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# SOME ANALOGUES OF HARTOGS' THEOREM IN AN ALGEBRAIC SETTING.

By RICHARD S. PALAIS.\*

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**1. Introduction.** Let  $K$  be a field, and let  $f$  be any function from  $K \times K$  into  $K$ . For each  $x_0 \in K$  define maps  $f_{x_0}$  and  $f^{x_0}$  from  $K$  to itself by  $x \mapsto f(x_0, x)$  and  $x \mapsto f(x, x_0)$  respectively, and call  $f$  *separately polynomial* if all the  $f_{x_0}$  and  $f^{x_0}$  are polynomial functions. Clearly, if  $f$  is a polynomial function, then it is separately polynomial. What additional assumptions, if any, are needed to ensure that a separately polynomial function is polynomial? For what fields  $K$  is it true that every separately polynomial function is polynomial? (The latter question has an answer that is elegantly simple but somewhat surprising. The reader is invited to frame a conjecture before referring to the theorem of Section 5.)

More generally, suppose  $X$ ,  $Y$ , and  $Z$  are affine algebraic varieties over  $K$  and  $f$  is a map from  $X \times Y$  to  $Z$  which is separately polynomial. Again we can ask for conditions on  $f$  that will ensure that it is a polynomial map, conditions on  $X$ ,  $Y$ , and  $Z$  that will ensure that all such  $f$  are polynomial, and conditions on  $K$  that ensure  $f$  is polynomial no matter what the varieties  $X$ ,  $Y$ , and  $Z$ . (The precise sense in which we use the terms "variety" and "polynomial map" will be made clear in Section 6.) We shall find complete answers to some of these questions and partial answers to the rest.

I am indebted to Jack Milnor, Mike Artin, and Henry King for a number of suggestions, comments, and examples that were very helpful in the preparation of what follows.

**2. Notations and Generalities.** Let  $K$  be any field. Given  $P = P(X_1, \dots, X_n)$  in the polynomial ring  $K[X_1, \dots, X_n]$ , we denote by  $\hat{P}: K^n \rightarrow K$  the map  $(\alpha_1, \dots, \alpha_n) \mapsto P(\alpha_1, \dots, \alpha_n)$ . Functions from  $K^n$  to  $K$  of the form  $\hat{P}$  will be called *polynomial functions*. In case  $K$  is finite, it is well known and trivial that all functions  $K^n \rightarrow K$  are polynomial and that  $P \mapsto \hat{P}$  is not one to one. If  $K$  is infinite, then  $P \mapsto \hat{P}$  is one to one (so we can safely ignore the distinction

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between polynomials and polynomial functions), and not all functions  $K^n \rightarrow K$  are polynomial.

If  $X$  is any set, we denote by  $K^X$  the vector space of all  $K$ -valued functions on  $X$ . We have a map  $\text{Ev}$  of  $X$  into the dual space  $(K^X)^*$  of  $K^X$ ; namely  $\text{Ev}(x)$  is the linear functional  $f \mapsto f(x)$ . More generally, if  $V$  is any linear subspace of  $K^X$ , we regard  $\text{Ev}(x)$  as a linear functional on  $V^*$  by restriction, so that we can regard  $\text{Ev}$  as a map  $\text{Ev}: X \rightarrow V^*$  (of course it is injective exactly when  $V$  "separates points" of  $X$ ).

Given sets  $X$ ,  $Y$ , and  $Z$ , a function  $f: X \times Y \rightarrow Z$ , and  $(x_0, y_0) \in X \times Y$ , we define  $f_{x_0}: Y \rightarrow Z$  and  $f^{y_0}: X \rightarrow Z$  by  $y \mapsto f(x_0, y)$  and  $x \mapsto f(x, y_0)$ , respectively.

Given  $f \in K^X$  and  $g \in K^Y$ , we define  $f \otimes g \in K^{X \times Y}$  by  $f \otimes g(x, y) = f(x)g(y)$ . Given subalgebras  $A$  and  $B$ , respectively, of  $K^X$  and  $K^Y$  (containing the constants), the finite linear combinations of  $f \otimes g$  with  $f \in A$  and  $g \in B$  are a concrete realization of the tensor product algebra  $A \otimes B$ . We regard  $A$  and  $B$  as embedded in  $A \otimes B$  via  $f \mapsto f \otimes 1$  and  $g \mapsto 1 \otimes g$ . If  $X = K^n$ ,  $Y = K^m$ ,  $A = K[X_1, \dots, X_n]$ , and  $B = K[Y_1, \dots, Y_m]$ , then  $A \otimes B = K[X_1, \dots, X_n, Y_1, \dots, Y_m]$ . In particular, if  $f$  and  $g$  are polynomial functions from  $K$  to  $K$ , then  $f \otimes g$  is a polynomial function from  $K \times K$  to  $K$ .

A sequence  $\{f_n\}$  in  $K^X$  will be said to be *pointwise eventually zero* if for each  $x \in X$  there is a positive integer  $N(x)$  such that  $f_n(x) = 0$  when  $n > N(x)$ . The sequence  $\{f_n\}$  will be said to be *eventually zero* if  $f_n = 0$  for all sufficiently large  $n$ . If we make  $K$  into a discrete metric space by defining  $\rho(k, k') = 1$  whenever  $k$  and  $k'$  are distinct points of  $K$ , then these two notions reduce simply to pointwise convergence to the zero function and uniform convergence to the zero function respectively.

The remainder of this section is meant to provide some motivations and a general setting for the kind of questions we shall be considering. The reader uninterested in abstract nonsense can safely skip to Example 4 at the end of the section.

Let  $C$  be a category of sets with extra structure, the morphisms being those set maps which "preserve" the structure (more precisely, we assume there is a "forgetful functor" from  $C$  to the category  $\text{ENS}$  of sets). Suppose that products exist in  $C$  and that the underlying set of the product of two objects is the cartesian product of their underlying sets (with the usual projection morphisms). Also suppose that for  $(x_0, y_0) \in X \times Y$ , the maps  $x \mapsto (x, y_0)$  and  $y \mapsto (x_0, y)$  of  $X$  and  $Y$  into  $X \times Y$  are morphisms of  $C$ . It then follows that, whenever  $f$  is a morphism of  $X \times Y$  into  $Z$ ,  $f_{x_0}$  and  $f^{y_0}$  are also morphisms; or, as we shall say, that  $f$  is "separately morphic." There are now three obvious questions one might ask. Given a set map  $X \times Y \rightarrow Z$  (where  $X$ ,  $Y$ , and  $Z$  are

objects of  $C$ ) which is separately morphic, what additional assumptions on  $f$  are needed to deduce that  $f$  is a morphism? For a given category  $C$ , what conditions on  $X$ ,  $Y$ , and  $Z$  imply that every set map  $f: X \times Y \rightarrow Z$  which is separately morphic is a morphism of  $C$ ? And for what categories  $C$  is it true that all separately morphic set maps  $f: X \times Y \rightarrow Z$  are morphisms? The initial response of those accustomed to thinking about categories is that these are basically unnatural questions. The product of two objects  $X$  and  $Y$  in a category  $C$  is designed to deal with pairs of morphisms  $Z \rightarrow X$  and  $Z \rightarrow Y$ , not with a single morphism  $X \times Y \rightarrow Z$ . One therefore has no reason to expect these questions to have reasonable, natural answers. As some of the examples below show, this is, generally speaking, a correct assessment of the situation. For the most part one has only counterexamples to reasonable conjectures. Still, as we shall see, in certain very nice and very rigid categories there are some interesting positive results.

*Example 1.* Complete metric spaces. The following is a classic counterexample from elementary calculus. We take  $X = Y = Z = \mathbf{R}$  and define a function  $f$  from  $X \times Y$  to  $Z$  by  $f(0, 0) = 0$  and  $f(x, y) = xy/(x^2 + y^2)$  for  $(x, y) \neq (0, 0)$ . Note that in terms of polar coordinates, except at the origin,  $f$  has the value  $(\sin 2\theta)/2$ . This  $f$  is clearly separately continuous, but it is not continuous. Since  $f$  is such a "good" function (e.g., it is rational, and the spaces  $X$ ,  $Y$ , and  $Z$  are so simple), it would seem almost hopeless to look for positive results (note that it would do no good to assume  $X$ ,  $Y$ , and  $Z$  compact; in the above example we could equally well replace  $\mathbf{R}$  by  $I = [0, 1]$ ). Nevertheless there is at least one very beautiful and classical positive result, due to Deane Montgomery [5]. Suppose  $X = Y = Z = G$  is a complete metric space whose underlying set is a group, and suppose that the group multiplication  $(x, y) \mapsto xy$  is a separately continuous function  $\mu: X \times Y \rightarrow Z$  (i.e., each left translation and each right translation is a self-homeomorphism of  $G$ ). Then  $\mu$  is automatically jointly continuous. Moreover  $x \mapsto x^{-1}$  is also automatically continuous, so that  $G$  is a topological group!

