C\(^1\) ACTIONS OF COMPACT LIE GROUPS ON COMPACT MANIFOLDS ARE C\(^\infty\)-EQUIVALENT TO C\(^\infty\) ACTIONS.

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Introduction. Let \(C_1\) and \(C_2\) be two categories and let \(F : C_1 \longrightarrow C_2\) be a weakening of structure ("forgetful") functor, i.e. for each pair of objects \(X_1\) and \(X_2\) of \(C_1\) the map of \(\text{Morph}(X_1, X_2)\) to \(\text{Morph}(F(X_1), F(X_2))\) is injective. Let \(\hat{C}_1\) and \(\hat{C}_2\) denote the equivalence classes of objects of \(C_1\) and \(C_2\) defined by the equivalence relations of isomorphisms in the respective categories. Then \(F\) induces a map \(\hat{F} : \hat{C}_1 \rightarrow \hat{C}_2\) and the question of whether \(\hat{F}\) is injective, surjective, or bijective is often of interest. For example for \(0 \leq k \leq \infty\) let \(\text{Man}_k\) denote the category of \(C^k\) paracompact manifolds and \(C^k\) maps, and for \(k > l\) let \(\hat{F}_{lk} : \text{Man}_k \longrightarrow \text{Man}_l\) be the obvious forgetful functor. Then for \(l = 0\) \(\hat{F}_{0k} : \text{Man}_k \longrightarrow \text{Man}_0\) is neither surjective nor injective, i.e. there exist topological manifolds which admit no \(C^k\) structure \([3]\) and there exist topological manifolds with non-isomorphic \(C^k\) structures \([5]\). However if \(l > 0\) then \(\hat{F}_{lk}\) is always bijective, i.e. every \(C^1\) manifold admits a compatible \(C^k\) structure \([10]\) and if two \(C^k\) manifolds are \(C^1\) diffeomorphic they are \(C^k\) diffeomorphic (the latter is trivial from standard approximation theorems).

Below we shall prove the analogous statements for the categories \(\text{Man}_k(G)\) of compact \(C^k\) \(G\)-manifolds where \(G\) is a compact Lie group (see §\(1\) for precise definitions). Our main results are summarized in the following theorems.

**Theorem A.** Let \(G\) be a compact Lie group and let \(M_1\) and \(M_2\) be closed \(C^k\) \(G\)-manifolds \((2 \leq k \leq \infty)\). If \(M_1\) and \(M_2\) are \(C^1\) equivariantly diffeomorphic \((1 \leq l < k)\) then they are \(C^k\) equivariantly diffeomorphic. In fact any \(C^1\) equivariant map \(f : M_1 \rightarrow M_2\) can be approximated arbitrarily well in the \(C^1\) topology by a \(C^k\) equivariant map.

**Theorem B.** Let \(G\) be a compact Lie group and \(M_0\) a closed \(C^k\) \(G\)-mani-
fold \( 1 \leq k \leq \infty \). Then there is a closed \( C^\infty \) \( G \)-manifold \( M_1 \) which is \( C^k \) equivariantly diffeomorphic to \( M_0 \).

The following is a more precise form of Theorem B.

**Theorem C.** Let \( M \) be a closed \( C^\infty \) manifold, \( G \) a compact Lie group, and for \( 1 \leq k \leq \infty \) let \( \mathcal{A}^k(G, M) \) denote the space of \( C^k \) actions \( \alpha : G \times M \to M \) of \( G \) on \( M \) with the \( C^k \) topology and let \( \text{Diff}^k(M) \) denote the group of \( C^k \) diffeomorphisms of \( M \) with the \( C^k \) topology. Given \( \alpha_0 \in \mathcal{A}^k(G, M) \) we can find \( \alpha \in \mathcal{A}^\infty(G, M) \) arbitrarily near \( \alpha_0 \) in \( \mathcal{A}^k(G, M) \) and \( f \) arbitrarily near the identity in \( \text{Diff}^k(M) \) such that \( \alpha f = \alpha_0 \); where \( \alpha f \in \mathcal{A}^k(G, M) \) is defined by \( \alpha f(g, x) = f^{-1} \alpha(g, fx) \).

