation group associated with X then every leaf of Θ^* is a Hausdorff manifold.

PROOF. Let Θ be as in (4). Then (1) means that Θ is univalent and hence there is a unique maximum local R-transformation group φ acting on M such that $\varphi^+(D) = \Theta(D) = X$ (corollary of theorem X) so (1) implies (2). That (2) implies (3) is obvious and that (3) implies (4) follows from theorem VIII of chapter II. If (4) holds, then by theorem IV of chapter II and lemma b above, Π_R maps each leaf of Θ^* diffeomorphically into R, hence by definition VI Θ is univalent, i.e. X is univalent so (4) implies (1).

COROLLARY. *If M is a Hausdorff differentiable manifold then every differentiable vector field on M is univalent, hence $\varphi \to \varphi^+(D)$ is a one-to-one correspondence between all* maximum local R-transformation groups acting on M and all differentiable vector fields on M. Thus given a differentiable vector field on M it makes sense to speak of THE maximum local R-transformation group it generates.

PROOF. Since M is Hausdorff condition (4) of the theorem is automatically satisfied.

This corollary is apparently a very old result and of course can be proved ab initio with very little trouble. Though almost everyone interested in transformation groups seems to be aware of it in one form or another (usually in the form that a vector field on a Hausdorff manifold generates a local transformation group satisfying (1) of definition VII) I do not know who first discovered it. Probably most people, like the author, rediscovered it for themselves as the natural global form of the classical existence and uniqueness theorem for first-order ordinary differential equations.

8. The Need for Non-Hausdorff Manifolds.

In chapter II we remarked that in view of the corollary of theorem XI it might have seemed well-advised, in the interest of elegance and simplicity, to have disallowed non-Hausdorff manifolds throughout the entire theory of transformation groups. Corollary 2 of theorem XII, corollaries 1 and 2 of theorem XVIII and the corollary of theorem XX seem to give additional weight to this view. There is, however, another side of this coin. The manifold of the universal globalization of a univalent infinitesimal G-transformation group acting on M, constructed in theorem IX, was essentially $G \times M/\Theta^{*}$. Now the quotient manifold of a Hausdorff manifold defined by a regular differential system is not necessarily Hausdorff. In particular, it is easy to construct a univalent infinitesimal G-transformation group acting on a Hausdorff manifold M such that $G \times M/\Theta^*$ is not Hausdorff. Thus if we were to disallow non-Hausdorff manifolds we would have to drop (1) from theorem X. Actually the situation would be even worse, we would have to drop (2) from theorem X also as the example below shows. Needless to say (3) and (4) of theorem X would still be equivalent, but to prove their equivalence without the intermediate steps involving (1) and (2) would be quite complicated. In fact, the whole theory of non-Hausdorff manifolds and of quotient manifolds developed in chapter I, and the theory of globalizations developed at the beginning of this chapter, were developed expressly for the purpose of simplifying and of making more transparent the author's original proof of the equivalence of (3) and (4) which used these concepts only implicitly.

We now exhibit a univalent infinitesimal R-transformation group acting on a Hausdorff manifold which admits no Hausdorff globalization.

Let $M = R \times R - \{(0,0)\}$ and let x and y be the usual coordinate functions on M, i.e. $x(s,t) = s$, $y(s,t) = t$. Let Θ be the infinitesimal R-transformation group acting on M such that $\Theta(D) = (1 - \cos\theta)\,\partial/\partial y$ where θ is the usual polar angle. By the corollary of theorem XX, Θ is univalent. Let φ be the maximum local R-transformation group it generates. It is easy to say what is φ fairly explicitly:

$D_\varphi p = R$ unless p is of the form $(0,t)$

$D_\varphi p = (-t, \infty)$ if p is of the form $(0,t)$ with $t > 0$

$D_\varphi p = (-\infty, t)$ if p is of the form $(0,t)$ with $t < 0$.

φ^p is uniquely determined by the conditions that $X = x \circ \varphi$ and $Y = y \circ \varphi$ are the solutions of the system of differential equations $dX/dt = 0$ $dY/dt = 1 - X/\sqrt{X^2 - Y^2}$ with domain $D_\varphi p$ and satisfying the initial conditions $X(0) = x(p)$, $Y(0) = y(p)$. It is easily verified from this that

$\lim_{n\to\infty} \varphi(2,(-1/n,-1)) = (0,+1)$ and that $\lim_{n\to\infty} y(\varphi(2,(1/n,-1)) = 0$.

We can now show that if (M^*,ψ) is any globalization of Θ then M^* is not Hausdorff. In fact supposing M^* is Hausdorff we will derive a contradiction. By the corollary of theorem VIII, $\varphi = \psi \restriction D_\varphi$ hence since limits are unique in a Hausdorff space

$$\psi(2,(0,-1)) = \psi(\lim_{n\to\infty}(2,(-1/n,-1))) = \lim_{n\to\infty} \psi(2,(-1/n,-1)) =$$

$$\lim_{n\to\infty} \varphi(2,(-1/n,-1)) = (0,1) \text{ . Hence } 1 = y(0,1) = y(\psi(2,(0,-1))) =$$

$$y(\lim_{n\to\infty} \psi(2,(1/n,-1)) = y(\lim_{n\to\infty} \varphi(2,(1/n,-1))) = \lim_{n\to\infty} y(\varphi(2,(1/n,-1))) = 0$$

which is the canonical mathematical contradiction.

The above counter-example is due jointly to the author and Professor A. M. Gleason. Professor Garett Birkhoff provided another counter-example at about the same time.

9. Can Theorem XX Be Generalized?

The author originally hoped that theorem XX could be generalized to say that if G is a simply connected Lie group and M a Hausdorff differentiable manifold then every infinitesimal G-transformation group Θ acting on M is univalent. If M is compact then corollary 2 of theorem XVIII provides a proof and in fact shows that Θ must be proper. After weeks of vain searching for a proof in the general case, we were rescued by Dr. Albert Nijenhuis who gave an elegant counter-example for the case $G = R^2$. As Dr. Nijenhuis himself realized, his example contained all the essential ideas for a counter-example in the general case such as we give below. It is interesting to note that R is characterized among all connected Lie groups by the property that each of its connected subsets is simply connected, and it is this fact that bars any generalization of theorem XX.

LEMMA. *If G is any connected Lie group of dimension $r > 0$ other than R then there is an open neighborhood V of the identity in G which is homeomorphic to $R^{r-1} \times S^1$ (where S^1 is the one-sphere = circle).*

PROOF. If $r = 1$ then G must be the circle group and we can take $V = G$. If $r > 1$ let $(x_1 \ldots x_r, \mathcal{O})$ be a cubical coordinate system of breadth 2 centered at e and take for V the hypervolume of revolution generated by rotating $\{p \varepsilon \mathcal{O} : x_1(p) = 0$ and $x_i(p) < 1/8 \quad i = 2 \ldots r\}$ about the $(r-2)$-plane $x_1 = 0$, $x_2 = 1/4$.

THEOREM XXI. *If G is a connected Lie group of dimension $r > 0$ other than R then there is a manifold M diffeomorphic to R^r and an infinitesimal G-transformation group Θ acting on M such that*

(1) *For each $p \varepsilon M$ $L \to \Theta(L)_p$ maps $\mathfrak{g}$ isomorphically onto the tangent space to M at p (so a fortiori Θ is a Lie algebra isomorphism).*

(2) *Θ is not univalent.*

PROOF. Let V be an open neighborhood of e in G homeomorphic to $R^{r-1} \times S^1$ and let (M, Π) be a universal covering manifold of V. Then M is diffeomorphic to R^r and if for each $L \varepsilon \mathfrak{g}$ we define $\Theta(L)$ by $\delta\Pi(\Theta(L)_p) = L_{\Pi(p)}$ (which is possible since Π is a local diffeomorphism) then Θ is an infinitesimal G-transformation group acting on M satisfying (1). Since V is not simply connected, we can find distinct points p and q in M such that $\Pi(p) = \Pi(q) = e$. Let $\bar{\sigma} : [0,1] \to M$ be a C^∞ arc in M with $\bar{\sigma}(0) = p$ and $\bar{\sigma}(1) = q$ and let $\sigma = \Pi \circ \bar{\sigma}$. Defining $\varphi : [0,1] \to G \times M$ by $\varphi(t) = (\sigma(t), \bar{\sigma}(t))$ it is clear that φ is an integral curve of Θ^*, hence its endpoints

$(e,p) = \varphi(0)$ and $(e,q) = \varphi(1)$ belong to the same leaf Σ of $\mathfrak{G}^*$. Since $\Pi_G(e,p) = e = \Pi_G(e,q)$, Π_G is not one-to-one on Σ and hence (theorem III) $\mathfrak{G}$ is not univalent.

