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A simple proof of the Banach contraction principle

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The author dedicates this work to two friends from long ago,
Professors Albrecht Dold and Ed Fadell

Abstract. We give a simple proof of the Banach contraction lemma.

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In what follows, X is a metric space with distance function ρ and $f : X \to X$ is a *contraction mapping*, i.e., we assume $\rho(fx_1, fx_2) \leq K\rho(x_1, x_2)$ for all $x_1, x_2 \in X$, with $0 < K < 1$, so by induction, if f^m denotes f composed with itself m times, then $\rho(f^m(x_1), f^m(x_2)) \leq K^m \rho(x_1, x_2)$. By the triangle inequality,

$$\rho(x_1, x_2) \leq \rho(x_1, f(x_1)) + \rho(f(x_1), f(x_2)) + \rho(f(x_2), x_2),$$

so $(1 - K)\rho(x_1, x_2) \leq \rho(x_1, f(x_1)) + \rho(f(x_2), x_2)$, and since $K < 1$, we have

Fundamental Contraction Inequality. *If $f : X \to X$ is a contraction mapping, with contraction constant K, then for all x_1 and x_2 in X,*

$$\rho(x_1, x_2) \leq \frac{1}{1-K}(\rho(x_1, f(x_1)) + \rho(x_2, f(x_2))).$$

In particular, if x_1 and x_2 are fixed points of f we get $\rho(x_1, x_2) = 0$, hence:

Corollary. *A contraction mapping can have at most one fixed point.*

Proposition. *If $f : X \to X$ is a contraction mapping then, for any x in X, the sequence $f^n(x)$ of iterates of x under f is a Cauchy sequence.*

Proof. Taking $x_1 = f^n(x)$ and $x_2 = f^m(x)$ in the Fundamental Inequality gives

$$\begin{aligned}\rho(f^n(x), f^m(x)) &\leq \frac{1}{1-K}(\rho(f^n(x), f^n(f(x))) + \rho(f^m(x), f^m(f(x))))\\ &\leq \frac{K^n + K^m}{1-K}\rho(x, f(x)).\end{aligned}$$

and since $K < 1$, $K^n \to 0$, so $\rho(f^n(x), f^m(x)) \to 0$ as n and m tend to infinity. $\square$

If X is complete, then this Cauchy sequence converges to a point p of X, and this p is clearly a fixed point of f. Then letting m tend to infinity in the latter inequality:

Banach Contraction Principle. *If X is a complete metric space and $f : X \to X$ is a contraction mapping, then f has a unique fixed point p, and for any x in X the sequence $f^n(x)$ converges to p. In fact,*

$$\rho(f^n(x), p) \leq \frac{K^n}{1-K}\rho(x, f(x)).$$

The importance of this latter inequality is as follows. Suppose we are willing to accept an "error" of ϵ, i.e., instead of the actual fixed point p of f we will be satisfied with a point p' of X satisfying $\rho(p, p') < \epsilon$, and suppose also that we start our iteration at some point x in X. Then from the inequality it is easy to specify an integer N so that $p' = f^N(x)$ will be a satisfactory answer. Since we want $\rho(f^N(x), p) \leq \epsilon$, we just have to pick N so large that $\frac{K^N}{1-K}\rho(x, f(x)) < \epsilon$. Now the quantity $d = \rho(x, f(x))$ is something that we can compute after the first iteration and we can then compute how large N has to be by taking the log of the above inequality and solving for N (remembering that $\log(K)$ is negative). The result is:

Stopping Rule. *If $d = \rho(x, f(x))$ and*

$$N > \frac{\log(\epsilon) + \log(1-K) - \log(d)}{\log(K)}$$

then $\rho(f^N(x), p) < \epsilon$.

From a practical programming point of view, this inequality allows us to express our iterative algorithm with a "for loop" rather than a "while loop", but it has another interesting interpretation. Suppose we take $\epsilon = 10^{-m}$ in our stopping rule inequality. What we see is that the growth of N with m is a constant plus $m/|\log(K)|$, or in other words, to get one more decimal digit of precision we have to do (roughly) $1/|\log(K)|$ more iteration steps. Stated a little differently, if we need N iterative steps to get m decimal digits of precision, then we need another N to double the precision to $2m$ digits.

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