As an immediate consequence of Theorems A and B we have the following

**Theorem D.** For \( 0 \leq k \leq \infty \) let \( \text{Man}_k(G) \) denote the category of closed \( C^k \) \( G \)-manifolds and \( C^k \) equivariant maps, and for \( l < k \) let \( F_{lk} : \text{Man}_k(G) \to \text{Man}_l(G) \) denote the obvious weakening of structure functor. Then if \( l \geq 1 \) \( \widehat{F}_{lk} : \text{Man}_k(G) \to \text{Man}_l(G) \) is bijective.

It is worth remarking that for the case \( l = 0 \) the situation is again quite the opposite and \( \widehat{F}_{0k} \) is in general neither injective nor surjective. For example there exist \( C^\infty \) actions of \( Z_2 \) on a sphere which are \( C^0 \) equivalent to the linear reflexion in a line but are not \( C^1 \) equivalent to a linear action [4]. Also whereas there are always at most countably many equivalence classes of smooth actions of a compact Lie group \( G \) on a compact smooth manifold \( M \) [8], so that in fact \( \text{Man}_k(G) \) is always countable for \( k \geq 1 \), \( Z_2 \) or \( S^1 \) can act in uncountably many topologically inequivalent ways on \( S^6 \), and more generally any non-trivial compact Lie group \( G \) can act uncountably many topologically inequivalent ways on some \( S^n \) [9], so that \( \text{Man}_0(G) \) is always uncountable and hence \( \widehat{F}_{0k} \) is never surjective for \( G \neq e \).

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1. **Notation and preliminaries.** \( G \) will always denote a compact Lie group. A (left) action of \( G \) on a space \( X \) is a continuous map \( \alpha : G \times X \to X \) such that the map \( \tilde{\alpha} : G \to X^X \) defined by \( \tilde{\alpha}(g)(x) = \alpha(g, x) \) is a homomorphism of \( G \) onto the group of homeomorphisms of \( X \). We call \( \tilde{\alpha}(g) \) the operation of \( g \) on \( X \) (defined by the action \( \alpha \)). A \( G \)-space is a completely regular space \( X \) together with a fixed action \( \alpha \) of \( G \) on \( X \) and generally we write \( gx \) for \( \alpha(g, x) \). If \( X \) and \( Y \) are \( G \)-spaces a map \( f : X \to Y \) is called equivariant if \( f(gx) = gf(x) \). A Fréchet \( G \)-module is a \( G \)-space \( V \) which is
a Fréchet space (complete, metrizable, locally convex, topological vector space) such that each operation of $G$ on $V$ is linear. If in addition $V$ is finite dimensional we call it a linear $G$-space. In this case by choosing an arbitrary orthogonal structure for $V$ and “averaging it over the group” (see Theorem 1.1) we can find an orthogonal structure for $V$ with respect to which each operation of $G$ is orthogonal, and with this extra structure we call $V$ a Euclidean $G$-space.

By a $C^k$ $G$-manifold $0 \leq k \leq \infty$ we shall mean a $C^k$ manifold $M$, possibly with boundary, which is a $G$-space in such a way that the action $\alpha: G \times M \to M$ is a $C^k$ map, so in particular each operation of $G$ on $M$ is a $C^k$ diffeomorphism. If $M$ is compact and without boundary we call it a closed $C^k$ $G$-manifold. In particular we regard $G$ itself as a closed $C^\infty$ $G$-manifold with the natural left translation action.