*Example 2.*  $C^k$  manifolds ( $k = 1, 2, \dots, \infty, \omega$ ). Here  $C^\omega$ , as usual, means real analytic. Again the general situation is apparently hopeless. Exactly the same counterexample as above in Example 1 is separately  $C^\omega$ . Moreover if we multiply that function by  $(x^2 + y^2)^e$  where  $e = (2m + 1)/4$  (so that in polar coordinates we have the function  $(r^{m+1/2} \sin 2\theta)/2$ ), then we get a function which is separately  $C^\omega$ , and is  $C^m$  but not  $C^{m+1}$ . It is not even hard to construct an example of a separately  $C^\omega$  function which is  $C^\infty$  but is not  $C^\omega$  [multiply the function of Example 1 by  $e^{-1/(x^2 + y^2)}$ ].

*Example 3.* Complex analytic manifolds and holomorphic maps. Superficially this seems not very different from the case  $C^\omega$  above. But it is. There is the remarkable and highly non-trivial theorem of Hartogs [1] which says that a function  $f: X \times Y \rightarrow \mathbf{C}$  which is separately holomorphic is in fact holomorphic. It is easy to fall into a trap at this point and to conclude, invalidly, that any separately holomorphic map  $f: X \times Y \rightarrow Z$  is holomorphic. If  $f$  is continuous, then each  $(x_0, y_0)$  in  $X \times Y$  has a product neighborhood which is mapped into a neighborhood of  $f(x_0, y_0)$  that can be identified with  $\mathbf{C}^n$ , and the desired conclusion does follow. However, T. J. Barth has pointed out [*Trans. Amer. Math. Soc.* **207** (1975), p. 182] that the function  $f(x, y)$  which is  $(x + y)^2 / (x - y)$  for  $x \neq y$ ,  $\infty$  for  $x = y \neq 0$ , and 0 for  $x = y = 0$  is a separately holomorphic map of  $\mathbf{C} \times \mathbf{C}$  into the Riemann sphere  $\mathbf{C} \cup \{\infty\}$  which is not continuous.

Now suppose that  $f: \mathbf{C}^2 \rightarrow \mathbf{C}$  is a separately polynomial function. By Hartogs' Theorem,  $f(x, y) = \sum_{m,n=0}^\infty a_{mn} x^m y^n$ , where the series converges absolutely and uniformly for all  $x$  and  $y$ . Define entire holomorphic functions  $b_n(x) = \sum_{m=0}^\infty a_{mn} x^m$ , and let  $X_N$  denote the closed subset of  $x$  in  $\mathbf{C}$  such that  $b_n(x) = 0$  for all  $n > N$ . Clearly,  $X_1 \subseteq X_2 \subseteq \dots$ . Now  $f_x(y) = \sum_{n=0}^\infty b_n(x) y^n$ , and since  $f_x$  is a polynomial, all the  $b_n(x)$  for  $n$  large enough are zero. That is,  $x$  belongs to some  $X_N$ , so that  $\mathbf{C}$  is the union of the closed sets  $X_N$ , and by the Baire category theorem, some  $X_N$  has non-empty interior. Since for  $n > N$  the entire function  $b_n$  vanishes on a non-empty open set, by the principle of analytic continuation it is identically zero. Since  $b_n(x) = \sum_{m=0}^\infty a_{mn} x^m$ , it follows from the uniqueness of power series that  $a_{mn} = 0$  for  $n > N$ . Interchanging the roles of  $x$  and  $y$ , it follows that  $a_{mn}$  is also zero provided  $m$  is sufficiently large. Hence there are only finitely many non-zero  $a_{mn}$ , and  $f$  is a polynomial.

We have now verified that, at least when  $K = \mathbf{C}$ , a separately polynomial function  $K^2 \rightarrow K$  is polynomial. Note that because of the case  $k = \omega$  of Example 2 this proof breaks down for  $K = \mathbf{R}$ .

*Example 4.* Let  $K$  be an infinite but countable field. Let  $\{x_n\}$  be a sequence which contains each element of  $K$  exactly once. Define a monic polynomial  $f_n$  of degree  $n$  by  $f_n(X) = (X - x_1)(X - x_2) \dots (X - x_n)$ . Note that  $f_n(x_m) = 0$  for  $n \geq m$ . It follows that for each  $(x, y) \in K \times K$  the sum  $\sum_{n=1}^\infty f_n(x) f_n(y)$  is effectively finite (i.e., all but a finite number of terms vanish), and so this sum defines a function  $f(x, y)$  from  $K \times K$  to  $K$ . For  $x = x_m$  we have  $f^x = f_x = \sum_{n=1}^{m-1} f_n(x_m) f_n$ , which is a polynomial of degree exactly  $m - 1$  [the coefficient of  $x^{m-1}$  is  $\prod_{n=1}^{m-1} (x_m - x_n) \neq 0$ ]. Thus  $f$  is separately polynomial. But since the degrees of the  $f_x$  are unbounded,  $f$  is not a polynomial function from  $K \times K$  to  $K$  [for if  $f(x, y) = \sum_{m,n=0}^{M,N} a_{mn} x^m y^n$ , then clearly  $\text{degree}(f_x) \leq N$  for all  $x$ , and  $\text{degree}(f^y) \leq M$  for all  $y$ ].

At this point we have the following information concerning the question of when all separately polynomial functions  $f: K \times K \rightarrow K$  are polynomial. It is true if  $K$  is a finite field (since every function from  $K \times K$  to  $K$  is a polynomial). It is false for all countably infinite fields  $K$  (and in particular for the rational field  $\mathbf{Q}$  and its algebraic closure  $\overline{\mathbf{Q}}$ ), and it is true for at least one uncountable field, namely the field  $\mathbf{C}$  of complex numbers.

3. A Useful Lemma.

LEMMA 3.1. *Let  $X$  be any set, let  $K$  be any field, and let  $V$  be a finite dimensional subspace of  $K^X$ . Then  $\{\text{Ev}(x)|x \in X\}$  spans  $V^*$ , so in particular there exist  $x_1, \dots, x_n$  in  $X$  such that  $\text{Ev}(x_1), \dots, \text{Ev}(x_n)$  is a basis for  $V^*$ . Choose such  $x_1, \dots, x_n$ , and let  $\xi_1, \dots, \xi_n$  be the dual basis for  $V$ . If  $Y$  is any set and  $f: X \times Y \rightarrow K$  is any function such that  $f^y \in V$  for all  $y \in Y$ , then*

$$f = \sum_{i=1}^n \xi_i \otimes f_{x_i}.$$

*Proof.* Let  $W$  be the linear span of  $\{\text{Ev}(x)|x \in X\}$  in  $V^*$ . Since  $V$  is finite dimensional and hence reflexive, to prove that  $W = V^*$  it will suffice to show that the annihilator of  $W$ ,  $W^0 = \{f \in V^* | l(f) = 0 \text{ for all } l \in W\}$ , contains only 0. But if  $f \in W^0$  and  $x \in X$ , then, since  $\text{Ev}(x) \in W$ ,  $f(x) = \text{Ev}(x)(f) = 0$ , so  $f = 0$ .

