Actes, Congrès intern. Math., 1970. Tome 2, p. 243 à 249.

BANACH MANIFOLDS OF FIBER BUNDLE SECTIONS

by Richard S. PALAIS *

1. Introduction.

In the past several years significant progress has been made in our understanding of infinite dimensional manifolds. Research in this area has split into two quite separate branches ; first a study of the theory of abstract Banach manifolds, and secondly a detailed study of the properties of certain classes of concrete manifolds that arise as spaces of differentiable maps or more generally as spaces of sections of fiber bundles. A survey of the remarkable progress made in the first mentioned area, (i.e. infinite dimensional differential topology) will be found in the reports to this Congress by N. Kuiper and R. Anderson. Here I would like to survey a part of the recent work in the second area, which for obvious reasons (made explicit in § 1 of [12]) has come to be called non-linear global analysis. I shall also attempt to indicate what in my opinion are fruitful directions of current research and hazard a few guesses for the near future. An attempt to be comprehensive would be futile since the subject shades off imperceptibly into many extremely active classical fields of mathematics for which in fact it plays the role of "foundations" (e.g. non-linear partial differential equations and continuum mechanics). I shall therefore concentrate on those few topics which have most engaged my personal interest, particularly the intrinsic structures of manifolds of sections and applications to the calculus of variations. I shall also not attempt to cover research prior to 1966 which is surveyed in the excellent and comprehensive review article [4] of James Eells Jr.

2. Manifold structures for spaces of bundle sections.

If ξ is a smooth ($= C^\infty$) vector bundle over a smooth compact manifold M we can define the Banach spaces $C^k(\xi)$ of C^k sections of ξ as well as many more exotic Banach spaces of distributional sections of ξ, such as the Sobolev spaces $L^p_k(\xi)$. Let us use the symbol Γ to denote a generic "differentiability class" such as C^k of L^p_k. We can regard Γ as a functor defined on the category $VB(M)$ of smooth vector bundles over M and taking values in the category of Banach spaces and continuous linear maps (if $f : \xi \to \eta$ is a smooth vector bundle morphism then $\Gamma(f) : \Gamma(\xi) \to \Gamma(\eta)$ is of course just $s \to f \circ s$). We shall assume that we have a continuous inclusion $\Gamma(\xi) \subset C^0(\xi)$ (e.g. if $\Gamma = L^p_k$ the condition for this is $k > n/p$ where $n = \dim M$). A central foundational question for many

----

(1) Research supported in part by USAF Grant No. AFOSR 68-1403.

problems of non-linear analysis is : when can we "extend" Γ to a functor from the category $FB(M)$ of smooth fiber bundles over M to the category of smooth Banach manifolds ? This is an abstract and general version of the question, "when is the technique of linearizing non-linear problems meaningful and "natural"? If E and E' are smooth fiber bundles over M, a smooth map $f : E \to E'$ is a morphism of $FB(M)$ if for each $x \in M$ $f(E_x) \subset E'_x$. A smooth vector bundle ξ over M is called an open vector sub bundle of E if ξ is open in E and the inclusion map $\xi \to E$ is a morphism of $FB(M)$. If $s_0 \in C^0(E)$ then such a ξ is called a vector bundle neighborhood (VBN) of s_0 in E if $s_0 \in C^0(\xi)$. The existence of such a ξ is a basic lemma [12, Theorem 12.10]. Let us say $s_0 \in \Gamma(E)$ provided $s_0 \in \Gamma(\xi)$. It is easily seen that a sufficient condition for this to be independent of the choice of such a ξ is that :

> $FB(\Gamma)$: Given objects ξ and η of $VB(M)$ and a morphism $f : \xi \to \eta$ of $FB(M)$, $s \to f \circ s$ defines a continuous map $\Gamma(f) : \Gamma(\xi) \to \Gamma(\eta)$.