Chapter IV

LIE TRANSFORMATION GROUPS

In this chapter M will denote an n-dimensional Hausdorff differentiable manifold and $G(M)$ the group of bi-differentiable homeomorphisms of M onto itself. We denote by $V(M)$ the set of proper differentiable vector fields on M. Each $L \varepsilon V(M)$ generates a global R-transformation group acting on M and it is natural to try to develop a Lie theory for $G(M)$ taking $V(M)$ as the analogue of the Lie algebra. If M is compact then every differentiable vector field on M is proper, so $V(M)$ is just the Lie algebra of all differentiable vector fields on M. If M is not compact, however, then although $V(M)$ is stable under multiplication by real scalars, it is not stable under addition and the bracket operation. For example, let $M = R \times R$ with x and y the usual coordinate system. Let $X = y\, \partial/\partial x$ and $Y = (x^2/2)\, \partial/\partial y$. Then X and Y are proper and in fact generate respectively the global R-transformation groups φ and ψ given by

$\varphi(t,(u,v)) = (u + vt, v)$ and $\psi(t,(u,v)) = (u, v + u^2 t/2)$.

On the other hand, $[X,Y] = xy(\partial/\partial y) - (x^2/2)(\partial/\partial x)$ is not proper and in fact the maximum local R-transformation group θ generated by $[X,Y]$ is given by

$\theta(t,(u,v)) = ((2u/(2 + ut)), v \exp(\int_0^t (2u/(2 + uz))dz))$ with

$D_\theta = \{ (t,(u,v)) : ut + 2 > 0 \}$.

Also $X + Y = y(\partial/\partial x) + (x^2/2)(\partial/\partial y)$ is not proper. In fact if σ is the maximum local R-transformation group generated by $X + Y$ and we put $f(t) = x \circ \sigma(t,(u,v))$ and $g(t) = y \circ \sigma(t,(u,v))$ then f and g are solutions of the differential equations $df/dt = g$, $dg/dt = f^2/2$, $f(0) = u, g(0) = v$. It is readily verified that these differential equations do not have solutions defined for all t unless $u = v = 0$.

However, there is a remarkable fact that makes $V(M)$ 'enough of a Lie algebra' to develop a useful Lie theory; namely, that if a set of proper vector fields on M generates a finite dimensional Lie algebra, then this Lie algebra consists entirely of proper vector fields. It is to the non-trivial proof of this fact that we now proceed (theorem III).

1. Two Theorems on Lie Groups.

We denote the adjoint representation of a Lie group G by ad. Thus for each $g \varepsilon G$ $ad(g)$ is the differential of the inner automorphism $h \to ghg^{-1}$ considered as acting on the Lie algebra $\mathfrak{g}$ of right invariant vector fields on G. We note that by [1, proposition 1, page 118] (taking $\tilde{\omega}$ to be $h \to ghg^{-1}$) that if $g \varepsilon G$ and $X \varepsilon \mathfrak{g}$ then $\exp(ad(g)X) = g \exp(X) g^{-1}$, and that from [1, page 124] for X and Y in $\mathfrak{g}$ we have $[X,Y] = \lim_{t \to 0} (1/t)(ad(\exp tX)Y - Y)$, the limit being taken in the unique topology with respect to which $\mathfrak{g}$ is a topological vector space. Since every subspace of $\mathfrak{g}$ is closed in the latter topology, it follows that if V is a subspace of $\mathfrak{g}$ such that $ad(\exp tX)Y \varepsilon V$ for all $t \varepsilon R$ (so in particular $Y = ad(\exp 0X)Y \varepsilon V$) then $[X,Y] \varepsilon V$.

A subset S of a Lie algebra $\mathcal{L}$ will be called a set of generators for $\mathcal{L}$ if there is no proper Lie subalgebra of $\mathcal{L}$ including S.

THEOREM I. <u>Let G be a connected Lie group and $\mathfrak{g}$ the Lie algebra of right invariant vector fields on G. If S is a set of generators for $\mathfrak{g}$ such that $X \varepsilon S$ implies $tX \varepsilon S$ for all real t then $\exp(S)$ is a set of generators for G.</u>

PROOF. Let H be the subgroup of G generated by $\exp(S)$, $U = \{X \varepsilon \mathfrak{g} : \exp tX \varepsilon H \text{ for all } t \varepsilon R\}$ and V the subspace of $\mathfrak{g}$ spanned by U. Clearly $S \subseteq U \subseteq V$. If $h \varepsilon H$ and $X \varepsilon U$ then since $\exp(t\, ad(h)X) = \exp(ad(h)tX) = h(\exp tX)h^{-1} \varepsilon H$ it follows that $ad(h)X \varepsilon U$. Thus $ad(H)U \subseteq U$ so by linearity $ad(H)V \subseteq V$. It follows that if $X \varepsilon U$ and $Y \varepsilon V$ then $ad(\exp tX)Y \varepsilon V$ for all $t \varepsilon R$, so by the remark just preceding the theorem $[X,Y] \varepsilon V$. Thus $[U,V] \subseteq V$ and by linearity $[V,V] \subseteq V$. Since by definition V is a subspace of $\mathfrak{g}$ this shows that it is actually a subalgebra, and since $S \subseteq V$ we have $V = \mathfrak{g}$. Since V is the linear span of U we can find $X_1 \dots X_r$ in U forming a basis for $\mathfrak{g}$. Then by the proof of [1, proposition 1, page 128] elements of the form $(\exp t_1X_1) \dots (\exp t_rX_r)$ cover a neighborhood of the identity in G and hence generate G. Since the X_i are in U all the latter elements lie in H so $H = G$ as was to be proved.

THEOREM II. *Let G be a Lie group and $\mathfrak{g}$ its Lie algebra of right invariant vector fields on G. If S is a set of generators for $\mathfrak{g}$ such that $X \varepsilon S$ implies $tX \varepsilon S$ for all real t and $X, Y \varepsilon S$ implies $ad(\exp X)Y \varepsilon S$ then S spans $\mathfrak{g}$.*

PROOF. Let V be the linear span of S. Since $ad(\exp(S))S \subseteq S$ if follows that $ad(\exp(S))V \subseteq V$ and since (theorem I) $\exp(S)$ generates the connected component of the identity in G, which includes $\exp(V)$, it follows that $ad(\exp(V))V \subseteq V$. By the remark immediately preceding theorem I, it follows that $[V,V] \subseteq V$. Then since V is a subspace of $\mathfrak{g}$ it is a subalgebra of $\mathfrak{g}$ and, since the set S of generators of $\mathfrak{g}$ is included in V, $V = \mathfrak{g}$ as was to be proved.

2. Infinitesimal Groups.

In the following sequence of lemmas, Θ will denote an infinitesimal G-transformation group acting on a Hausdorff differentiable manifold M. For each $p \varepsilon M$ we denote by Σ_p the leaf of Θ^* which contains (e,p).

LEMMA a. *A necessary and sufficient condition for a local R-transformation group φ acting on M to be generated by $\Theta(X)$ is that $(\exp tX, \varphi_t(p)) \varepsilon \Sigma_p$ for all $(t,p) \varepsilon D_\varphi$.*

PROOF. Let $E(t) = \exp tX$ and for each $p \varepsilon M$ define $\Phi^p : D_{\varphi^p} \rightarrow G \times M$ by $\Phi^p(t) = (E(t), \varphi^p(t))$. Then $\delta E(D_t) = X_{E(t)}$ and so by theorem II of chapter II $\delta\Phi^p(D_t) = (X_{E(t)}, \varphi^+(D)_{\varphi(t,p)})$. Thus if φ is generated by $\Theta(X)$, i.e. $\varphi^+(D) = \Theta(X)$, then $\delta\Phi(D_t) = (X_{E(t)}, \Theta(X)_{\varphi(t,p)}) \varepsilon \Theta^*_{\Phi^p}$ so Φ^p is an integral curve of Θ^*. Since $\Phi^p(0) = (e,p)$ it will follow that the range of Φ^p is included in the leaf of Θ^* containing (e,p), which is Σ_p. Conversely if Φ^p maps into Σ_p then since Φ^p is a differentiable map into $G \times M$ it is also, by the corollary of theorem IV, chapter II, a differentiable map into Σ_p and therefore $(X_e, \varphi^+(D)_p) = \delta\Phi^p(D_e)$ belongs to the tangent space to Σ_p at (e,p), which is $\Theta^*_{(e,p)}$. By definition of Θ^* it follows that $\varphi^+(D)_p = \Theta(X)_p$ so $\varphi^+(D) = \Theta(X)$, i.e. φ is generated by $\Theta(X)$.