Since  $f^y \in V$  and the  $\text{Ev}(x_i)$  are dual to the  $\xi_i$ , we have

$$f^y = \sum_{i=1}^n \text{Ev}(x_i)(f^y) \xi_i.$$

Now  $\text{Ev}(x_i)(f^y) = f^y(x_i) = f(x_i, y) = f_{x_i}(y)$ . Thus  $f(x, y) = f^y(x) = \sum_{i=1}^n f_{x_i}(y) \xi_i(x)$ , which says  $f = \sum_{i=1}^n \xi_i \otimes f_{x_i}$ . ■

PROPOSITION 3.2. *Let  $\{f_n\}$  be a sequence in  $K^X$  which is pointwise eventually zero. A necessary and sufficient condition for  $\{f_n\}$  to be eventually zero is that the  $f_n$  span a finite dimensional subspace of  $K^X$ .*

*Proof.* Necessity is trivial. If, conversely, the subspace  $V$  spanned by the  $f_n$  is finite dimensional, choose  $x_1, \dots, x_d$  in  $X$  such that  $\text{Ev}(x_1), \dots, \text{Ev}(x_d)$  is a basis for  $V^*$  (cf. Lemma 3.1), and choose  $N(x_i)$  such that  $f_n(x_i) = 0$  for  $n > N(x_i)$ . If  $n > \text{Max}\{N(x_1), \dots, N(x_d)\}$ , then  $\text{Ev}(x_j)(f_n) = f_n(x_j) = 0$  for  $j = 1, \dots, d$ , and hence  $f_n = 0$ ; so  $\{f_n\}$  is eventually zero. ■

COROLLARY 3.3. *If  $\{f_n\}$  is a sequence in  $K^X$  which is pointwise eventually zero but is not eventually zero, then there is a subsequence of  $\{f_n\}$  (which of course has the same properties) whose elements are linearly independent.*

*Proof.* Inductively, delete any element which depends linearly on the preceding elements. By 3.2 there will be arbitrarily far out elements of the sequence which are not deleted. ■

#### 4. A Condition for Separately Polynomial Functions to be Polynomial.

**THEOREM.** *Let  $K$  be a field, let  $f: K \times K \rightarrow K$  be a separately polynomial function, and suppose there exists an infinite subset  $Y$  of  $K$  such that for all  $y \in Y$  the polynomials  $f^y$  lie in some fixed finite-dimensional subspace  $V$  of the space  $K[X]$  of polynomial functions from  $K$  to  $K$ . Then  $f$  is a polynomial function  $K \times K \rightarrow K$ .*

*Proof.* Restricted to  $K \times Y$ ,  $f$  is a function to which we can apply Lemma 3.1. Choosing  $x_1, \dots, x_n \in K$  such that  $\text{Ev}(x_1), \dots, \text{Ev}(x_n)$  is a basis for  $V^*$ , and letting  $\xi_1, \dots, \xi_n$  be the dual basis for  $V$ , we have

$$\begin{aligned} f|(K \times Y) &= \sum_{i=1}^n \xi_i \otimes (f_{x_i}|Y) \\ &= \left( \sum_{i=1}^n \xi_i \otimes f_{x_i} \right) \Big| (K \times Y). \end{aligned}$$

We claim that in fact  $f = \sum_{i=1}^n \xi_i \otimes f_{x_i}$ . It will of course suffice to show that  $f_x = \sum_{i=1}^n \xi_i(x) f_{x_i}$  for every  $x \in K$ . But by assumption both sides are polynomials, and since they agree on the infinite set  $Y$ , they must in fact be the same polynomial. Now  $\xi_i(X) \in V \subseteq K[X]$  and  $f_{x_i}(Y) \in K[Y]$ , so that  $f = \sum_{i=1}^n \xi_i \otimes f_{x_i} \in K[X] \otimes K[Y] = K[X, Y]$ . ■

We note that the condition of the theorem is also necessary provided  $K$  is indeed infinite. In fact, as pointed out in Example 4 of Section 2, if  $f: K \times K \rightarrow K$  is a polynomial function, then for  $d$  sufficiently large,  $f^y$  belongs to the  $d$ -dimensional space of all polynomials of degree less than  $d$  for all  $y \in K$ .

#### 5. Fields for Which All Separately Polynomial Maps Are Polynomial.

**THEOREM.** *If  $K$  is a field, then a necessary and sufficient condition for every separately polynomial function from  $K \times K$  to  $K$  to be a polynomial is that  $K$  be either finite or uncountable.*

*Proof.* By the remark at the end of Section 2 we need only consider the case that  $K$  is uncountable. Let  $V_d$  denote the finite dimensional space of polynomial functions from  $K$  to  $K$  of degree less than  $d$ , and let  $Y_d = \{y \in K | f^y$

$\in V_d$ . Since  $K[X] = \bigcup_{d=1}^{\infty} V_d$  and  $f^y \in K[X]$  for all  $y \in K$ , it follows that  $K = \bigcup_{d=1}^{\infty} Y_d$ . But  $K$  is uncountable, and there are only countably many  $Y_d$ , so some  $Y_d$  must be infinite (and in fact uncountably infinite). The theorem is now an immediate corollary of that of the preceding section. ■

This elementary proof verifies what the reader no doubt suspected all along. The special proof for  $\mathbf{C}$  (using the deep theorem of Hartogs) in Example 3 of Section 2 is a bit of a fraud.

**6. Affine Algebraic Varieties and their Morphisms.** In all that follows,  $K$  will denote an infinite field. Most of what we say would be equally true (and trivial) for  $K$  finite, but there would be annoying special cases. An algebra will always mean commutative algebra over  $K$  with unit. In fact the algebras we shall consider will generally be finitely generated algebras of functions on some set  $X$  (i.e., subalgebras of  $K^X$ ) which contain the constant functions. Homomorphisms of algebras are assumed to preserve the unit. The set of algebra homomorphisms of an algebra  $\mathcal{A}$  into the one dimensional algebra  $K$  will be denoted by  $\hat{\mathcal{A}}$ . If  $\mathcal{A}$  is a subalgebra of  $K^X$ , then we have the map  $\text{Ev}: X \rightarrow \hat{\mathcal{A}}$  already considered above, and we recall that  $\mathcal{A}$  is said to separate points when this map is injective.

*Definition.* An (affine) algebraic variety (over  $K$ ) is a set  $X$  together with a finitely generated algebra  $\mathcal{P}(X)$  of  $K$ -valued functions on  $X$  (called the polynomial functions on  $X$ ) such that:

- (1)  $\mathcal{P}(X)$  separates points of  $X$  and
- (2) each algebra homomorphism  $\mathcal{P}(X) \rightarrow K$  is of the form  $f \mapsto f(x)$  for some  $x \in X$ ,

or in other words such that  $\text{Ev}: X \rightarrow \hat{\mathcal{P}(X)}$  is bijective.

If  $Y$  is a second algebraic variety, then a morphism  $\varphi: X \rightarrow Y$  is a set mapping such that the homomorphism  $f \mapsto f \circ \varphi$  of  $K^Y$  into  $K^X$  maps  $\mathcal{P}(Y)$  into  $\mathcal{P}(X)$ , and we denote by  $\varphi^*$  the restricted homomorphism  $\mathcal{P}(Y) \rightarrow \mathcal{P}(X)$ .

We shall review the basic facts about algebraic varieties which we shall need. They are all well known. The reader not familiar with these results is referred to one of the following for a fuller discussion with proofs: [2], [3], [4], [6], [8]. (These sources usually assume  $K$  is algebraically closed, but for the facts we need this assumption is irrelevant.)

Algebraic varieties and their morphisms form a category. Products exist in this category. The categorical product of  $X$  and  $Y$  is the cartesian product  $X \times Y$  together with the algebra  $\mathcal{P}(X \times Y) = \mathcal{P}(X) \otimes \mathcal{P}(Y)$ . Given  $(x_0, y_0) \in$

$X \times Y$ , the maps  $y \mapsto (x_0, y)$  and  $x \mapsto (x, y_0)$  of  $Y$  and  $X$  into  $X \times Y$  are morphisms; so we are in the situation considered in Section 2.

$K$  itself is an algebraic variety with  $\mathcal{P}(K)$  the ring of polynomial functions  $K[X]$  (since  $K$  is infinite, we can identify  $P(X) \in K[X]$  with the function  $\hat{P}: \alpha \mapsto P(\alpha)$  in  $K^K$ ). Its  $n$ -fold product with itself is  $K^n$  with  $\mathcal{P}(K^n) = K[X_1, \dots, X_n]$ .