Equally obvious is the fact that if we define the "natural atlas" for $\Gamma(E)$ to be the collection $\{\Gamma(\xi)\}$ of Banach spaces indexed by the open vector subbundles ξ of E, then this same condition $FB(\Gamma)$ is just what is required to make these charts C^0-related and hence for the natural atlas to define $\Gamma(E)$ as a C^0 Banach manifold. More surprising perhaps is the observation that $FB(\Gamma)$ implies that the maps $\Gamma(f)$ of its statement are C^∞, whence the natural atlas defines $\Gamma(E)$ as a smooth Banach manifold and we have our desired extension of Γ. Given a smooth fiber bundle $\pi : E \to M$ its "tangent bundle along the fiber" is a smooth vector bundle over E, $p : TF(E) \to E$, but may also be regarded as a smooth fiber bundle $\tilde{p} = \pi \circ p : TF(E) \to M$ over M and it is easily seen that there is a canonical identification of $\Gamma(p) : \Gamma(TF(E)) \to \Gamma(E)$ with the tangent bundle of $\Gamma(E)$. If $f : E \to E'$ is a smooth fiber bundle morphism then its "differential along the fiber" is a smooth fiber bundle morphism $\delta f : TF(E) \to TF(E')$ over M and with the above identification $\Gamma(\delta f)$ is the differential of $\Gamma(f)$. For further details see [12]. A similar treatment will be found in Eliasson [5], [6]. Recently J.P. Penot has given a detailed and comprehensive treatment of this problem including several new approaches to the manifold structure of $\Gamma(E)$, [13], and Mike Field has shown that when G is a compact Lie group, M a G-manifold and E is a G-fiber bundle over M, then $\Gamma_G(E)$, the equivariant G-sections of E, is a smooth submanifold of $\Gamma(E)$ [8]. One should also mention here the important related work of A. Douady [1] and Kijowski [9] concerning manifold structures for spaces of submanifolds of a given manifold.

3. Extra structures for manifolds of sections.

The manifolds $\Gamma(E)$ have aside from their differentiable structure much added structure whose properties are of the utmost importance in dealing with concrete problems in non-linear analysis. The "essence" of this extra structure is as yet not fully understood and manifests itself in differing though related guises in varying circumstances. The elucidation and axiomatization of this additional structure I regard as one of the most intriguing and important foundational questions of non-linear global analysis and I shall remark here on the current status of such research.

In dealing with the manifolds $\Gamma(E)$ it is appropriate to use only the charts $\Gamma(\xi)$ of the natural atlas ; the "extra structure" whatever it is, gets lost in passing to the maximal atlas. We should therefore look at the coordinate transformations between two such charts. These are of the form $\Gamma(f) : \Gamma(\xi) \to \Gamma(\eta)$ where $f : \xi \to \eta$ is a fiber bundle morphism of vector bundles. It makes sense to speak of bounded sets in the Banach spaces $\Gamma(\xi)$ and $\Gamma(\eta)$ and with mild conditions on Γ one can prove that the following condition holds :

$BP(\Gamma)$: If $f : \xi \to \eta$ is a fiber bundle morphism of vector bundles over M , $\Gamma(f)$ maps bounded sets to bounded sets.

(see e.g. [12, 19.12] for the case $\Gamma = L^p_k$). From this observation Karen Uhlenbeck in her thesis [17] developed a notion of intrinsically bounded (*IB*) *subsets of* $L^p_k(E)$ and used them very effectively to prove that certain wide classes of calculus of variations problems satisfied Condition (*C*) (See § 6 below). Perhaps the simplest of many diverse descriptions of *IB* sets is "a finite union of subsets of $L^p_k(E)$, each a bounded set in some $L^p_k(\xi)$". What gives them their usefulness (aside from their being preserved by induced morphisms $L^p_k(f)$) is that they are relatively compact in $C^0(E)$, by Rellich's theorem. U. Koschorke has investigated an abstract axiomatic notion of "boundedness structure" suggested by *IB* sets and made several interesting applications (unpublished). About a year ago J. Dowling and K. Uhlenbeck independently made what I consider a very surprising and important observation ; namely that if $f : \xi \to \eta$ is a smooth fiber bundle morphism of vector bundles over M, then for $p > 1$ $L^p_k(f)$ maps weakly convergent sequences to weakly convergent sequences, or equivalently $L^p_k(f)$ is weakly continuous on bounded sets. What makes this so remarkable is that $L^p_k(f)$ is highly non-linear and usually even the mildest non-linearity destroys weak continuity. [For example, consider the quadratic map φ of Hilbert space H to itself, $\varphi(x) = x + \|x\|^2 e$ where e is a non-zero vector in H. $D\varphi_0$ = identity so φ maps some ball (say of radius $2r$) diffeomorphically. If $\{e_n\}$ is an orthonormal base then $re_n \to 0$ weakly, $\varphi(re_n) \to r^2 e$ weakly, but $\varphi(0) = 0 \neq r^2 e$]. As a result it makes no sense to speak of the "weak topology" of an infinite dimensional manifold in general, yet the theorem of Dowling and Uhlenbeck shows that it does make sense for the $L^p_k(E)$, and moreover the *IB* sets turn out to be just the relatively compact sets of this topology. Quite recently Richard Graff has found a simple and elegant proof of this theorem which moreover works whenever $BP(\Gamma)$ is satisfied, Γ is reflexive (i.e. each $\Gamma(\xi)$ is), and Γ satisfies "Rellich's condition" (i.e. $\Gamma(\xi) \subset C^0(\xi)$ is a compact map), hence for any such Γ one can define the "weak topology" for the manifolds $\Gamma(E)$. This weak topology is certainly a part of the extra structure we seek. How big a part is not yet clear.