LEMMA b. *If $X_1 \ldots X_k$ are k elements of $\mathfrak{g}$ such that $\Theta(X_i)$ is proper, $i = 1 \ldots k$, and if ψ^i is the global R-transformation group generated by $\Theta(X_i)$, then for each $p \varepsilon M$ the mapping $\Phi^p : (t_1 \ldots t_k) \rightarrow ((\exp t_k X_k) \ldots (\exp t_1 X_1), \psi^k_{t_k} \ldots \psi^1_{t_1}(p))$ is a differentiable map of R^k into Σ_p.*

PROOF. Since the mappings $(t,p) \to \psi_t^i(p)$ are jointly differentiable in (t,p) it is clear at any rate that each Φ^p is a differentiable map of R^k into $G \times M$, so by the corollary of theorem IV, chapter II, it suffices to show that Φ^p has its range included in Σ_p. Now for $k = 1$ this follows from lemma a so we proceed by induction on k and assume the lemma holds for $k = m$. Let $g = (\exp t_m X_m) \ldots (\exp t_1 X_1)$ and let $q = \psi_{t_m}^m \ldots \psi_{t_1}^1(p)$. By the induction hypothesis $(g,q) \varepsilon \Sigma_p$ so $(e,q) \varepsilon \bar{R}_g \Sigma_p$ and hence by the corollary of theorem VI of chapter II $\Sigma_q = \bar{R}_g \Sigma_p$. Now by lemma a $(\exp t_{m+1} X_{m+1}, \psi_{t_{m+1}}^{m+1}(q)) \varepsilon \Sigma_q = \bar{R}_g \Sigma_p$ so $((\exp t_{m+1} X_{m+1})g, \psi_{t_{m+1}}^{m+1}(q)) \varepsilon \Sigma_p$ which is the desired result for $k = m + 1$.

LEMMA c. Let X and Y be elements of $\mathfrak{g}$ such that $\Theta(X)$ and $\Theta(Y)$ are proper. Then $\Theta(\mathrm{ad}(\exp X)Y)$ is also proper.

PROOF. Let φ and ψ be the global R-transformation groups acting on M generated by $\Theta(X)$ and $\Theta(Y)$ respectively and let $\lambda_t = \varphi_1 \circ \psi_t \circ \varphi_1^{-1}$. Then $\lambda : (t,p) \to \lambda_t(p)$ is clearly a global R-transformation group acting on M hence it will suffice to show that it is generated by $\Theta(\mathrm{ad}(\exp X)Y)$. By lemma a it will be enough to show for all $(t,p) \varepsilon R \times M$ that $(\exp t(\mathrm{ad}(\exp X)Y), \lambda_t(p)) \varepsilon \Sigma_p$. But as $\exp t(\mathrm{ad}(\exp X)Y) = \exp(\mathrm{ad}(\exp X)tY) = (\exp X)(\exp tY)(\exp -X)$ and $\lambda_t = \varphi_1 \circ \psi_t \circ \varphi_{-1}$ this is an immediate consequence of lemma b.

LEMMA d. If there is a basis $X_1 \ldots X_r$ for $\mathfrak{g}$ such that $\Theta(X_i)$ is proper $i = 1 \ldots r$ then Θ is uniform.

PROOF. Let ψ^i be the global R-transformation group generated by $\Theta(X_i)$ and let $(x_1 \ldots x_r, V)$ be a canonical coordinate system of the second kind in G with respect to $X_1 \dots X_r$, so that for each $g\varepsilon V$ $g = (\exp x_1(g)X_1) \dots (\exp x_r(g)X_r)$. Then by lemma b for each $p\varepsilon M$ the map $\Phi^p : g \to (g, \psi^1_{x_1(g)} \dots \psi^r_{x_r(g)}(p))$ is a differentiable map of V into Σ_p. Let V^p be the image of Φ^p. Then clearly Π_G maps V^p one-to-one onto V. Moreover V^p is open in Σ_p. In fact given $q\varepsilon V^p$ let U be a neighborhood of q in Σ_p such that Π_G maps U diffeomorphically into G (theorem IV of chapter II) and let W be a neighborhood of $\Pi_G(q)$ in V such that $\Phi^p(W) \subseteq U$. Then $\Phi^p(W) = (\Pi_G \restriction U)^{-1}(W)$ is an open set of Σ_p containing q and included in V^p. By theorem V of chapter II, it follows that V^p is a component of $\Sigma_p \cap \Pi_G^{-1}(V)$ and in fact since $\Phi^p(e) = (e,p)$ it is the component of (e,p) in $\Sigma_p \cap \Pi_G^{-1}(V)$. Thus for each $p\varepsilon M$ Π_G maps V^p = (the component of (e,p) in $\Sigma_p \cap \Pi_G^{-1}(V)$) one-to-one onto V and hence by definition VIII of Chapter III, V is a uniform neighborhood for M with respect to Θ and so Θ is uniform.

THEOREM III. <u>Let $\mathcal{L}$ be a finite dimensional Lie algebra of vector fields on the Hausdorff differentiable manifold</u> M. <u>Then the following three conditions are equivalent.</u>

(1) <u>Every</u> $L\varepsilon \mathcal{L}$ <u>is proper</u>.

(2) <u>The set of</u> $L\varepsilon \mathcal{L}$ <u>which are proper generate the Lie algebra</u> $\mathcal{L}$.

(3) <u>There is a connected Lie group</u> G <u>and a global</u> G-<u>transformation group</u> φ <u>such that</u>

(a) $g \to \varphi_g$ is an isomorphism of G into $G(M)$.

(b) φ^+ is an isomorphism onto $\mathcal{L}$.

PROOF. That (1) implies (2) is trivial. Suppose that (2) holds. By a classical theorem there is a simply connected Lie group $\widetilde{G}$ with Lie algebra $\widetilde{\mathfrak{g}}$ isomorphic to $\mathcal{L}$. Let Θ be an isomorphism of $\widetilde{\mathfrak{g}}$ onto $\mathcal{L}$. Then Θ is an infinitesimal $\widetilde{G}$-transformation group acting on M. Let S be the set of $X \varepsilon \widetilde{\mathfrak{g}}$ such that $\Theta(X)$ is proper. Then by (2) S generates $\widetilde{\mathfrak{g}}$. If $X \varepsilon S$ and s is a real number then $sX \varepsilon S$ (in fact if φ is the global R-transformation group generated by X then $(t,p) \to \varphi(ts,p)$ is the global R-transformation group generated by sX). By lemma c if X and Y belong to S so does $\mathrm{ad}(\exp X)Y$. It follows from theorem II that S spans $\widetilde{\mathfrak{g}}$ and hence we can choose a base $X_1 \ldots X_r$ for $\widetilde{\mathfrak{g}}$ with each $X_i \varepsilon S$, i.e. such that each $\Theta(X_i)$ is proper. By lemma d Θ is uniform and hence by the corollary of theorem XVII of chapter III, Θ is proper. Let ψ be the global $\widetilde{G}$-transformation group acting on M generated by Θ and let K be the kernel of $g \to \psi_g$. Then K is a closed normal subgroup of $\widetilde{G}$. If X is in the Lie algebra of K (which is a Lie subgroup of $\widetilde{G}$ by [1, corollary, page 135]) then for any $p \varepsilon M$ $\psi^p(\exp tX) = p$ and hence putting $E(t) = \exp tX, \Theta(X)_p = \psi^+(X)_p = \delta\psi^p(X_e) = \delta\psi \circ \delta E(D_\circ) = \delta(\psi \circ E)(D_\circ) = 0$ and so $\Theta(X) = 0$ and therefore, as Θ is an isomorphism, $X = 0$. Thus K is a zero dimensional Lie subgroup of $\widetilde{G}$ and so discrete and hence the natural homomorphism h of $\widetilde{G}$ onto $G = \widetilde{G}/K$ is a local diffeomorphism. It is immediate that we get a global G-transformation group φ with properties (a) and (b) of (3) by taking $\varphi(h(g),p) = \psi(g,p)$. Thus (2) implies (3).

Finally if (3) holds and $L \varepsilon \mathcal{L}$ then L generates the global R-transformation group $(t,p) \to \varphi(\exp t\varphi^{+-1}(L),p)$ so L is proper. Thus (3) implies (1).

DEFINITION I. A finite dimensional Lie algebra of differentiable vector fields on a Hausdorff differentiable manifold M will be called an <u>infinitesimal group</u> of M if it satisfies any one and hence all of the three equivalent conditions of theorem III.

COROLLARY. <u>If $\mathcal{L}$ is a finite dimensional Lie algebra of differentiable vector fields on a Hausdorff differentiable manifold M then the proper vector fields contained in $\mathcal{L}$ form an infinitesimal group of M.</u>

3. Connected Lie Transformation Groups.

DEFINITION II. Let $\mathcal{J}$ be a connected Lie group whose underlying group is a subgroup of the group $G(M)$ of diffeomorphisms of the Hausdorff differentiable manifold M. We shall call $\mathcal{J}$ a <u>connected Lie transformation group of</u> M if the mapping $\varphi : (t,p) \rightarrow t(p)$ of $\mathcal{J} \times M \rightarrow M$ is differentiable and hence a global $\mathcal{J}$-transformation group acting on M. We call φ the <u>natural global</u> $\mathcal{J}$-transformation group and the range of φ^+ is called the <u>infinitesimal group</u> of $\mathcal{J}$.