If  $X$  is any variety, then a function  $x \mapsto (f_1(x), \dots, f_n(x))$  from  $X$  to  $K^n$  is a morphism if and only if each  $f_i$  belongs to  $\mathcal{P}(X)$ . In particular,  $\mathcal{P}(X)$  is the set of all morphisms  $X \rightarrow K$ . In view of this we shall also call morphisms  $X \rightarrow Y$  polynomial maps. A set map  $\varphi: X \rightarrow Y$  between varieties is a polynomial map if and only if  $f \circ \varphi \in \mathcal{P}(X)$  for all  $f \in \mathcal{P}(Y)$  (the "only if" part being of course the definition). Note that a set map

$$(x_1, \dots, x_n) \mapsto (f_1(x_1, \dots, x_n), \dots, f_m(x_1, \dots, x_n))$$

of  $K^n$  to  $K^m$  is a polynomial map if and only if each  $f_i$  is actually a polynomial. There is a natural  $T_1$ -topology for varieties (with respect to which morphisms are continuous), called the Zariski topology, or Z-topology for short. If  $X$  is a variety, then a basis for the Z-topology for  $X$  consists of the sets  $X_f = \{x \in X \mid f(x) \neq 0\}$ , where  $f \in \mathcal{P}(X)$ . The Z-closed subsets of  $X$ , also called subvarieties of  $X$ , are the sets  $V$  with the property that given any  $x$  not in  $V$  there exists  $f \in \mathcal{P}(X)$  vanishing on  $V$  but not at  $x$ . It turns out that this is equivalent to  $V$  being a variety if we define  $\mathcal{P}(V) = \{(f|_V) \mid f \in \mathcal{P}(X)\}$ . A variety  $X$  is called irreducible if  $\mathcal{P}(X)$  is an integral domain (or equivalently if  $X$  cannot be decomposed into the union of two proper subvarieties). In this case the algebraic dimension,  $\text{Dim}(X)$ , of  $X$  is defined to be the transcendence degree of  $\mathcal{P}(X)$  (or its field of quotients) over  $K$ . In general a variety  $X$  can be written (uniquely except for order) as the union of irreducible subvarieties  $X_1, \dots, X_n$ , with no  $X_i$  included in the union of the others. These  $X_i$  are called the irreducible components of  $X$ , and the maximum of the  $\text{Dim}(X_i)$  is defined to be the algebraic dimension,  $\text{Dim}(X)$ , of  $X$ . If  $X$  is irreducible, then any proper subvariety has strictly smaller dimension. Zero dimensional varieties are finite sets. Irreducible one dimensional varieties are called curves, and their proper subvarieties are precisely their finite subsets. A curve  $C$  is said to be rational, or of genus zero, if the field of fractions of  $\mathcal{P}(C)$  is isomorphic to the field  $K(X)$  of rational functions in one indeterminate (so, for example,  $K$  is a rational curve); otherwise  $C$  is said to have genus greater than zero. Each variety  $X$  is isomorphic to a subvariety of some  $K^n$ . Specifically, let  $\xi = (\xi_1, \dots, \xi_n)$  generate  $\mathcal{P}(X)$ , and let  $\mathcal{I}^\xi$  denote the ideal of polynomials  $P(X_1, \dots, X_n)$  in  $K[X_1, \dots, X_n]$  such that  $P(\xi_1, \dots, \xi_n) = 0$ . Let  $X^\xi = \{x \in K^n \mid P(x) = 0 \text{ for all } P \in \mathcal{I}^\xi\}$ . Then  $X^\xi$  is

a subvariety of  $K^n$ , and  $x \mapsto (\xi_1(x), \dots, \xi_n(x))$  is a morphism  $E^\xi: X \rightarrow K^n$  which maps  $X$  isomorphically onto the variety  $X^\xi$ . If  $Y$  is another variety and  $\eta = (\eta_1, \dots, \eta_m)$  generate  $\mathcal{P}(Y)$ , then we have  $E^\eta: Y \rightarrow K^m$  mapping  $Y$  isomorphically onto the subvariety  $Y^\eta$  of  $K^m$ . If  $f: X \rightarrow Y$  is a morphism, then  $\eta_i \circ f \in \mathcal{P}(X)$ , so that  $\eta_i \circ f = F_i(\xi_1, \dots, \xi_n)$  for  $i = 1, \dots, m$ , where  $F_i \in K[X_1, \dots, X_n]$ . Thus we have a polynomial map  $F: K^n \rightarrow K^m$ ,  $x \mapsto (F_1(x), \dots, F_m(x))$ . Moreover,  $F$  clearly maps  $X^\xi$  into  $Y^\eta$ , and in fact we have  $F \circ E^\xi = E^\eta \circ f$ . This gives a very concrete mechanism for dealing with varieties and their morphisms, which was in fact the classical technique for dealing with them.

If  $F$  is an extension field of  $K$  and  $X$  is a variety over  $K$ , we get a variety  $X_F$  over  $F$  which includes  $X$  as a  $Z$ -dense subset. Namely, let  $X_F$  be the set of all ( $K$ -algebra) homomorphisms  $\mathcal{P}(X) \rightarrow F$ . Since these are in natural bijective correspondence with the  $F$ -algebra homomorphisms  $\mathcal{P}(X) \otimes_K F \rightarrow F$ , we can regard  $\mathcal{P}(X) \otimes_K F$  as an  $F$ -algebra  $\mathcal{P}(X_F)$  of  $F$ -valued functions on  $X_F$ . Any morphism of varieties over  $K$ ,  $\varphi: X \rightarrow Y$ , extends uniquely to a morphism of varieties over  $F$ ,  $\varphi_F: X_F \rightarrow Y_F$ . If, as above,  $\xi = (\xi_1, \dots, \xi_n)$  generate  $\mathcal{P}(X)$ , etc., then  $E_F^\xi: X_F \rightarrow F^n$  maps  $X_F$  isomorphically onto  $X_F^\xi = \{x \in F^n \mid P(x) = 0 \text{ for all } P \in \mathcal{P}^\xi\}$ .

### 7. Varieties Algebraically of the Second Category.

PROPOSITION 7.1. *If  $X$  is an irreducible algebraic variety, then the following are equivalent:*

- (1) *If  $X$  is decomposed as the countable union of subsets  $\{S_n\}$ , at least one  $S_n$  is  $Z$ -dense in  $X$ .*
- (2)  *$X$  cannot be decomposed as the countable union of proper subvarieties.*
- (3) *If  $\{X_n\}$  is an increasing sequence of subvarieties of  $X$  whose union is  $X$ , then some  $X_n$  is equal to  $X$ .*
- (4)  *$X$  is of the second category relative to its  $Z$ -topology.*

*Proof.* The equivalence of (1), (2), and (3) is immediate and left to the reader. Suppose (4) holds, and let  $X = \cup_{n=1}^\infty X_n$ , where  $X_n$  is a subvariety (i.e.,  $Z$ -closed subset) of  $X$ . Since  $X$  is of the second category in its  $Z$ -topology, some  $X_n$  has non-empty interior. But in an irreducible variety, any non-empty open set is dense. Hence  $X_n$  is dense as well as closed, so  $X_n = X$ , and (4) implies (2). Conversely, if (2) holds and  $X$  is decomposed as the union of countably many closed sets  $\{X_n\}$ , one of these  $X_n$  must be equal to all of  $X$ , and so in particular it has non-empty interior; so (2) implies (4). ■

*Definition 7.2.* An irreducible variety  $X$  is said to be *algebraically of the second category* if it satisfies any one and hence all of the four properties of the preceding proposition. More generally, an arbitrary variety is said to be algebraically of the second category if each of its irreducible components is algebraically of the second category.

**PROPOSITION 7.3.** *A curve (or more generally an algebraic variety of dimension one) is algebraically of the second category if and only if it is uncountable. In particular,  $K$  is algebraically of the second category if and only if it is uncountable.*

*Proof.* Immediate, from the fact that the only proper subvarieties of a curve are its finite subsets, and that every variety of dimension one is a finite union of curves and points. ■

**PROPOSITION 7.4.** *Each of the following is necessary and sufficient for a variety  $X$  to be algebraically of the second category:*

- (1) *If  $X$  is the increasing union of a sequence of subsets  $\{S_n\}$ , then some (and so all sufficiently large)  $S_n$  are  $Z$ -dense in  $X$ .*
- (2)  *$X$  is not the increasing union of a sequence of proper subvarieties.*
- (3) *If a sequence  $\{f_n\}$  in  $\mathcal{P}(X)$  is pointwise eventually zero (cf. Section 2), then it is eventually zero.*

*Proof.* The equivalence of (1) and (2) is trivial. We show next that (2) implies (3). Let  $\{f_n\}$  in  $\mathcal{P}(X)$  be eventually zero, and let  $X_n = \bigcap_{k>n} f_k^{-1}(0)$ . The  $X_n$  are clearly  $Z$ -closed and increasing, and since  $\{f_n\}$  is pointwise eventually zero, every  $x$  in  $X$  belongs to some  $X_n$ . It follows from (2) that some  $X_n$  equals  $X$ , which means  $f_k = 0$  for  $k > n$ , so (3) holds. Conversely, we shall show that if (2) does not hold, then neither does (3), which will show that (1), (2), and (3) are equivalent. For suppose  $X$  is the union of an increasing sequence  $\{X_n\}$  of proper  $Z$ -closed subsets, and let  $\{f_n\}$  be a non-zero element of  $\mathcal{P}(X)$  which vanishes on  $X_n$ . Since any  $x$  in  $X$  is in all  $X_n$  from some point on,  $\{f_n\}$  is pointwise eventually zero, and since no  $f_n$  is zero  $\{f_n\}$  is not eventually zero.