If $M$ is a $C^k$ manifold we denote by $\mathcal{A}^k(G, M)$ the space of $C^k$ actions of $G$ on $M$ and by $\text{Diff}^k(M)$ the group of $C^k$ diffeomorphisms of $M$, both with the $C^k$ topology. There is a natural (right) action of $\text{Diff}^k(M)$ on $\mathcal{A}^k(G, M)$, $(f, \alpha) \mapsto \alpha^f$, where $\alpha^f(g, x) = f^{-1} \alpha(g, fx)$. This action is easily seen to be continuous. Note that $f$ is equivariant from $M$ with the action $\alpha^f$ to $M$ with the action $\alpha$. If $M_1$ and $M_2$ are $C^k$ $G$-manifolds we denote by $C^k(M_1, M_2)$ the space of $C^k$ maps of $M_1$ to $M_2$ with the $C^k$ topology and by $C_{G^k}(M_1, M_2)$ the subspace of equivariant maps. We note that $C^k(M_1, M_2)$ is in a natural way a $G$-space; namely if $f \in C^k(M_1, M_2)$ then $gf \in C^k(M_1, M_2)$ is defined by $(gf)(x) = g(f(g^{-1}x))$, and $C_{G^k}(M_1, M_2)$ is just the set of $f \in C^k(M_1, M_2)$ left fixed by each operation of $G$. If $M$ is a $C^k$ $G$-manifold and $W$ is a linear $G$-space then $C^k(M, W)$ is a Fréchet $G$-module. In particular if $W$ is any finite dimensional vector space we can regard it as a linear $G$-space by letting $G$ act trivially and $C^k(M, W)$ becomes a Fréchet $G$-module, the action of $g \in G$ on $f \in C^k(M, W)$ being given by $(gf)(x) = f(g^{-1}x)$. In particular $C^k(M) = C^k(M, \mathbb{R})$ is a Fréchet $G$-module and more particularly still $C^k(G)$ is a Fréchet $G$-module.

The following is a classical and trivial remark ("averaging over the group ").

1.1. Theorem. If $V$ is a Fréchet $G$-module then $A: V \to V$ defined by $A(v) = \int gvd\mu(g)$, where $\mu$ is normalized Haar measure on $G$, is a continuous linear projection of $V$ onto the subspace $V^G$ of $V$ consisting of elements left fixed by each operation of $G$.

As a corollary we have:
1.2. **Theorem.** If \( M \) is a \( C^k \) \( G \)-manifold and \( W \) is a linear \( G \)-space then \( A : C^k(M, W) \to C^k(M, W) \) defined by \( (Af)(x) = \int g(f(g^{-1}x))d\mu(g) \) is a continuous linear projection onto \( C_{G}^k(M, W) \).

1.3. **Corollary.** \( C_{G}^k(M, W) \) is dense in \( C_{G}^1(M, W) \) for \( 0 \leq l \leq k \).

**Proof.** Given \( f \in C_{G}^1(M, W) \) choose a sequence \( \{f_n\} \) in \( C^k(M, W) \) converging to \( f \) in \( C^1(M, W) \). Then \( \{Af_n\} \) is a sequence in \( C_{G}^k(M, W) \) converging to \( Af = f \) in \( C_{G}^1(M, W) \).

Q.E.D.

For each Fréchet \( G \)-module \( V \) we define a continuous bilinear map \( (f, v) \mapsto f^*v \) of \( C^0(G) \times V \) into \( V \) by \( (f^*v)(x) = \int f(g)gvd\mu(g) \). It is trivial that for fixed \( v \in V \) \( f \mapsto f^*v \) is a continuous, linear and equivariant (i.e. \( (gf)^*v = g(f^*v) \)) map of \( C^0(G) \) into \( V \) and \( v \) is in the closure of its image (in fact if \( \{f_n\} \) is a sequence in \( C^0(G) \) of positive functions with integral one whose supports shrink to the identity, then \( f_n^*v \to v \)). An element of a Fréchet \( G \)-module \( V \) is called almost invariant if its orbit spans a finite dimensional subspace of \( V \). From the equivariance of \( f \mapsto f^*v \) it follows that if \( f \in C^0(G) \) is almost invariant in \( C^0(G) \) then \( f^*v \) is almost invariant in \( V \). According to the Peter-Weyl Theorem [1] the almost invariant elements of \( C^0(G) \) are dense, so since \( f \mapsto f^*v \) is continuous and has \( v \) in its closure we can find almost invariant \( f_n \) in \( C^0(G) \) such that \( f_n^*v \to v \). Since each \( f_n^*v \) is almost invariant in \( V \) this proves the following classical fact:

1.4. **Theorem.** In any Fréchet \( G \)-module \( V \) the almost invariant elements are dense.