Another approach to "extra structure" starts with the observation that the functors Γ do not exist in isolation ; there is a vast collection of them (the various $L^{p'}_k$s, $C^{k+a'}$s etc.) related by various "embedding theorems" (e.g. $L^p_k \subset C^r$ if $k > \dfrac{n}{p} + r$; $L^p_k \subset L^q_l$ for $k > l$ and $k - \dfrac{n}{p} > l - \dfrac{n}{p}\Big)$. These relationships are known to be absolutely crucial in the analysis of concrete linear and non-linear problems and it is quite plausible to me that it is to this family of relationships that we must look to fully understand the "extra structure". H. Omori has axiomatized at least part of this structure with his notion of ILH and ILB manifolds [10]. While too complicated to explain here it is clear from the applications already made by Omori and others that this is an important concept ; it is also probably the natural setting for some eventual abstract form of the Nash-Moser implicit function theorem.

Additional structures for the $\Gamma(E)$ deserving of special attention are the geometric structures (Finsler metrics, affine connections etc.) induced from similar structures for E and M. These are of course intimately related to numerous classical non-linear problems, particularly in the calculus of variations. Interesting work has been done in this area by Eliasson ([5], foundations), Dowling ([2], Hopf-Rinow theorem) and Ebin ([3], differential geometry of manifolds of Riemannian metrics). One should mention also Uhlenbeck's theorem [17] that lB sets in $L^p_k(E)$ are just those which are bounded sets for any one of a certain natural class of "admissible" Finsler structures. For a while there was a hope that the $\Gamma(E)$ would carry "natural" layer or Fredholm structures (see [7] and also the report to this congress by J. Eells). This could have important consequences (e.g. a degree theory and Leray-Schauder type fixed point theorems). Unfortunately, despite considerable effort there is little evidence to support such a conjecture.

4. Partial differential operators.

Let E be a C^∞ fiber bundle over M. Then $J^r(E)$, the bundle of r-jets of sections of E, is a C^∞ fiber bundle over E ; if $s \in C^\infty(E)$ then $j_r(s)_x$ its r-jet at $x \in M$ lies in $J^r(E)_{s(x)}$. Let F be another C^∞ fiber bundle over E and let $\Phi : J^r(E) \to F$ be a fiber bundle morphism over E. Given $s \in C^\infty(E)$ define $\Phi_*(s) \in C^\infty(s^*F)$ by

$$\Phi_*(s)\ (x) = \Phi\,(j_r(s)_*) \in F_{s(x)} = (s^*F)_x .$$

If we define $F_E(C^\infty, C^\infty)$ to be

$$\{(\sigma, s) \in C^\infty(M, F) \times C^\infty(E) \mid \sigma(x) \in F_{s(x)} \text{ all } x \in M\}$$

then $F_E(C^\infty, C^\infty)$ is a fiber bundle over $C^\infty(E)$ whose fiber over $s \in C^\infty(E)$ is just $C^\infty(s^*F)$, and Φ_* is a section of this bundle. For this reason K. Uhlenbeck, who introduced such operators in [17], called them differential section operators. This concept seems to capture the notion of partial differential operator in its full generality. Consider the special case when $F = \pi^*E'$ is induced from a bundle E' over M ($\pi : E \to M$ being the projection). Then we may regard $J^r(E)$ as a bundle over M also and Φ then becomes a bundle morphism $J^r(E) \to E'$ over M. Moreover $s^*F = s^*\pi^*E' = (\pi s)^*E' = E'$ for any $s \in C^\infty(E)$ so $F_E(C^\infty, C^\infty)$ is the trivial bundle $C^\infty(E') \times C^\infty(E)$ and Φ_* a map $C^\infty(E) \to C^\infty(E')$. This is just the class of non-linear partial differential operators defined in [12, § 15]. As a natural example of a differential section operator D which is not a partial differential operator in the latter more restricted sense, let $M = I = [0, 1]$, $E = W \times I$ where W is a Riemannian manifold, and let $F = TW \times I$. Given $\sigma \in C^\infty(E) = C^\infty(I, W)$ let $D\sigma \in \sigma^*(TW) = \sigma^*(F)$ denote the covariant derivative of σ' along σ (so $D\sigma = 0$ is the condition that σ be a geodesic). More generally the Euler-Lagrange operator for a calculus of variations problem can in general only be interpreted globally as a differential section operator.