Note that if $\mathcal{J}$ is a connected Lie transformation group and φ the natural global $\mathcal{J}$-transformation group then $\varphi_t = t$ so of course $t \rightarrow \varphi_t$ is one-to-one. If X is in the Lie algebra of right invariant vector fields on $\mathcal{J}$ then clearly $(t,p) \longrightarrow (\exp tX)(p) = \varphi(\exp tX, p)$ is the global R-transformation group generated by $\varphi^+(X)$. Thus if $\varphi^+(X) = 0$ then $\exp tX$ is the identity of $\mathcal{J}$ for all real t so $X = 0$. Hence φ^+ is an isomorphism onto the infinitesimal group $\mathcal{L}$ of $\mathcal{J}$. Thus conditions (3a) and (3b) of theorem III are satisfied and $\mathcal{L}$ is an infinitesimal group in the sense of definition I. Conversely:

THEOREM IV. *Every infinitesimal group of the Hausdorff manifold* M *is the infinitesimal group of a unique connected Lie transformation group of* M.

PROOF. If $\mathcal{L}$ is an infinitesimal group of M then by property (3) of theorem III there is a connected Lie group G and a global G-transformation group φ acting on M such that the range of φ^+ is $\mathcal{L}$ and $g \to \varphi_g$ is an isomorphism of the underlying group of G onto a subgroup T of $G(M)$. If wecarry the topology of G over to T via $g \to \varphi_g$ then T becomes a connected Lie transformation group $\mathfrak{J}$ with infinitesimal group $\mathcal{L}$.

Let $\mathfrak{J}'$ be any other connected Lie transformation group with infinitesimal group $\mathcal{L}$ and let ψ and σ be the natural global $\mathfrak{J}$ and $\mathfrak{J}'$-transformation groups. Since ψ^+ and σ^+ are isomorphisms onto $\mathcal{L}$, $\sigma^{+-1} \circ \psi^+$ is an isomorphism of the Lie algebra of $\mathfrak{J}$ onto the Lie algebra of $\mathfrak{J}'$. By [1, the italicized remark at the bottom of page 113] there is a local isomorphism h of a neighborhood V of the identity in $\mathfrak{J}$ onto a neighborhood V' of the identity in $\mathfrak{J}'$ such that $\delta h = \sigma^{+-1} \circ \psi^+$. The map $(t,p) \to \sigma(h(t),p)$ of $V \times M \to M$ is a local $\mathfrak{J}$ -transformation group acting on M with infinitesimal generator $\sigma^+ \circ \delta h = \psi^+$ and hence by chapter III, theorem VII, it is a restriction of ψ. Thus for $t \varepsilon V$ and $p \varepsilon M$ $t(p) = \psi(t,p) = \sigma(h(t),p) = h(t)(p)$, i.e. $h(t) = t$ so h is the identity map. It follows in particular that V is a neighborhood of the identity in $\mathfrak{J}'$. Since $\mathfrak{J}$ and $\mathfrak{J}'$ are each connected they are each the subgroup of $G(M)$ generated by V and so have identical underlying groups. Since they coincide in a neighborhood of the identity as topological groups, they are identical as topological groups.

4. Lie Transformation Groups.

When should we call a subgroup T of $G(M)$ a Lie transformation group? One natural requirement is that T be a Lie group in a topology which is not too weak, namely stronger than the compact-open topology (see section 1 of appendix) so that $(t,p) \to t(p)$ is a continuous map of $T \times M$ into M. If this were all we were to require then we could simply give T the discrete topology. However, there is a second natural requirement for a topology of T that goes in the other direction, namely that if φ is a global R-transformation group acting on M such that $\varphi_t \varepsilon T$ for all $t \varepsilon R$ then $t \to \varphi_t$ should be continuous, i.e. a one-parameter subgroup of T. As we shall see, these two conditions determine, if any, a unique Lie group topology for T.

DEFINITION III. Let T be a subgroup of $G(M)$ and L a proper vector field on M with associated global R-transformation group φ. We shall say that L is <u>tangent to</u> T if $\varphi_t \varepsilon T$ for all $t \varepsilon R$.

DEFINITION IV. Let $\mathcal{J}$ be a topological group whose underlying group T is a subgroup of $G(M)$. We shall call the topology of $\mathcal{J}$ a <u>Lie topology</u> for T if

(1) $\mathcal{J}$ is a Lie group,

(2) the map $(t,p) \to t(p)$ of $\mathcal{J} \times M \to M$ is differentiable,

(3) if L is a proper vector field on M tangent to T with associated global R-transformation group φ then $t \to \varphi_t$ is a one-parameter subgroup of $\mathcal{J}$ (and hence by (1) $\varphi_t = \exp tX$ for some X in the Lie algebra of $\mathcal{J}$).

We note that by virtue of [7, theorem, page 212] it follows that (1) could be replaced by (1') $\mathcal{J}$ is a locally compact group, and (2) could be replaced by (2') the map $(t,p) \to t(p)$ of $\mathcal{J} \times M \to M$ is continuous, and that the stronger statements (1) and (2) would then follow. However, this depends on some very deep and recent discoveries about the structure of locally compact groups and we feel that it is preferable to frame definition IV with the stronger statements so as not to obscure the elementary nature of the present theory.

The following well-known result is an immediate consequence of the existence of canonical coordinate systems of the second kind in a Lie group and the fact [1, remark, page 128] that every one-parameter subgroup of a Lie group is of the form $t \to \exp tX$ for some X in the Lie algebra.

LEMMA. *Let* G *and* H *be Lie groups and let* φ *be a homomorphism of the underlying group of* G *into the underlying group of* H. *If* $\varphi \circ \psi$ *is a one-parameter subgroup of* H *whenever* ψ *is a one-parameter subgroup of* G *then* φ *is continuous. In particular, if* G *and* H *have the same underlying groups and the same one-parameter subgroups they are identical as topological groups.*

THEOREM V. *A subgroup* T *of* $G(M)$ *admits at most one Lie topology.*

PROOF. Let $\mathcal{J}$ be a topological group with underlying group T whose topology is a Lie topology for T. If $t \to \varphi_t$ is a one-parameter subgroup of $\mathcal{J}$ then by [1, proposition 1, page 128] $t \to \varphi_t$ is an analytic map of R into $\mathcal{J}$ and so, by (2) of definition IV, $\varphi : (t,p) \to \varphi_t(p)$ is a differentiable map of $R \times M \to M$ and hence a global R-transformation group acting on M. Clearly the infinitesimal generator of φ is tangent to T. Conversely, if L is a proper vector field on M tangent to T and φ the global R-transformation group it generates, then by (3) of definition IV $t \to \varphi_t$ is a one-parameter subgroup of $\mathcal{J}$. Thus the one-parameter subgroups of $\mathcal{J}$ are uniquely determined by T and the properties listed in definition IV as the mappings $t \to \varphi_t$ where φ is a global R-transformation group generated by a vector field tangent to T. The theorem follows directly from this and the lemma.

DEFINITION V. A subgroup T of $G(M)$ will be called a Lie transformation group of M if it admits a Lie topology. By theorem VI it makes sense to speak of the Lie topology of a lie transformation group of M. Properties having significance for a Lie group when used in connection with a Lie transformation group are to be interpreted with respect to its Lie topology. In particular, if T is a Lie transformation group of M, then we can speak of the connected component of the identity T_o of T. It is trivial from definition III that T_o is a connected Lie transformation group of M (in the sense of definition II). By the infinitesimal group of T we shall mean the infinitesimal group of T_o. It is clearly just the set of all proper vector fields on M tangent to T.

THEOREM VI. *Let $\mathcal{J}$ be a Lie group satisfying the second axiom of countability (or equivalently, with only countably many components) whose underlying group T is a subgroup of $G(M)$. If the mapping $(t,p) \to t(p)$ of $\mathcal{J} \times M \to M$ is differentiable then T is a Lie transformation group of M and moreover the given topology of $\mathcal{J}$, the Lie topology of T, and the modified compact-open topology of T (definition A, section*

3 of appendix) are all the same.

PROOF. Properties (1) and (2) in definition IV hold by hypothesis and property (3) of definition IV follows from theorem G of the appendix. Thus T is a Lie transformation group of M and the topology of $\mathfrak{T}$ its unique Lie topology. It also follows from theorem G of the appendix that the topology of $\mathfrak{T}$ is the modified compact-open topology of T .

COROLLARY. *If $\mathfrak{T}$ is a connected Lie transformation group of M in the sense of definition II then the underlying group of $\mathfrak{T}$ is a Lie transformation group of M and the topology of $\mathfrak{T}$ is its unique Lie topology.*

We note that in theorem VI it would have been sufficient to assume that $\mathfrak{T}$ was a locally compact topological group satisfying the second axiom of countability and that $(t,p) \to t(p)$ of $\mathfrak{T} \times M \to M$ was continuous. See the remark following definition IV.

We now develop a necessary and sufficient condition for a subgroup T of G(M) to be a Lie transformation group which is very useful for applications.

LEMMA a. *Let G be a topological group whose underlying group is a normal subgroup of a group T and suppose that for each $t \varepsilon T$ the map $\varphi_t : g \to tgt^{-1}$ of G onto itself is continuous. Then there is a unique topology for T which makes T into a topological group in which G is an open subgroup.*

PROOF. Obvious.