2. **The \( C^k \) equivariant embedding theorem.** The following theorem is proved in [6], and [7] for the cases \( k = 1 \) and \( k = \infty \) respectively. While either proof extends easily enough to the case of general \( k \) we give here the appropriate generalization of Mostow's proof, which is the easier.

2.1. **Theorem.** If \( M \) is a compact \( C^k \) \( G \)-manifold with \( 1 \leq k \leq \infty \) then there exists an equivariant \( C^k \) embedding of \( M \) in some Euclidean \( G \)-space.

**Proof.** Let \( W \) be a Euclidean space in which \( M \) admits a \( C^k \) embedding (e.g. \( R^{2n+1}, n = \dim M \)), regarding as a Euclidean \( G \)-space by letting \( G \) act trivially. Since the space of embeddings of \( M \) in \( W \) is open in the Fréchet \( G \)-module \( C^k(M, W) \) we can by Theorem 1.4 find a \( C^k \) embedding \( f : M \to W \) with \( f \) almost invariant in \( C^k(M, W) \). Let \( U \) be the linear span of the orbit of \( f \) in \( C^k(M, W) \), a finite dimensional invariant linear subspace of \( C^k(M, W) \) and hence a linear \( G \)-space. Choose a \( G \)-invariant positive definite product
for $U$, making $U$ a Euclidean $G$-space. By a change of scale we can assume $f = f_1$ is a unit vector and we extend $f_1$ to an orthonormal basis for $U$; $f_2, \ldots, f_m$. For $g \in G$ $gf_i = \sum a_{ij}(g)f_j$ where, since $G$ acts orthogonally on $U$ and the $f_i$ are orthonormal, $a_{ij}(g^{-1}) = a_{ji}(g)$. Now $U \otimes W$ is a Euclidean $G$-space (the action of $G$ being defined by $g(u \otimes w) = (gu) \otimes w)$ and we will define a $C^k$ equivariant embedding $F: M \to U \otimes W$.

Every element of $U \otimes W$ can be written uniquely as

$$f_1 \otimes w_1 + \cdots + f_m \otimes w_m$$

for $w_1, \ldots, w_m$ in $W$, i.e. the choice of basis $f_1, \ldots, f_m$ for $U$ identifies $U \otimes W$ with the direct sum of $m$ copies of $W$. Define $F: M \to U \otimes W$ by $F(x) = f_1 \otimes f_1(x) + \cdots + f_m \otimes f_m(x)$. Since the first component of $F$ is the $C^k$ embedding $f: M \to W$, a fortiori $F$ is a $C^k$ embedding of $M$ in $U \otimes W$ so it remains only to show that $F$ is equivariant. But

$$F(gx) = \sum f_i \otimes f_i(gx) = \sum f_i \otimes (g^{-1}f_i)(x)$$

$$= \sum f_i \otimes \sum a_{ij}(g^{-1})f_j(x)$$

$$= \sum (\sum a_{ji}(g)f_i) \otimes f_j(x)$$

$$= \sum (gf_j) \otimes f_j(x) = g \sum f_j \otimes f_j(x) = gF(x).$$

Q. E. D.

3. The tubular neighborhood theorem. Let $\pi: \xi \to M$ be a $C^k$ vector bundle. If both $\xi$ and $\eta$ are $C^k$ $G$-manifolds, $\pi$ is equivariant, and each operation of $G$ on $\xi$ is a bundle map (i.e. $g$ maps $\xi_x$ linearly onto $\xi_{gx}$) then we call $\xi$ a $C^k$-vector bundle. Given a linear $G$-space $V$ the $C^\infty$ manifold $G_m(V)$ (the Grassmannian of $m$ dimensional linear subspaces of $V$) is in an obvious and natural way a $C^\infty$ $G$-space. There is moreover a natural vector bundle $\xi_m^V$ over $G_m(V)$, $\xi_m^V = \{(v,W) \in V \times G_m(V) \mid v \in W\}$, the projection being of course $(v,W) \mapsto W$. If we define $g(v,W) = (gv,gW)$ then $\xi_m^V$ becomes a $C^\infty$ $G$-vector bundle.