The elucidation of the general properties and structure of differential section operators is clearly a very difficult problem but deserves considerable effort since it is the very core of the foundations of global non-linear analysis.

One of the first natural questions that comes to mind, and an extremely important one for applications, is the following. Given E and F as above and section functors Γ and Γ' we can define analogously to $F_E(C^\infty, C^\infty)$ the bundle $F_E(\Gamma', \Gamma) = \{(\sigma, s) \in \Gamma'(M, F) \times \Gamma(E) \mid \sigma(x) \in F_{s(x)},\ \text{all } x \in M\}$ over $\Gamma(E)$ whose fiber over s is $\Gamma'(s^*F)$. Given a fiber bundle morphism $\Phi : J^r(E) \to F$ when does the differential section operator Φ_* extend to a continuous or differentiable section of $F_E(\Gamma', \Gamma)$. For the case that Γ and Γ' are Sobolev functors (L_r^p) and Φ_* a "polynomial" differential operator (almost the only case ever arising in practice) this is now fairly well understood ([12, § 16], [6], [17] and the thesis of Mark Schmidt [15] which is devoted to this question).

5. The Calculus of variations.

Let M have a smooth measure μ and let $\mathcal{L} : C^\infty(E) \to C^\infty(M, \mathbf{R})$ be a differential operator which extends to a smooth map $\mathcal{L} : \Gamma(E) \to L^1(M, \mathbf{R})$. Then we have a smooth real valued function $J : \Gamma(E) \to \mathbf{R}$ defined by $J(s) = \int \mathcal{L}(s)(x)\, d\mu(x)$. The calculus of variations is concerned with the study of the "extremals" or critical points of functionals such as J on certain submanifolds of $\Gamma(E)$ (where in general Γ is a Sobolev functor). Given $f \in C^\infty(E)$ let $\Gamma_{\partial f}(E)$ denote the closure in $\Gamma(E)$ of the set of $s \in \Gamma(E)$ which agree with f in a neighborhood of ∂M. Then $\Gamma_{\partial f}(E)$ is a smooth submanifold of $\Gamma(E)$ called the Dirichlet space of f and of particular interest is the "Dirichlet problem" of describing the critical points of $J \mid \Gamma_{\partial f}(E)$. An account of this will be found in [12, § 19] with the simplifying assumption that E is a sub-bundle of a trivial vector bundle and that $\mathcal{L}$ is of a special form relative to this embedding. Two more general intrinsic treatments will be found in [6] and [17]. The major concern of this research has been two-fold. First to find conditions for $\mathcal{L}$ that will guarantee $J \mid \Gamma_{\partial f}(E)$ satisfies Condition (C) of [11] and [16] (which in turn implies existence theorems for extremals) and secondly to prove that extremals have greater smoothness than is a priori evident ; under appropriate conditions on $\mathcal{L}$. The restrictions on $\mathcal{L}$ assume the form of "coerciveness" or "ellipticity" conditions familiar from linear theory. While the present state of affairs is far from definitive and much remains to be done there has been considerable progress in this area.

An important question in the case $\Gamma = L_k^2$ is "when are all the critical points of $J \mid \Gamma_{\partial f}(E)$ non-degenerate for most choices of f". For the case of geodesics on a Riemannian manifold V, where $M = I$, $E = V \times I$, $\mathcal{L}(\sigma) = \frac{1}{2} \|\sigma'\|^2$, $\Gamma = L_1^2$. Morse showed this was so. In this case $\Gamma_{\partial f}(E)$ consists of all $g \in \Gamma(E)$ having the same endpoints (p, q) as f and Morse's theorem is equivalent to the statement that for almost all $(p, q) \in V \times V$, q is not a conjugate point of p, a result which follows fairly directly from the Sard-Brown theorem or the more general Thom transversality theorem. For calculus of variations problems with several independent variables it has long been suspected that with appropriate conditions on $\mathcal{L}$ similar results could be proved, however only very recently have such conditions been found [18] by Uhlenbeck. Her proof involves regarding $\Gamma(E)$ as a smooth bundle of Hilbert manifolds with the $\Gamma_{\partial f}(E)$ as fibers and depends on F. Quinn's generalization of Smale's transversality theorem for Fredholm maps [14].