LEMMA b. *Let G be a connected Lie group whose underlying group is a subgroup of a group T and suppose that for each* $t \varepsilon T$

and each one-parameter subgroup ψ of G $\varphi_t \bullet \psi$ is a one-parameter subgroup of G, φ_t being the map $g \rightarrow tgt^{-1}$. Then there is a unique Lie group topology for T with respect to which G is the connected component of the identity.

PROOF. Since G is connected it is generated by the images of its one-parameter subgroups; hence, the hypothesis implies in particular that the underlying group of G is normal in T. By the lemma preceding theorem V, each of the mappings φ_t is continuous and lemma b now follows from lemma a.

THEOREM VII. *A necessary and sufficient condition that a subgroup T of $G(M)$ be a Lie transformation group of M is that the set S of proper vector fields on M tangent to T generate a finite dimensional Lie algebra $\mathcal{L}$. If this condition is fulfilled then $S = \mathcal{L}$ = the infinitesimal group of T.*

PROOF. The necessity of the condition is obvious. Conversely, suppose that $\mathcal{L}$ is finite dimensional. By definition I $\mathcal{L}$ is an infinitesimal group of M. Let G be the connected Lie transformation group of M with infinitesimal group $\mathcal{L}$ (theorem IV) and let φ be the natural global G-transformation group, so that φ^+ is an isomorphism of the Lie algebra $\mathfrak{g}$ of G onto $\mathcal{L}$. If $X \varepsilon \mathfrak{g}$ then clearly $\exp tX = \psi_t$ where ψ is the global R-transformation group generated by $\varphi^+(X)$. In particular, $\exp(\varphi^{+-1}(S)) \subseteq T$. Now clearly $L \varepsilon S$ implies $tL \varepsilon S$ for all real t so by theorem I $\exp(\varphi^{+-1}(S))$ generates G. Since a set of generators of G is included in T it follows that G itself is included in T, from which it follows that $\mathcal{L} = S$.

Now let $t \varepsilon T$ and let $s \to \psi_s$ be a one-parameter subgroup of G. Then $\psi : (s,p) \to \psi_s(p)$ is a global R-transformation group acting on M and since t is a diffeomorphism of M onto itself so also is $\varphi : (s,p) \to (t \circ \psi_s \circ t^{-1})(p)$. Now since for all $s \varepsilon R$ $\psi_s \varepsilon G \subseteq T$ and since $t \varepsilon T$ it follows that $\varphi_s = t \circ \psi_s \circ t^{-1} \varepsilon T$ for all s and hence the proper vector field L generating φ is tangent to T and so belongs to $S = \mathcal{L}$. Thus $s \to \varphi_s$ is a one-parameter subgroup of G.

It now follows from lemma b that there is a unique Lie group $\mathfrak{J}$ with underlying group T such that G is the component of the identity in $\mathfrak{J}$. Since G is a connected Lie transformation group of M the map $(g,p) \to g(p)$ of $G \times M \to M$ is differentiable. If $t_o \varepsilon T$ then Gt_o is the component of t_o in $\mathfrak{J}$. Now $t \to tt_o^{-1}$ maps Gt_o diffeomorphically onto G and as $t_o \varepsilon G(M)$ $p \to t(p)$ maps M diffeomorphically onto M. Thus
$(t,p) \to (tt_o^{-1}, t_o(p)) \to tt_o^{-1}(t_o(p)) = t(p)$ is a differentiable map of $Gt_o \times M \to M$. Since Gt_o is a neighborhood of t_o in $\mathfrak{J}$ it follows that $(t,p) \to t(p)$ of $\mathfrak{J} \times M \to M$ is differentiable.

Thus (1) and (2) of definition II hold. If L is a proper vector field on M tangent to T and φ the global R-transformation group it generates, then $L \varepsilon S = \mathcal{L}$ and hence $t \to \varphi_t$ is a one-parameter subgroup of G and a fortiori of $\mathfrak{J}$ so condition (3) of definition IV is also satisfied and the topology of $\mathfrak{J}$ is a Lie topology for T.

We note that if G_1 and G_2 are two Lie transformation groups of M then the subgroup G of $G(M)$ that they generate need not be. For example, let $M = R \times R$,
$G_1 = \{ \varphi_t : (u,v) \to (u+vt, v) \}$ $G_2 = \{ \psi_t : (u,v) \to (u, v+(u^2 t/2)) \}$.
Then G_1 and G_2 are clearly 1-dimensional Lie transformation groups of M but the group G they generate is not. For $X = y\ \partial / \partial x$ is tangent to G_1 and $Y = (x^2/2)\ \partial / \partial y$ is tangent to G_2 so both X

and Y are tangent to G. But the Lie algebra generated by X and Y is infinite-dimensional as is easily checked directly or as follows from theorem III since we saw that $X + Y$ and $[X,Y]$ were not proper at the beginning of the chapter.

DEFINITION VI. Let $\mathcal{L}$ be a Lie algebra of differentiable vector fields on a differentiable Hausdorff manifold M. We shall call $\mathcal{L}$ a Kobayashi Lie algebra if for each $p \varepsilon M$ the mapping $L \to L_p$ of $\mathcal{L}$ onto M_p is non-singular. If moreover $\mathcal{L}$ consists entirely of proper vector fields we shall call a Kobayashi infinitesimal group.

THEOREM VIII. Let $\mathcal{L}$ be a Kobayashi Lie algebra on the n-dimensional Hausdorff differentiable manifold M. Then $\mathcal{L}$ has dimension $\leqslant n$ and if $\mathfrak{A}$ is the set of proper vector fields in $\mathcal{L}$ then $\mathfrak{A}$ is a Kobayashi infinitesimal group of dimension say $r \leqslant n$. Define $\Theta : p \to \Theta_p$ by $\Theta_p = \{L_p : L \varepsilon \mathfrak{A}\}$. Then Θ is an involutive r-dimensional differential system on M. Let A be the connected Lie transformation group with infinitesimal group $\mathfrak{A}$ (theorem IV). Then for any $p \varepsilon M$ the orbit of p under A coincides with the leaf Σ_p of Θ containing p and in fact $\varphi_p : a \to a(p)$ is a local diffeomorphism of A onto Σ_p. A necessary and sufficient condition that φ^p be a diffeomorphism is that the isotropy group $A_p = \{ a \varepsilon A : a(p) = p \}$ reduce to the identity; so that every φ^p is a diffeomorphism if and only if A operates without fixed points.

PROOF. Since M had dimension n and $L \to L_p$ is non-singular $\mathcal{L}$ had dimension $\leqslant n$. By the corollary of theorem III $\mathfrak{A}$ is an infinitesimal group of M. Since a subalgebra of a Kobayashi algebra is clearly a Kobayashi algebra, $\mathfrak{A}$ is a Kobayashi infinitesimal group say of dimension r. That Θ is an r-dimensional involutive differential system on M follows from [1, proposition 1, page 88]. Let φ be the natural global A-transformation group. Then φ^+ is an isomorphism of the Lie algebra of A onto $\mathfrak{A}$. Since by theorem II of chapter II we have $\delta\varphi^p(X_a) = \varphi^+(X)_{a(p)}$ it follows that $(\delta\varphi^p)_a$ is an isomorphism of the tangent space to A at a onto $\Theta_{a(p)}$. Since $\varphi^p(e) = p$ and A is connected φ^p is a local diffeomorphism of A into Σ_p. In particular, the range of φ^p which is the orbit of p under A is open in Σ_p. Similarly any other orbit under A which intersects Σ_p is an open subset of Σ_p, and since orbits are disjoint and Σ_p is connected, the orbit of p under A must fill up Σ_p, i.e. φ^p is onto Σ_p. The last conclusions of the theorem are obvious.

5. Tensor Structures and Their Automorphism Groups.

Suppose a manifold M comes to us equipped with some extra structure, that is one or more fields of geometrical objects in the sense of Nijenhuis [9] and others. Then it is of interest to discover the automorphism group of this structure, that is, the subgroup of $G(M)$ leaving the given fields of geometrical objects fixed. It is of particular interest to be able to discern when such a group is a Lie transformation group. This has been settled in certain particular cases by Myers and Steenrod [10] (Riemannian structure), Nomizu [11] (complete affine connection), and Kobayashi [12] (absolute parallelism).

In this section we will be content with showing how the previous results of this chapter can be used to get very general theorems in this direction for the case of tensor structures. At a later time we hope to treat this whole question in considerably greater generality and detail.

DEFINITION VII. A <u>tensor structure</u> on a Hausdorff differentiable manifold M is a set S of differentiable tensor fields on M. We denote by $A(S)$ the group of all diffeomorphisms φ of M onto itself such that for all $T \varepsilon S$ we have $\delta\varphi \circ T = T \circ \varphi$ (i.e. $\delta\varphi(T_p) = T_{\varphi(p)}$ for all $p \varepsilon M$, see [8] for the meaning of this notation). $A(S)$ is called the <u>automorphism group of the structure</u> S. We denote by $\mathcal{L}(S)$ the set of all differentiable vector fields L on M such that $L[T] = 0$ for all $T \varepsilon S$ (here $L[T]$ is the Lie derivative of T with respect to L, see [8]). We define $\mathfrak{A}(S)$ to be the set of all proper vector fields in $\mathcal{L}(S)$..