Let  $S_1, \dots, S_k$  be the irreducible components of  $X$ . Suppose now that  $X$  is algebraically of the second category (i.e., each  $S_k$  is algebraically of the second category), and let  $\{X_n\}$  be an increasing sequence of  $Z$ -closed subsets of  $X$  whose union is  $X$ . Since  $\{X_n \cap S_j\}$  is a sequence of  $Z$ -closed subsets of  $S_j$  whose union is  $S_j$ , it follows from (3) of Proposition 7.1 that there is an  $N(j)$  such that  $S_j \subseteq X_n$  for  $n > N(j)$ . If  $N = \max\{N(1), \dots, N(k)\}$ , then since  $X$  is the union of the  $S_j$  it follows that  $X_n = X$  for  $n > N$ . Thus if  $X$  is algebraically of the second category, then (2) holds.

Finally, to complete the proof, we shall show that if (2) holds, then each  $S_j$  satisfies (3) of Proposition 7.1. Indeed let  $\{T_n\}$  be an increasing sequence of  $Z$ -closed subsets of  $S_j$  with union  $S_j$ , and let

$$X_n = T_n \cup \bigcup_{m \neq j} S_m,$$

so that  $\{X_n\}$  is an increasing sequence of  $Z$ -closed subsets of  $X$  with union  $X$ . Assuming (2) holds, we have that some  $X_n$  is all of  $X$ , which means that  $S_j$  is the union of  $T_n$  and  $S_j \cap \bigcup_{m \neq j} S_m$ , both of which are  $Z$ -closed subsets of  $S_j$ . Now one component of a space is never included in the union of the others, so

$$S_j \neq S_j \cap \bigcup_{m \neq j} S_m,$$

and since  $S_j$  is irreducible, it must equal  $T_n$ . ■

### 8. When Separately Polynomial Maps are Polynomial.

**THEOREM 8.1.** *Let  $X$  and  $Y$  be algebraic varieties over  $K$ , and let  $f: X \times Y \rightarrow K$  be separately polynomial. If there exists a  $Z$ -dense subset  $S$  of  $Y$  such that for all  $y \in S$  the polynomial maps  $f^y: X \rightarrow K$  lie in some fixed finite dimensional subspace  $V$  of  $\mathcal{P}(X)$ , then in fact  $f \in \mathcal{P}(X \times Y) = \mathcal{P}(X) \otimes \mathcal{P}(Y)$ . [Conversely, if  $f \in \mathcal{P}(X \times Y)$ , then in fact the set of all  $f^y, y \in Y$ , spans a finite dimensional subspace of  $\mathcal{P}(X)$ .]*

*Proof.* The proof is virtually identical to that of the theorem of Section 4, and we leave it to the reader to make the necessary changes. ■

**PROPOSITION 8.2.** *Let  $X$  and  $Y$  be algebraic varieties over  $K$ . If neither  $X$  nor  $Y$  is algebraically of the second category, then there exists a function  $f: X \times Y \rightarrow K$  which is separately polynomial but is not polynomial.*

*Proof.* By Proposition 7.4 we can find a sequence  $\{f_n\}$  in  $\mathcal{P}(X)$  which is pointwise eventually zero but is not eventually zero. By Corollary 3.3 we can assume that the  $f_n$  are linearly independent. Let  $\{g_n\}$  be a sequence in  $\mathcal{P}(Y)$  with these same properties. Just as in Example 4 of Section 2, we see that  $f(x, y) = \sum_{n=1}^{\infty} f_n(x) g_n(y)$  is a well-defined separately polynomial map from  $X \times Y$  to  $K$ . For each positive integer  $d$  we will find  $y_1, \dots, y_d$  in  $Y$  such that  $f^{y_1}, \dots, f^{y_d}$  are linearly independent, which by Theorem 8.1 implies that  $f$  is not polynomial. Since  $f^y = \sum_{n=1}^{\infty} g_n(y) f_n$ , and the  $f_n$  are linearly independent in  $\mathcal{P}(X)$ , it will suffice to choose  $y_1, \dots, y_d$  so that the sequences  $(g_1(y_i), g_2(y_i), \dots)$  are linearly independent in  $K^\infty$ . In particular it will suffice

to choose them so that the  $(g_1(y_i), \dots, g_d(y_i))$  are a basis for  $K^d$ . Let  $V$  be the  $d$ -dimensional subspace of  $\mathcal{P}(Y)$  spanned by  $g_1, \dots, g_d$ , and (by Lemma 3.1) choose  $y_1, \dots, y_d$  in  $Y$  such that  $\text{Ev}(y_1), \dots, \text{Ev}(y_d)$  is a basis for  $V^*$ . If  $\gamma_1, \dots, \gamma_d$  is the basis for  $V^*$  dual to  $g_1, \dots, g_d$ , then  $(g_1(y_i), \dots, g_d(y_i))$  is just the vector of components of  $\text{Ev}(y_i)$  with respect to  $\gamma_1, \dots, \gamma_d$ . ■

**THEOREM 8.3.** *Let  $X, Y$ , and  $Z$  be algebraic varieties over  $K$ , and let  $f: X \times Y \rightarrow Z$  be a separately polynomial map. If at least one of  $X$  and  $Y$  is algebraically of the second category, then  $f$  is polynomial.*

*Proof.* Recall that  $f$  is polynomial if and only if  $g \circ f$  is polynomial for every  $g \in \mathcal{P}(Z)$ . It follows that without loss of generality we can assume that  $Z = K$ . Also, by symmetry, we can suppose it is  $Y$  which is algebraically of the second category. Let  $\xi_1, \dots, \xi_m$  generate  $\mathcal{P}(X)$  as a  $K$ -algebra, and let  $V_d$  denote the finite dimensional subspace of  $\mathcal{P}(X)$  consisting of functions which can be expressed as polynomials of degree less than or equal to  $d$  in  $(\xi_1, \dots, \xi_m)$ . Since  $\mathcal{P}(X)$  is the increasing union of the subspaces  $V_d$ , and  $f^y \in \mathcal{P}(X)$  for all  $y \in Y$ , it follows that  $Y$  is the increasing union of the sequence of subsets  $S_d = \{y \in Y \mid f^y \in V_d\}$ . By (1) of Proposition 7.4, it follows that  $S_d$  is  $Z$ -dense for  $d$  large enough, and Theorem 8.1 completes the proof. ■

### 9. Fields over Which All Varieties are Algebraically of the Second Category

**PROPOSITION 9.1.** *Let  $X$  be an irreducible algebraic variety such that each subvariety of  $X$  of dimension  $\text{Dim}(X) - 1$  is algebraically of the second category, but such that  $X$  itself is not. Then  $X$  has at most countably many distinct irreducible subvarieties of dimension  $\text{Dim}(X) - 1$ .*

*Proof.* Let  $X = \cup_{n=1}^{\infty} S_n$ , where the  $S_n$  are proper subvarieties of  $X$  (cf. Proposition 7.1 (2) and Definition 7.2). Since each  $S_n$  is the finite union of its irreducible components (each of which is a subvariety of  $X$ ), we can assume that the  $S_n$  are themselves irreducible. Then we shall see that any irreducible subvariety  $Y$  of  $X$  with  $\text{Dim}(Y) = \text{Dim}(X) - 1$  must be one of the countably many  $S_n$ . In fact  $Y = \cup_{n=1}^{\infty} (Y \cap S_n)$ , and since  $Y$  is algebraically of the second category, it follows that for some  $n$ ,  $Y \subseteq S_n$ , so that  $\text{Dim}(S_n) \geq \text{Dim}(Y)$ . On the other hand, since  $S_n$  is a proper subvariety of the irreducible variety  $X$ ,  $\text{Dim}(S_n) \leq \text{Dim}(X) - 1 = \text{Dim}(Y)$ , so that  $\text{Dim}(S_n) = \text{Dim}(Y)$ . Since  $S_n$  is irreducible and includes  $Y$ , it follows that  $Y = S_n$ . ■