There is an application of the above results which should be fairly easy to carry out and would have interesting connections with topology. Assume M is a compact Riemannian symmetric space of rank one and V another Riemannian symmetric space of rank one. Among the natural Lagrangians for maps $M \to V$ there is the well-known higher order "energy" function whose extremals are the so called polyharmonic maps. Condition (C) and the regularity theorem are satisfied for this functional and moreover, because of the high degree of symmetry involved, the problem of finding explicitly the critical submanifolds and their indices should reduce to reasonably straightforward calculations. Using standard results of Morse theory this would lead to information about the homotopy type of $C^0(M, V)$ and in particular, taking M to be S^n, of the higher loop spaces of V.

Let me close by saying that I have only been able to give a small sample of the many promising lines of current research in non-linear global analysis. In particular I have not even mentioned here what I consider one of the most interesting and promising such programs, namely that initiated by Arnold in continuum mechanics and developed considerably in the past several years. For this I refer to *D*. Ebin's report to this Congress.

REFERENCES

[1] DOUADY A. — Le problème des modules pour les sous-espaces analytiques compacts d'un espace analytique donné, *Sem. Collège de France*, 1964-65.

[2] DOWLING J. — Finsler geometry on Sobolev manifolds, *Proc. of Symposia in Pure Math.*, Vol. 15, Amer. Math. Soc., 1970, p. 1-10.

[3] EBIN D. — The manifold of Riemannian metrics, *Proc. of Symposia in Pure Math* Vol. 15, Amer. Math. Soc., 1970, p. 11-40.

[4] EELLS J. — A setting for Global Analysis, *Bull. of A.M.S.*, Vol. 72. No. 5, 1966, p. 751-807.

[5] ELIASSON H. — Geometry of manifolds of maps, *J. Differential Geometry*, Vol. 1, 1967, p. 169-194.

[6] ELIASSON H. — Variational integrals in fiber bundles, *Proc. of Symposia in Pure Math.* Vol. 16, Amer. Math. Soc., 1970, p. 67-90.

[7] ELWORTHY K.D. and TROMBA T. — Differentiable structures and Fredholm maps on Banach manifolds, *Proc. of Symposia in Pure Math.*, Vol. 15, Amer. Math. Soc., 1970, p. 45-94.

[8] FIELD M. — *Thesis*, Univ. of Warwick, England, 1970.

[9] KOJOWSKI J. — Existence of differentiable structures in the set of submanifolds, *Studia Math.*, Vol. 33, p. 93-108.

[10] OMORI H. — On the group of diffeomorphisms of a compact manifold, *Proc. of Symposia in Pure Math.*, Vol. 15, Amer. Math. Soc., 1970, p. 167-184.

[11] PALAIS R.S. — Morse theory in Hilbert manifolds, *Topology*, Vol. 2, 1963,, p. 299-340.

[12] PALAIS R.S. — *Foundations of global non-linear analysis,* W. A. Benjamin Inc., New York, 1968.
[13] PENOT J.P. — *Diverse methodes de construction des varietes des applications,* (in preparation).
[14] QUINN F. — Transversal approximations on Banach manifolds, *Proc. of Symposia in Pure Math.,* Vol. 15, Amer. Math. Soc., 1970, p. 213-222.
[15] SCHMIDT M. — Extension of differential operators to Sobolev spaces, *Thesis,* Brandeis Univ., 1969.
[16] SMALE S. — Morse theory and a non-linear generalization of the Dirichlet Problem, *Ann. of Math.,* Sol. 80, 1964, p. 382-396.
[17] UHLENBECK K. — The Calculus of variations and global analysis, *Thesis,* Brandeis University, 1968.
[18] UHLENBECK K. — Integrals with non-degenerate critical points, *Bull. A.M.S.,* Vol 76, 1970, p. 125-128.

Brandeis University
Dept. of Mathematics,
Waltham
Massachusetts 02 154 (USA)