THEOREM IX. <u>If</u> S <u>is a tensor structure on</u> M <u>then</u> $\mathfrak{A}(S)$ <u>is the set of all proper vector fields on</u> M <u>tangent to</u> $A(S)$. <u>Thus a necessary and sufficient condition that</u> $A(S)$ <u>be a Lie transformation group is that</u> $\mathfrak{A}(S)$ <u>generate a finite dimensional Lie algebra. If this condition is fulfilled then</u> $\mathfrak{A}(S)$ <u>itself is a Lie algebra and hence an infinitesimal group and in fact it is the infinitesimal group of</u> $A(S)$.

PROOF. That $\mathfrak{A}(S)$ is the set of all proper vector fields on M tangent to $A(S)$ follows from [8, corollary of theorem VI]. The rest of the theorem follows from theorem VII.

LEMMA. <u>If</u> T <u>is a tensor field on the differentiable manifold</u> M <u>and</u> L <u>and</u> L' <u>are differentiable vector fields on</u> M <u>then</u> $[L,L'][T] = L[L'[T]] - L'[L[T]]$.

PROOF. It follows easily from [8, lemmas a, b, c] that the lemma is true for T a differentiable function, the differential of a differentiable function, or a differentiable vector field. From [8, lemma d] if the theorem holds for two tensor fields T and T' it holds for $T \otimes T'$. An argument like that preceding [8, theorem II] shows that the lemma holds for all tensor fields T.

THEOREM X. *If S is a tensor structure on M then $\mathcal{L}(S)$ is a Lie algebra.*

PROOF. Immediate from the above lemma and [8, theorem III].

DEFINITION VIII. Let S be a tensor structure on M. We shall call S *almost rigid* if $\mathcal{L}(S)$ is finite dimensional and a *Kobayashi structure* if $\mathcal{L}(S)$ is a Kobayashi Lie algebra.

THEOREM XI. *If S is an almost rigid tensor structure then $\mathfrak{A}(S)$ is an infinitesimal group. If S is a Kobayashi structure then $\mathfrak{A}(S)$ is a Kobayashi infinitesimal group.*

PROOF. The first statement follows from theorem XI and the corollary of theorem III. The second statement is a consequence of theorem VIII.

COROLLARY 1. *The automorphism group of an almost rigid tensor structure, S, is a Lie transformation group with infinitesimal group $\mathfrak{A}(S)$.*

PROOF. Immediate from theorems IX and XI.

COROLLARY 2. *If S is a Kobayashi structure then $A(S)$ is a Lie transformation group with infinitesimal group $\mathfrak{A}(S)$.*

For each $p \varepsilon M$ *the orbit,* Σ_p *, of* p *under* $A(S)$ *is a union of leaves of the differential system* Θ *defined by* $\Theta_p = \{ L_p : L \varepsilon \mathfrak{A}(S) \}$ *, and* $a \rightarrow a(p)$ *is a local diffeomorphism of* $A(S)$ *onto* Σ_p .

PROOF. The first statement is obvious from corollary, since a Kobayashi structure is clearly almost rigid. Let $A_o(S)$ be the component of the identity in $A(S)$. If b is any element of $A(S)$ then $a \rightarrow ab^{-1}$ is a diffeomorphism of the component of b in $A(S)$ onto $A_o(S)$. It follows from theorem VIII that $a \rightarrow a(p) = ab^{-1}(b(p))$ is a local diffeomorphism of the component of b in $A(S)$ onto the leaf of Θ containing $b(p)$.

The property of being an almost rigid structure or a Kobayashi structure is infinitesimal in nature and is usually very easily checked. For example, it is quite easy to show that the structure given by a Riemannian tensor is almost rigid and part of the results of Myers and Steenrod in [10] follow from this and corollary 1 above.

DEFINITION IX. An absolute parallelism on an n-dimensional differentiable manifold M is a tensor structure S consisting of n differentiable vector fields $\{ X_1 \ldots X_n \}$ on M which are everywhere linearly independent.

The following is a well-known result.

LEMMA. *If* $S = \{ X_1 \ldots X_n \}$ *is an absolute parallelism on* M *and* p *is any point of* M *then there is a cubical co-ordinate system centered at* p , $(x_1 \ldots x_n, V)$ *say of breadth* 2a, *such that if* $|u_i| < a$ $i = 1 \ldots n$ *and* φ *is the maximum local* R-*transformation group acting on* M *generated by*

$\sum_{i=1}^{n} u_i X_i$ then $(1,p) \varepsilon D_\varphi$, $\varphi(1,p) \varepsilon V$ and $x_i(\varphi(1,p)) = u_i$.

PROOF. If we write $\varphi(t,p) = \exp t(\sum_{i=1}^{n} u_i X_i)$ and $p = \varepsilon$ then the proof is to be found on pages 116 and 117 of [1].

THEOREM XII. <u>An absolute parallelism</u> $S = \{X_1 \ldots X_n\}$ <u>on a connected Hausdorff manifold</u> M <u>is a Kobayashi structure</u>.

PROOF. Let $L \varepsilon \mathcal{L}(S)$. Then if $u_1 \ldots u_n$ are any constants we have by [8, lemma c] $0 = -\sum_{i=1}^{n} u_i L[X_i] = -[L, \sum_{i=1}^{n} u_i X_i] = [\sum_{i=1}^{n} u_i X_i, L] = (\sum_{i=1}^{n} u_i X_i)[L]$. Thus if φ is the maximum local R-transformation group generated by $\sum_{i=1}^{n} u_i X_i$ then by [8, corollary of theorem VI] $\delta\varphi_t(L_p) = L_{\varphi_t(p)}$. In particular if $L_q = 0$ then $L_{\varphi_t(q)} = 0$. Now by the lemma points of the form $\varphi_1(q)$, for varying choices of $u_1 \ldots u_n$, cover a neighborhood of q so that the points where L vanishes is open. Since it is clearly also closed it follows that L does not vanish anywhere unless it is identically zero. This together with theorem IX shows that $\mathcal{L}(S)$ is a Kobayashi Lie algebra.

The theorem proved by Kobayashi in [12] is actually stronger than what is evident from theorem XIII and corollary 2 of theorem XII. In the first place, the group G considered by Kobayashi is <u>a priori</u> larger than $A(S)$. By definition G is the set of homeomorphisms of M onto itself which commute with all maximum local R-transformation groups generated by vector fields of the form $\sum_{i=1}^{n} u_i X_i$, $u_1 \ldots u_n$ being arbitrary real numbers. But it is immediate from the lemma above that any $\varphi \varepsilon G$ is differentiable and it then follows that $\varphi \varepsilon A(S)$ [12, lemma 3]. It is also evident from the lemma above that the set of points left fixed by any $a \varepsilon A(S)$ is open and closed, and so is either void or all of M. Thus $A(S)$ acts without fixed points and so (in the terminology of corollary 2 of theorem XI) $a \to a(p)$ is actually a diffeomorphism of $A(S)$ onto Σ_p. Moreover it can be shown that Σ_p is closed in M [12, lemma 7] and that $a \to a(p)$ is bicontinuous. It follows easily that the Lie topology of $A(S)$ is the compact-open topology and that Σ_p is a regularly imbedded closed submanifold of M diffeomorphic to $A(S)$. In [10] Myers and Steenrod prove roughly similar results for the group of isometries of a Riemannian manifold and it is natural to try to prove corresponding theorems for the automorphism groups of general almost rigid structures.

We end this chapter with a conjecture whose positive solution would make available a powerful tool for the further study of the Lie structure of $G(M)$, and would in itself give a much clearer insight into this structure than we now have.

Conjecture: Let M be a connected differentiable manifold and G a connected Lie transformation group of M. Denote by $\bar{G}$ the closure of G in $G(M)$ taken relative to the compact-open topology. Then $\bar{G}$ is a Lie transformation group of M and the Lie topology of $\bar{G}$ is its compact-open topology.

An equivalent formulation is that every connected Lie transformation group of M is an analytic subgroup of a Lie transformation group whose Lie topology is its compact-open topology.

It would be of interest to know if this were true even for one-parameter Lie transformation groups G. For this special case it would suffice to show that either $\bar{G} = G$ or else $\bar{G}$ is compact (or even locally compact) in the compact-open topology.

Appendix to Chapter IV

1. Compact-Open Topology.

We recall here a few facts about the compact-open topology for homeomorphism groups. See [13] for details.