**COROLLARY 9.2.** *If  $X$  is as in the above proposition and  $f \in \mathcal{P}(X)$ , then the set of  $\alpha$  in  $K$  for which  $f^{-1}(\alpha)$  has algebraic dimension greater than  $\text{Dim}(X) - 2$  is at most countable.*

*Proof.* If  $f^{-1}(\alpha)$  has dimension  $\text{Dim}(X)$ , then it must equal  $X$ , i.e.,  $f$  is constant. If  $\text{Dim}(f^{-1}(\alpha)) = \text{Dim}(X) - 1$ , then one of the irreducible components of  $f^{-1}(\alpha)$  is one of the at most countably many irreducible subvarieties of  $X$  of dimension  $\text{Dim}(X) - 1$ . Since  $f^{-1}(\alpha)$  and  $f^{-1}(\beta)$  are disjoint when  $\alpha \neq \beta$ , there can be at most countably many such  $\alpha$ . ■

**THEOREM 9.3.** *If  $K$  is an uncountable algebraically closed field, then every affine algebraic variety over  $K$  is algebraically of the second category.*

*Proof.* If not, pick an algebraic variety  $X$  over  $K$  of smallest dimension which is not algebraically of the second category. By Definition 7.2 we can assume  $X$  is irreducible. Since zero dimensional irreducible varieties are points (and so trivially algebraically of the second category), it follows that  $X$  has positive dimension, and hence there exists a non-constant  $f$  in  $\mathcal{P}(X)$ . Then since  $K$  is algebraically closed, it follows that  $\text{Dim}(f^{-1}(\alpha)) = \text{Dim}(X) - 1$  for all  $\alpha \in K$  (see for example the theorem of Section 3.3 of [4] or Theorem 2 of Chapter I, Section 6 of [8]). But clearly  $X$  satisfies the hypotheses of Proposition 9.1, so by Corollary 9.2  $K$  is countable, a contradiction. ■

The above theorem and the idea of the proof are due to M. Artin.

In the remainder of this section we will be considering *valued fields*; that is to say fields  $K$  together with an *absolute value*. The latter is a map  $x \mapsto |x|$  of  $K$  into the non-negative reals such that  $|x| = 0$  if and only if  $x = 0$ ,  $|1| = 1$ ,  $|xy| = |x| \cdot |y|$ , and  $|x + y| \leq |x| + |y|$ . Clearly  $K$  becomes a metric space if we define the distance from  $x$  to  $y$  to be  $|x - y|$ , and we call  $K$  a *complete* valued field if this metric space is complete. If  $K$  is any field, then we can define a "trivial" absolute value on  $K$  by  $|0| = 0$  and  $|x| = 1$  if  $x \neq 0$ . The resulting metric space is discrete (and hence complete). If the absolute value on a field  $K$  is not this trivial one, then there is an  $x$  in  $K$  with  $|x| \neq 0$  and  $|x| \neq 1$ . Then either  $|x| < 1$  or  $|x^{-1}| < 1$ . In the first case  $x^n \rightarrow 0$ , while in the second  $x^{-n} \rightarrow 0$ . Thus *only* the trivial absolute value gives the discrete topology. The facts we shall need about valued fields will be found in [7].

We shall call the metric topology for  $K$  its  $M$ -topology, to distinguish it from its  $Z$ -topology. Clearly the  $M$ -topology is finer than the  $Z$ -topology. If  $V$  is any variety over  $K$ , then it is easily seen that the  $Z$ -topology for  $V$  is the weakest topology for  $V$ , making each  $f \in \mathcal{P}(V)$  continuous when  $K$  is given its  $Z$ -topology. It follows that if we define the  $M$ -topology for  $V$  to be the weakest topology for  $V$ , making all  $f \in \mathcal{P}(V)$  continuous when  $K$  is given its  $M$ -topology, then this is finer than the  $Z$ -topology for  $V$ , i.e., every subvariety of  $V$  is  $M$ -closed in  $V$ . Moreover it is immediate from the definition that the  $M$ -topology of  $V$  induces the  $M$ -topology of any subvariety  $X$  of  $V$  [because  $\mathcal{P}(X) = \{(f|_X) | f \in \mathcal{P}(V)\}$ ], so that  $X$  with its  $M$ -topology is a *closed subspace*

of  $V$  with its  $M$ -topology. It follows easily from the fact that multiplication is an  $M$ -continuous map  $K \times K \rightarrow K$  that (contrary to what is the case for the  $Z$ -topology) the  $M$ -topology for the product of two  $K$ -varieties is just the product of their  $M$ -topologies. In particular the  $M$ -topology for  $K^n$  is that derived from the "norm"

$$\|(x_1, \dots, x_n)\| = \max |x_i|.$$

In particular,  $K^n$  is metrizable and is complete when  $K$  is. Now morphisms (or polynomial maps) of  $K$ -varieties are clearly  $M$ -continuous. Since every  $K$ -variety is isomorphic (and hence  $M$ -homeomorphic) to a subvariety  $X$  of some  $K^n$ , and  $X$  with its  $M$ -topology is a closed subspace of  $K^n$ , it follows that every  $K$ -variety is  $M$ -metrizable and is complete if  $K$  is complete. In particular, if  $K$  is complete, then every  $K$ -variety  $X$  is a Baire space in its  $M$ -topology, and so is every  $M$ -open set of  $X$ .

**PROPOSITION 9.4.** *Let  $K$  be a complete valued field,  $X$  an algebraic variety over  $K$ , and  $\emptyset$  a non-empty  $M$ -open subset of  $X$ . If  $X$  is the union of a sequence of subvarieties  $\{X_n\}$ , then some  $X_n$  includes a non-void  $M$ -open subset of  $\emptyset$ .*

*Proof.* As remarked above,  $\emptyset$  is a Baire space in its  $M$ -topology. Since  $X_n$  is  $Z$ -closed (and hence  $M$ -closed) and  $\emptyset = \bigcup_{n=1}^{\infty} (\emptyset \cap X_n)$ , some  $\emptyset \cap X_n$  has non-empty  $M$ -interior relative to  $\emptyset$ , hence relative to  $X$ . ■

If  $y \in K^n$  and  $\epsilon > 0$ , we denote by  $B(y, \epsilon)$  the  $\epsilon$ -ball about  $y$  in  $K^n$ , i.e.,  $\{x \in K^n \mid |x_i - y_i| < \epsilon; i = 1, \dots, n\}$ . Note that the product of an  $\epsilon$ -ball in  $K^n$  with one in  $K^m$  is an  $\epsilon$ -ball in  $K^{n+m}$ . Also, translation by  $y_0$  maps the  $\epsilon$ -ball about  $y$  onto the  $\epsilon$ -ball about  $y + y_0$ .

**LEMMA 9.5.** *If  $K$  is a non-discrete valued field, then any non-empty  $M$ -open subset of  $K^n$  is  $Z$ -dense in  $K^n$ .*

*Proof.* It will suffice to show that any  $\epsilon$ -ball  $B(x_0, \epsilon)$  in  $K^n$  is  $Z$ -dense, and we can suppose  $x_0 = 0$ . If  $n = 1$ , choose  $x$  with  $0 < |x| < \min(1, \epsilon)$ . Then  $\{x^n\}$  is a sequence of distinct points of  $K$  all in  $B(x_0, \epsilon)$ , so no non-zero polynomial can vanish on  $B(x_0, \epsilon)$ , i.e.,  $B(x_0, \epsilon)$  is  $Z$ -dense. We proceed by induction. Let  $P(X_1, \dots, X_n)$  be a non-zero polynomial, say  $a_0(X_1, \dots, X_{n-1}) + \dots + a_k(X_1, \dots, X_{n-1})X_n^k$ , with  $a_k(X_1, \dots, X_{n-1}) \neq 0$ . By inductive hypothesis we can find  $(y_1, \dots, y_{n-1}) \in K^{n-1}$  with  $|y_i| < \epsilon$  such that  $a_k(y_1, \dots, y_{n-1}) \neq 0$ . Let  $Q(X) = a_0(y_1, \dots, y_{n-1}) + \dots + a_k(y_1, \dots, y_{n-1})X^k$ . By the case  $n = 1$  we can find  $y \in K$  with  $|y| < \epsilon$  such that  $Q(y) \neq 0$ . But then  $P(y_1, \dots, y_{n-1}, y) \neq 0$ , and since  $(y_1, \dots, y_{n-1}, y)$  is in the  $\epsilon$ -ball about 0 in  $K^n$ , we are done. ■