If G is a group of homeomorphisms of a topological space X then the *compact-open* topology for G is the topology having as a basis all sets of the form

$(K_1 \ldots K_n, O_1 \ldots O_n) = \{ g \varepsilon G : g(K_i) \subsetneq O_i \quad i = 1 \ldots n \}$

where the K_i are compact and the O_i open subsets of X. If X is locally compact then the compact-open topology is the weakest topology for G making the map $(g,p) \to g(p)$ of $G \times X \to X$ continuous. If X is locally compact and locally connected, then G becomes a topological group with the compact-open topology.

2. Making a Topology Locally Arcwise Connected.

The following theorem of general topology is known but perhaps not well enough known.

THEOREM A. *Let* $(X, \mathcal{J})$ *be a topological space and let* $\mathcal{B}$ *be the set of arc components of open sets of* $(X, \mathcal{J})$. *Then*

(1) $\mathcal{B}$ *is a base for a topology* $\mathcal{J}'$ *for* X.

(2) *If* Z *is a locally arcwise connected* (*l.a.c.*) *space and* $f : Z \to (X, \mathcal{J})$ *is continuous then* $f : Z \to (X, \mathcal{J}')$ *is continuous*.

(3) $(X, \mathcal{J}')$ *is l.a.c. and in fact* $\mathcal{J}'$ *is the weakest l.a.c. topology for* X *which is stronger than* $\mathcal{J}$ (*stronger* = *more open sets*).

(4) *If* X *is a group and* $(X, \mathcal{J})$ *a topological group then* $(X, \mathcal{J}')$ *is a topological group and every one-parameter*

<u>subgroup of</u> $(X, \mathcal{J})$ (= <u>continuous homomorphism of</u> R <u>into</u> $(X, \mathcal{J})$) <u>is a one-parameter subgroup of</u> $(X, \mathcal{J}')$.

PROOF. Let $B_1, B_2 \varepsilon \mathcal{B}$ and let B_i be an arc component of $O_i \varepsilon \mathcal{J}$. Then if $p \varepsilon B_1 \cap B_2$ the arc component of p in $O_1 \cap O_2$, which belongs to $\mathcal{B}$, is clearly included in $B_1 \cap B_2$. Thus $B_1 \cap B_2$ is the union of sets from $\mathcal{B}$ so $\mathcal{B}$ is a base for a topology $\mathcal{J}'$.

Let Z be a l.a.c. and let $f : Z \rightarrow (X_1, \mathcal{J})$ be continuous. Given $B \varepsilon \mathcal{B}$ let $O \varepsilon \mathcal{J}$ with B an arc component of O. Given $p \varepsilon f^{-1}(B)$ let W be the arc component of p in $f^{-1}(O)$. Then $f(W)$ is arcwise connected, intersects B at $f(p)$, and is included in O and hence $f(W) \subseteq B$ and so $W \subseteq f^{-1}(B)$. Since Z is l.a.c. and $f^{-1}(O)$ is open W is open. It follows that $f^{-1}(B)$ is open. Since $\mathcal{B}$ is a base for $\mathcal{J}'$ $f : Z \rightarrow (X, \mathcal{J}')$ is continuous.

Next, let $B \varepsilon \mathcal{B}$. Given p and q in B then since B is an arc component in the topology $\mathcal{J}$ we can find a map $\sigma : [0,1] \longrightarrow B$ continuous in the topology $\mathcal{J}$ with $\sigma(0) = p$, $\sigma(1) = q$. By (2) σ is continuous with respect to the topology $\mathcal{J}'$ and it follows that B is arcwise connected as a subspace of $(X, \mathcal{J}')$. Since $\mathcal{B}$ is a base for $\mathcal{J}'$ it follows that $\mathcal{J}'$ is l.a.c. Since any element of $\mathcal{J}$ is the union of its arc components and the latter are open in $\mathcal{J}'$ it is clear that $\mathcal{J}'$ is stronger than $\mathcal{J}$. Suppose $\mathcal{J}''$ is a l.a.c. topology stronger than $\mathcal{J}$. Then the identity map of $(X, \mathcal{J}'')$ onto $(X, \mathcal{J})$ is continuous and hence by (2) the identity map of $(X, \mathcal{J}'')$ onto $(X, \mathcal{J}')$ is continuous, i.e. $\mathcal{J}''$ is stronger than $\mathcal{J}'$ so $\mathcal{J}'$ is the weakest l.a.c. topology stronger than $\mathcal{J}$.

If X is a group let $f : X \times X \longrightarrow X$ be defined by $f(x,y) = xy^{-1}$. If $(X, \mathcal{J})$ is a topological group then $f : (X, \mathcal{J}) \times (X, \mathcal{J}) \rightarrow (X, \mathcal{J})$ is continuous and hence since $\mathcal{J}'$ is stronger than $\mathcal{J}$

$f : (X, \mathcal{T}') \times (X, \mathcal{T}') \to (X, \mathcal{T})$ is continuous. Since $(X, \mathcal{T}') \times (X, \mathcal{T}')$ is l.a.c. it follows from (2) that $f : (X, \mathcal{T}') \times (X, \mathcal{T}') \longrightarrow (X, \mathcal{T}')$ is continuous, i.e. $(X, \mathcal{T}')$ is a topological group. Since R is l.a.c. it also follows from (2) that any continuous homomorphism of R into $(X, \mathcal{T})$ is a continuous homomorphism of R into $(X, \mathcal{T}')$.

3. The Modified Compact-Open Topology.

DEFINITION A. Let G be a group of homeomorphisms of the topological space X onto itself. The *modified compact-open topology* for G is the weakest locally arcwise connected topology for G which is stronger than the compact-open topology, that is (theorem A) it is the topology which has as a base the arc components of open sets in the compact-open topology.

THEOREM B. *Let G be a group of homeomorphisms of the locally compact, locally connected space X onto itself and let $\tilde{G}$ be G with the modified compact-open topology. Then $\tilde{G}$ is a locally arcwise connected topological group, the map $(g,p) \longrightarrow g(p)$ of $\tilde{G} \times X \to X$ is continuous, and if $t \to \varphi_t$ is a homomorphism of R into G such that $(t,p) \longrightarrow \varphi_t(p)$ is a continuous map of $R \times X \longrightarrow X$ then it is a one-parameter subgroup of $\tilde{G}$.*

PROOF. An immediate consequence of theorem A and the remarks in section 1.

4. Weakening the Topology of a Lie Group.

The theorems of this section are due to Professor A. M. Gleason. Though theorems C and D are quite possibly known, the result contained in the statement of theorem E (which states in essence that a locally arcwise connected group topology cannot be strictly weaker than a locally compact group topology satisfying the second axiom of countability) and the elegant proof of this theorem seem to be new.

THEOREM C. *Let* X *be a connected, locally connected, and locally compact Hausdorff space. Let* $\{E_n\}$ *be a sequence of disjoint closed subsets of* X *such that* $E_1 \neq \emptyset$ *and* $\bigcup_{n=1}^{\infty} E_n = X$. *Then* $E_1 = X$ *and hence* $E_n = \emptyset$ *for* $n \geqslant 2$.

PROOF. Suppose $E_1 \neq X$. We shall construct by induction an increasing sequence $\{n_k\}$ of integers and a sequence $\{V_k\}$ of subsets of X with the following properties: (1) V_k is open and connected and $\bar{V}_k$ is compact, (2) V_k meets the frontier of E_{n_k} , (3) $\bar{V}_k$ is included in V_{k-1} and is disjoint from $E_1 \cup E_2 \cup \ldots \cup E_{n_k-1}$.

Let $n_1 = 1$. Since E_1 is a proper subset of X it has a frontier point p_1 . Since X is locally connected and locally compact, we can find a connected open neighborhood V_1 of p_1 with $\bar{V}_1$ compact. Suppose $n_1 \ldots n_k$ and $V_1 \ldots V_k$ are chosen satisfying the required properties. Since V_k meets the frontier of E_{n_k} and is open it contains points not belonging to E_{n_k} . Since it contains no point of $E_1 \cup \ldots \cup E_{n_k-1}$ and $\bigcup_{n=1}^{\infty} E_n = X$ it must contain a point of E_m for some integer $m > n_k$. Let n_{k+1} be the least such m . Then $E_{n_{k+1}} \cap V_k$ is a proper subset of V_k (for V_k also meets E_{n_k} and E_{n_k} and $E_{n_{k+1}}$ are disjoint) and since V_k is connected $E_{n_{k+1}} \cap V_k$ has a frontier point p_{k+1} relative to V_k . *A fortiori* p_{k+1} is a frontier point of $E_{n_{k+1}}$ relative to X. Now $O = V_k - (E_1 \cup E_2 \cup \ldots \cup E_{n_{k+1}-1})$ is an open set containing p_{k+1}. Since X is locally compact and Hausdorff, it is regular and, since it is also locally connected, we can find an open, connected neighborhood V_{k+1} of p_{k+1} such that $\bar{V}_{k+1} \subseteq O$. Clearly $n_{k+1} > n_k$ and V_{k+1} satisfies the required properties and the induction is complete.