**PROPOSITION 9.6.** *Let  $X$  be an algebraic variety over a non-discrete valued field  $K$ . Suppose a polynomial map  $f: X \rightarrow K^n$  maps a non-empty  $M$ -open set  $\emptyset$  of  $X$  homeomorphically (with respect to the  $M$ -topologies) onto an  $M$ -open set of  $K^n$ . Then the  $Z$ -closure of  $\emptyset$  in  $X$  has algebraic dimension at least  $n$ .*

*Proof.* Let  $X_1, \dots, X_r$  be the irreducible components of the  $Z$ -closure of  $\emptyset$ . Since  $X_i$  is  $Z$ -closed (hence  $M$ -closed)  $\emptyset$  is the finite union of  $M$ -closed subsets  $\emptyset \cap X_1, \dots, \emptyset \cap X_r$ . Thus at least one of the  $X_i$  (say  $X_1$ ) must include a non-empty open subset  $U$  of  $\emptyset$ . If  $g = f|_{X_1}$ , then  $g(X_1) \supseteq g(U)$  which is  $M$ -open, and hence by Lemma 9.5 is  $Z$ -dense in  $K^n$ . It follows that  $g^*: \mathcal{P}(K^n) \rightarrow \mathcal{P}(X_1)$  is injective. Thus  $\mathcal{P}(X_1)$  includes a subring isomorphic to  $K[Y_1, \dots, Y_n]$ , and hence  $\mathcal{P}(X_1)$  has transcendence degree at least  $n$ ; i.e.,  $\text{Dim}(X_1) \geq n$ . ■

**LEMMA 9.7.** *Let  $X$  be an irreducible algebraic variety of algebraic dimension  $n$  over a complete, non-discrete valued field  $K$ . Let  $\emptyset$  denote the set of points  $x_0 \in X$  for which there exists a polynomial map  $f: X \rightarrow K^n$  mapping an open  $M$ -neighborhood of  $x_0$   $M$ -homeomorphically onto an  $M$ -open set in  $K^n$ . If  $\emptyset$  is non-empty, then  $X$  is algebraically of the second category.*

*Proof.* Suppose  $X$  is the union of a sequence  $\{X_k\}$  of subvarieties of  $X$ . We must show that some  $X_k$  is all of  $X$ , and since  $X$  is irreducible it will suffice to show that  $\text{Dim}(X_k) \geq n$  for some  $k$ . By Proposition 9.6 it will suffice to find in some  $X_k$  a non-empty  $M$ -open set  $U$  which is mapped  $M$ -homeomorphically onto an  $M$ -open set of  $K^n$ . Now  $\emptyset$  is by its definition  $M$ -open in  $X$ , and clearly any sufficiently small non-empty  $M$ -open set of  $\emptyset$  is a suitable candidate for  $U$ . Thus Proposition 9.4 completes the proof. ■

We recall that a field  $K$  is called perfect either if it is of characteristic zero or if it is of characteristic  $p > 0$ , and every element of  $K$  has a  $p$ th root in  $K$ . The only property of a perfect field  $K$  we shall need is that if  $P(X_1, \dots, X_n) \in K[X_1, \dots, X_n]$  is non-constant and irreducible, then at least one of the formal partial derivatives  $\partial P / \partial X_i$  is non-zero.

**THEOREM 9.8.** *If  $K$  is a perfect, complete, non-discrete valued field, then every affine algebraic variety  $X$  over  $K$  is algebraically of the second category.*

*Proof.* By Definition 7.2 we can assume  $X$  is irreducible and, say, of algebraic dimension  $n > 0$ . By the preceding lemma it will suffice to show that the set  $\emptyset$  of  $x_0 \in X$  for which there exists a polynomial map  $f: X \rightarrow K^n$  mapping an open  $M$ -neighborhood of  $x_0$   $M$ -homeomorphically onto an  $M$ -open set in  $K^n$  is non-empty. Consider first the case when  $X$  is isomorphic to a subvariety  $Y$  of  $K^{n+1}$  such that the ideal of polynomials vanishing on  $Y$  is a principal ideal  $(F)$ .

Since  $Y$  is irreducible, so is the polynomial  $F$ , and since  $\text{Dim}(Y) = n$ ,  $F$  is not constant, so some formal partial derivative  $\partial F / \partial X_i$  ( $i = 1, \dots, n+1$ ) is not zero. By a linear change of coordinates we can assume  $\partial F / \partial X_{n+1} \neq 0$ . Now  $\partial F / \partial X_{n+1}$  has degree less than that of  $F$ , as a polynomial in  $X_{n+1}$ , so it is certainly not in the ideal  $(F)$ . Thus the set  $U$  of  $y \in Y$ , where  $(\partial F / \partial X_{n+1})(y) \neq 0$  is a non-empty (hence  $Z$ -dense)  $Z$ -open subset of  $Y$ . Consider the polynomial map  $\tilde{F}$ :

$$(x_1, \dots, x_{n+1}) \mapsto (x_1, \dots, x_n, F(x_1, \dots, x_{n+1}))$$

of  $K^{n+1} \rightarrow K^{n+1}$ . By the analytic inverse function theorem (Theorem 2, Chapter 3, Section 9 of [7]), for each  $u_0 \in U$  there is an  $\epsilon > 0$  and an  $M$ -neighborhood  $N$  of  $u_0$  in  $K^{n+1}$  such that  $\tilde{F}$  maps  $N$   $M$ -homeomorphically (in fact "bi-analytically") onto an  $\epsilon$ -ball in  $K^{n+1}$ . But then clearly  $N \cap Y = \{x \in N \mid F(x) = 0\}$  is an  $M$ -neighborhood of  $u_0$  in  $Y$  which is mapped by  $\tilde{F}$   $M$ -homeomorphically onto an  $\epsilon$ -ball in  $K^n$ . Now in general,  $X$  is not isomorphic to such a  $Y$ , but is birationally isomorphic to such a  $Y$ . That is (cf. Chapter I, Section 3 of [8] and in particular Theorem 6), if we identify  $X$  with a subvariety in  $K^m$ , there is a linear map  $T: K^m \rightarrow K^{n+1}$  and non-empty  $Z$ -open sets  $G_X$  and  $G_Y$  of  $X$  and  $Y$ , respectively, such that  $T$  maps  $G_X$  one to one onto  $G_Y$ . Moreover the inverse map  $(T|_{G_X})^{-1}: G_Y \rightarrow G_X$  is a "rational map," i.e., there are polynomials  $P_i(X_1, \dots, X_{n+1})$ ,  $Q_i(X_1, \dots, X_{n+1})$ ,  $i = 1, \dots, m$ , with the  $Q_i$  vanishing nowhere on  $G_Y$ , such that for all  $y \in G_Y$

$$(T|_{G_X})^{-1}(y) = (P_1(y)/Q_1(y), \dots, P_m(y)/Q_m(y)).$$

It follows in particular that  $T$  maps  $G_X$   $M$ -homeomorphically onto  $G_Y$ . Since  $U$  and  $G_Y$  are both  $Z$ -open (and hence  $Z$ -dense) subsets of the irreducible variety  $Y$ , their intersection is a non-empty  $Z$ -open (and hence  $M$ -open) subset of  $Y$ . Then  $\emptyset' = (T|_{G_X})^{-1}(U \cap G_Y)$  is a non-empty  $M$ -open subset of  $X$  which is included in  $\emptyset$ . For if  $x_0 \in \emptyset'$  and  $u_0 = T(x_0) \in U$ , then, choosing  $N$  as above,  $(T|_{G_X})^{-1}(N)$  is an  $M$ -neighborhood of  $x_0$  in  $X$  which is mapped  $M$ -homeomorphically by  $f = \tilde{F} \circ T$  onto an  $M$ -open set in  $K^n$ . ■

If we look for necessary conditions on an infinite field  $K$  in order that every  $K$ -variety should be algebraically of the second category, the most obvious perhaps is that  $K$  itself, considered as a curve over  $K$ , should be algebraically of the second category, i.e., (cf. Proposition 7.3)  $K$  must be uncountable. At first glance one might hope that this condition was also sufficient. It is not. We shall next describe a class of examples (suggested by M. Artin) of uncountable fields  $K$ , of arbitrary characteristic, for which there exist

curves  $C$  over  $K$  which are countable and hence (by Proposition 7.3 again) not algebraically of the second category. We start with a countable field  $k$  of characteristic  $p$  which for simplicity we assume to be algebraically closed. For example,  $k$  might be the algebraic closure of the prime field of characteristic  $p$ . For  $K$  we will take the field  $k(\{X_\alpha\})$  of rational functions in uncountably many independent transcendentals  $X_\alpha$ . We shall see that if  $C_k$  is any non-rational plane algebraic curve over  $k$  then  $C_K = C_k$ ; hence, since  $C_k$  is clearly countable, so is  $C_K$ . We start with an arbitrary plane algebraic curve  $C_k \subseteq k^2$ . That is  $C_k = \{(x, y) \in k^2 \mid f(x, y) = 0\}$ , where  $f$  is an irreducible element of  $k[X, Y]$  and  $f$  generates the ideal of polynomials vanishing on  $C_k$ . We let  $\mathfrak{T}(C_k)$  denote the field of quotients of  $\mathfrak{P}(C_k)$ , and we recall that  $C_k$  is called *rational* if and only if  $\mathfrak{T}(C_k)$  is isomorphic to the field  $k(T)$  of rational functions in one variable. By Lüroth's theorem (Section 63 of [9]), if  $\mathfrak{T}(C_k)$  is isomorphic over  $k$  to a subfield of  $k(T)$ , then  $C_k$  is rational. Finally we recall that if  $F$  is any extension field of  $k$ , then  $C_F$  denotes the  $F$ -subvariety of  $F^2$  defined by  $\{(x, y) \in F^2 \mid f(x, y) = 0\}$ .

LEMMA 1. *If there exists  $(A, B) \in C_{k(T)}$  with  $(A, B) \notin k^2$ , then  $C_k$  is a rational curve.*

*Proof.* Write  $A = P_1(T)/Q_1(T)$ ,  $B = P_2(T)/Q_2(T)$ , where  $P_i$  is relatively prime to  $Q_i$  and  $Q_i \neq 0$ . Except for the finite set  $S$  of  $t \in k$  where  $Q_1(t)Q_2(t) = 0$ , we know that  $P_1(t)/Q_1(t)$  has a value  $A(t) \in k$ ,  $P_2(t)/Q_2(t)$  has a value  $B(t) \in k$ , and  $(A(t), B(t)) \in C_k$ . Given  $(x_0, y_0) \in C_k$ , the set of  $t \in k - S$  where  $(A(t), B(t)) = (x_0, y_0)$  is clearly finite [since otherwise  $P_1(T)/Q_1(T)$  and  $P_2(T)/Q_2(T)$  would be the constants  $x_0$  and  $y_0$ ]; hence, since  $k - S$  is infinite,  $\{(A(t), B(t)) \mid t \in k - S\}$  is an infinite and hence  $Z$ -dense subset of the curve  $C_k$ , so any polynomial  $Q(X, Y) \in k[X, Y]$  vanishing on  $\{(A(t), B(t)) \mid t \in k - S\}$  vanishes on all of  $C_k$  and so is a multiple of  $f$ . Now an element  $g$  of  $\mathfrak{T}(C_k)$  is of the form  $P(X, Y)/Q(X, Y)$ , where  $P, Q \in k[X, Y]$  and  $Q$  is not a multiple of  $f$ . It follows that  $Q(P_1(T)/Q_1(T), P_2(T)/Q_2(T))$  is not the zero rational function, so  $P(P_1(T)/Q_1(T), P_2(T)/Q_2(T))/Q(P_1(T)/Q_1(T), P_2(T)/Q_2(T))$  is a well-defined element  $g(A, B)$  of  $k(T)$ . Clearly  $g \mapsto g(A, B)$  is a field embedding  $\mathfrak{T}(C_k) \rightarrow k(T)$  over  $k$ , so that as remarked above, by Lüroth's theorem  $C_k$  is a rational curve. ■

LEMMA 2. *If there exists  $(\alpha, \beta) \in C_{k(T_1, \dots, T_n)}$  with  $(\alpha, \beta) \notin k^2$ , then  $C_k$  is rational.*

*Proof.*

$$\alpha = p_1(T_1, \dots, T_n)/q_1(T_1, \dots, T_n) \quad \text{and} \quad \beta = p_2(T_1, \dots, T_n)/q_2(T_1, \dots, T_n),$$

and since  $\alpha, \beta$  are not both in  $k$ , there exist points  $x, y$  in  $k^n$  where neither  $q_1$  nor  $q_2$  vanishes and where at least one of  $\alpha$  and  $\beta$  take distinct values. Let  $P_i(T) = p_i(x + T(y - x))$  and  $Q_i(T) = q_i(x + T(y - x))$ . Then  $Q_i(T) \neq 0$  [in fact  $Q_i(0)$  and  $Q_i(1)$  are non-zero], so  $A = P_1(T)/Q_1(T)$  and  $B = P_2(T)/Q_2(T)$  are in  $k(T)$ , and clearly  $(A, B) \in C_{k(T)}$ . Also, at least one of  $A$  and  $B$  takes distinct values at 0 and 1, so  $(A, B) \notin k^2$  and Lemma 1 completes the proof. ■

It is now easy to see that if  $K = k(\{X_\alpha\})$  and  $C_k$  is properly included in  $C_K$ , then  $C_k$  must be a rational curve. For suppose  $(\alpha, \beta) \in C_K$  with  $(\alpha, \beta) \notin k^2$ . If we denote by  $T_1, \dots, T_n$  those finitely many of the  $X_\alpha$  that actually occur in some representation of  $\alpha$  and  $\beta$  as quotients of polynomials in the  $X_\alpha$ , then  $(\alpha, \beta) \in k(T_1, \dots, T_n)$ , and we are back to the case of Lemma 2. Finally we should remark that since  $k$  is algebraically closed, it is easy to see that there exist non-rational plane curves over  $k$ . For example, in characteristic different from two it is elementary that  $y^2 = x^3 + Ax + B$  is rational if and only if the right hand side has repeated roots.

There still remains the following interesting question: if every curve over a field  $K$  is uncountable (and hence algebraically of the second category), does it follow that all varieties over  $K$  are algebraically of the second category? This would be a very pretty fact if true (and would greatly simplify the proof of the theorems of this section).

**10. An Open Question.** Let  $G$  be a set with both the structure of an abstract group and that of an affine algebraic variety over some infinite field  $K$ . Let  $\mu : G \times G \rightarrow G$  and  $m : G \times G \rightarrow G$  denote the maps  $(x, y) \mapsto xy^{-1}$  and  $(x, y) \mapsto xy$  respectively. Let  $L_x$  and  $R_x$  denote respectively the left and right translation maps  $G \rightarrow G$ , and let  $\mathcal{G}$  denote the inversion map  $x \mapsto x^{-1}$  of  $G$  into  $G$ . By definition,  $G$  is an affine algebraic group if  $\mu$  is a polynomial map. A complete analogue of Montgomery's theorem [5] would say that if each  $L_x$  and each  $R_x$  are polynomial maps, then  $\mu$  is a polynomial map. This is probably too much to hope for in the algebraic category, though it does seem reasonable to conjecture that  $m$  is a polynomial map in this case (even if  $G$  is not algebraically of the second category). Thus a conjectural reasonable algebraic analogue of Montgomery's theorem would be the following: if  $\mathcal{G}$  is a polynomial map and each  $L_x$  is a polynomial map, then  $G$  is an algebraic group. [Note that since  $\mathcal{G} \circ L_x \circ \mathcal{G} = R_{x^{-1}}$ , if  $\mathcal{G}$  is polynomial and each left (right) translation is polynomial, then so also is each right (left) translation.] Of course, if  $G$  is algebraically of the second category, then Theorem 8.3 settles this question in the affirmative. In particular if  $K$  is uncountable and algebraically closed or a perfect, non-discrete, complete valued field (e.g.,  $\mathbf{R}$  or a  $p$ -adic

completion of  $Q$ ), then the conjecture is true (Theorems 9.3, 9.8). (It was the question of whether Montgomery's theorem was true in the algebraic category over  $\mathbf{R}$  that originally started the author thinking about these questions.)

However, the fact that  $\mu$  is a group law should make it unnecessary to assume  $G$  is algebraically of the second category.

*Added in Proof:* In a paper in the same number of this journal entitled "Separately algebraic group laws," A. Magid has verified this conjecture of Sec. 10 in case  $K$  is an infinite perfect field. Moreover he shows that, as I had originally suspected, the complete analogue of Montgomery's theorem is true in the algebraic category, in the sense that if each  $L_x$  and  $R_x$  are polynomial then so is  $I$ .

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