Now $\{\bar{V}_k\}$ is a decreasing sequence of non-empty compact sets and hence the $\bar{V}_k$ have a common point p. By (3) and the fact that $\{n_k\}$ is strictly increasing p does not belong to any E_n, contradicting $X = \bigcup_{n=1}^{\infty} E_n$.

LEMMA. *A Hausdorff space W which is the continuous image of a compact locally connected space X is locally connected.*

PROOF. Let f map X continuously onto W. We note that to show $K \subseteq W$ is open it suffices to show that $f^{-1}(K)$ is open, for then $f^{-1}(CK) = Cf^{-1}(K)$ is closed in X and hence compact so that $CK = ff^{-1}(CK)$ is compact and hence closed in W. Let K be a component of an open set O of W. If a component B of $f^{-1}(O)$ meets $f^{-1}(K)$ then $f(B)$ is a connected subset of O meeting K and therefore is included in K so that $B \subseteq f^{-1}(K)$. Thus $f^{-1}(K) = \bigcup \{B : B \text{ is a component of } f^{-1}(O) \text{ and } B \cap f^{-1}(K) \neq \emptyset\}$. Now as X is locally connected and $f^{-1}(O)$ is open, each component of $f^{-1}(O)$ is open so that $f^{-1}(K)$ is open. By the remark at the beginning of the proof K is open. Thus every component of an open set of W is open and so W is locally connected.

THEOREM D. *Let X be an arcwise connected Hausdorff space* *and let $\{E_n\}$ be a sequence of disjoint closed subsets of X such that $E_1 \neq \emptyset$ and $\bigcup_{n=1}^{\infty} E_n = X$. Then $E_1 = X$ and hence $E_n = \emptyset$ for $n \geqslant 2$.*

PROOF. Let $p \varepsilon E_1$. Given $q \varepsilon X$ let W be the image of an arc joining p to q. Then $\{W \cap E_n\}$ is a sequence of disjoint closed subsets of W with $W \cap E_1 \neq \emptyset$ and $\bigcup_{n=1}^{\infty}(W \cap E_n) = W$. Now W is locally compact (in fact compact), connected, and by the lemma locally connected, so by theorem C $W \cap E_1 = W$. It follows that $q \varepsilon E_1$ so as q was an arbitrary point of X it follows that $E_1 = X$.

THEOREM E. *Let φ be a continuous one-to-one homomorphism of a locally compact group G satisfying the second axiom of countability onto a locally arcwise connected group H. Then φ^{-1} is also continuous.*

PROOF. Let V be a compact neighborhood of e_G. It will suffice to show that $\varphi(V)$ is a neighborhood of e_H. Let U be an open, symmetric neighborhood of e such that $\bar{U}^2 \subseteq V$. Then $V - U$ and hence $\varphi(V - U)$ are compact so the complement of $\varphi(V - U)$ is a neighborhood of e_H. Let X be an arcwise connected, open neighborhood of e_H such that XX^{-1} does not meet $\varphi(V - U)$.

Given g_1 and g_2 in $\varphi^{-1}(X)$ we put $g_1 \sim g_2$ if and only if $g_1 g_2^{-1} \varepsilon \bar{U}$. Since $\bar{U}$ is a symmetric neighborhood of e_G, $\sim$ is symmetric and reflexive. If $g_1 \sim g_2$ and $g_2 \sim g_3$ then $g_1 g_3^{-1} = (g_1 g_2^{-1})(g_2 g_3^{-1}) \varepsilon \bar{U}^2 \subseteq V$. But $\varphi(g_1 g_3^{-1}) = \varphi(g_1)\varphi(g_3)^{-1} \varepsilon XX^{-1}$ which is disjoint from $\varphi(V - U)$ so $g_1 g_3^{-1} \not\varepsilon V - U$ and hence $g_1 g_3^{-1} \varepsilon \bar{U}$ so $g_1 \sim g_3$. Thus $\sim$ is transitive also and hence an equivalence relation. Let $\{g_\alpha\}$ be a complete set of representatives of $\varphi^{-1}(X)$ under $\sim$, one of which we can take to be e_G. Given $g \varepsilon \varphi^{-1}(X)$ we can find a $g_\alpha \sim g$ so

$g \varepsilon \bar{U} g_\alpha$. Thus $\{\bar{U} g_\alpha\}$ is a covering of $\varphi^{-1}(X)$. If $g \varepsilon \bar{U} g_\alpha \cap \bar{U} g_\beta$ then $g_\alpha g^{-1} \varepsilon \bar{U}^{-1} = \bar{U}$ and $g g_\beta^{-1} \varepsilon \bar{U}$ so $g_\alpha g_\beta^{-1} \varepsilon \bar{U}^2 \subseteq V$. But $\varphi(g_\alpha g_\beta^{-1}) \varepsilon X X^{-1}$ which is disjoint from $\varphi(V - U)$ so $g_\alpha g_\beta^{-1} \notin V - U$ and so $g_\alpha g_\beta^{-1} \varepsilon \bar{U}$. Thus $g_\alpha \sim g_\beta$ so $\alpha = \beta$ and it follows that the $\bar{U} g_\alpha$ are disjoint. Since the $\bar{U} g_\alpha$ have non-empty interiors and G satisfies the second axiom of countability it follows that $\{\bar{U} g_\alpha\}$ is a countable set. Now as φ is one-to-one $\{X \cap \varphi(\bar{U} g_\alpha)\}$ is a countable disjoint covering of X . Moreover as each $\bar{U} g_\alpha$ is compact so is each $\varphi(\bar{U} g_\alpha)$ so each $X \cap \varphi(\bar{U} g_\alpha)$ is closed in X . Since $e_H \varepsilon X \cap \varphi(\bar{U} e_G)$ it follows from theorem D that $X = X \cap \varphi(\bar{U})$. Thus $X \subseteq \varphi(\bar{U}) \subseteq \varphi(V)$ so $\varphi(V)$ is a neighborhood of e_H .

THEOREM F. *Let G be a Lie group satisfying the second axiom of countability (or equivalently, with only countably many components). Suppose the underlying group X of G is a topological group $H = (X, \mathfrak{J})$ in a topology $\mathfrak{J}$ weaker than the topology of G . Then the topology of G has the arc components of open sets of H for a basis and any one-parameter subgroup of H is also a one-parameter subgroup of G .*

PROOF. Let $\mathfrak{J}'$ be the topology for X having the arc components of open sets of H as a basis. By theorem A it follows that $(X, \mathfrak{J}')$ is a locally arcwise connected topological group, that any one-parameter group of H is a one-parameter group of $(X, \mathfrak{J}')$, and that the topology of G is stronger than $\mathfrak{J}'$. By theorem E it follows that the topology of G is actually equal to $\mathfrak{J}'$.

THEOREM G. *Let $\mathfrak{G}$ be a Lie group satisfying the second axiom of countability (or equivalently, with only countably many components) whose underlying group G is a group of* *homeomorphisms of a locally compact, locally connected space* X. *If the map* $(g,p) \to g(p)$ *of* $\mathfrak{G} \times X \to X$ *is continuous then the topology of* $\mathfrak{G}$ *is the modified compact-open topology for* G *and any homomorphism* $t \to \varphi_t$ *of* R *into* G *such that* $(t,p) \to \varphi_t(p)$ *is continuous as a map of* $R \times X \to X$ *is a one-parameter subgroup of* $\mathfrak{G}$.

PROOF. If we take H to be G with the compact-open topology then the assumption that the map $(g,p) \to g(p)$ of $\mathfrak{G} \times X \to X$ is continuous implies that the topology of $\mathfrak{G}$ is stronger than the topology of H. The hypothesis that X is locally compact and locally connected implies that H is a topological group, so the first conclusion follows from theorem F. The second conclusion is then a consequence of theorem B.

Theorem G is a consequence of the statement of [13, theorem 9] for the important case that X is a manifold and G acts differentiably. Unfortunately, the proof of that theorem is vitiated by an invalid application of the implicit function theorem. In fact in the generality that [13, theorem 9] is stated (i.e. without countability restrictions on G) it is false. For example, let H be a Lie group with a proper analytic normal subgroup N. Then (lemma b of theorem VII) the underlying group of H can be made into a Lie group G (*not* satisfying the second axiom of countability) in which N is the connected component of the identity. If we let G act on H by left translation then the compact-open topology is easily seen to be the given topology on H. Since H is locally connected [13, theorem 9] would give in this case the incorrect result that $G = H$.

Fixed Notations

Notations introduced on pages 1-5 are not noted here.

G, e, $\mathfrak{g}$, M, Π_G, Π_M, R_g, $\bar{R}_g$, and $\bar{R}$....................see page 32

D_φ, $D_\varphi p$, D_{φ_g}, φ^p, and φ_g..............................see page 34

φ^{+} = the infinitesimal generator of the local G-transformation φ......................see page 34

Θ^{*} = the infinitesimal graph of an infinitesimal G-transformation group Θ...........see page 38

$\Theta \upharpoonright O$ = the restriction of the infinitesimal G-transformation group Θ acting on M to an open submanifold O of M..................see page 59

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