If $\pi: \xi \to M$ is a $C^k$ $G$-vector bundle, $N$ a $C^k$ $G$-manifold, and $f: N \to M$ is a $C^k$ equivariant map then the induced vector bundle $f^*\xi$ over $N$ is clearly a $C^k$ $G$-vector bundle (recall that $f^*\xi = \{(x,v) \in N \times \xi \mid fx = \pi v\}$ the action of $G$ on $f^*\xi$ is just the restriction of the “product” action on $N \times \xi$).

If $M$ is a $C^k$ $G$-manifold ($k \geq 1$) and $f: M \to V$ is a $C^k$ equivariant immersion of $M$ in a linear $G$-space $V$ then a $C^1$ $G$-normal bundle to $f$ ($l \leq k$)
is a $C^1$ $G$-vector bundle $\nu$ over $M$ of the form $\nu = h^*e_m^V$ ($m = \dim V - \dim M$) where $h: M \to G_m(V)$ is a $C^1$ equivariant map such that for each $x \in M$ $h(x) = \nu_x$ is a linear complement to $\text{im} (Df_x)$. We note that such an $f$ always has a $C^{k-1}$ $G$-normal bundle: namely give $V$ a $G$-invariant positive definite inner product (i.e. the structure of an orthogonal $G$-space) and define $h: M \to G_m(V)$ by $h(x) = (\text{im} (Df_x))^\perp$. We shall see shortly that indeed $f$ has a $C^k$ $G$-normal bundle.

Assume now that $1 \leq l \leq k$ and that $\pi: \nu \to M$ is a $C^l$ $G$-normal bundle to the $C^k$ $G$-equivariant embedding $f: M \to V$. Identify $M$ with its image under $f$ and also with the zero section of $\nu$ and define $E: \nu \to V$ by $E(w) = \pi(w) + w$. Then $E$ is clearly an equivariant $C^l$ map and it is classical that if $\partial M = \partial$ then $E$ maps a neighborhood $N$ of $M$ in $\nu$ diffeomorphically onto a “tubular neighborhood” $U$ of $M$ in $V$. If we define $\tilde{\pi}: U \to M$ by $\tilde{\pi} = \pi \circ (E|N)^{-1}$ then $\tilde{\pi}$ is a $C^l$ equivariant retraction of $U$ onto $M$ and we call $\tilde{\pi}: U \to M$ the $C^l$ $G$-tubular neighborhood of $M$ defined by $\nu$.

3.1. Proposition. Let $M$ be a $C^k$ $G$-manifold without boundary ($k \geq 1$) which admits a $C^k$ equivariant embedding in a linear $G$-space $V$ with a $C^k$ $G$-normal bundle $\nu$. If $M'$ is a compact $C^k$ $G$-manifold then there is a neighborhood $\tilde{G}_0$ of $C_0^0(M', M)$ in $C^0(M', M)$ and a continuous retraction $\tilde{A}: \tilde{G}_0 \to C_0^0(M', M)$ such that for $0 \leq r \leq k$ $\tilde{A}_r$ restricts to a continuous retraction $\tilde{A}_r: \tilde{G}_r \to C_0^r(M', M)$ where $\tilde{G}_r = \tilde{G}_0 \cap C^r(M', M)$.

Proof. Let $\tilde{G}_0 = \{f \in C^0(M', V) | \text{im} f \subseteq U\}$ where $\tilde{\pi}: U \to M$ is a $C^k$ $G$-tubular neighborhood of $M$ defined by $\nu$ as above. Note that $C^0(M', M) \subseteq C^0(M', V)$ so we can define $\tilde{G}_0 = \{f \in C^0(M', M) | Af \in \tilde{G}_0\}$ where

$$A: C^0(M', V) \to C_0^0(M', V)$$

is the continuous linear projection of Theorem 1.2. Since $\tilde{G}_0$ is open in $C^0(M', V)$, $\tilde{G}_0$ is open in $C^0(M', M)$. Moreover if $f \in C_0^0(M', M)$ then $Af = f$ and so $\text{im} (Af) = \text{im} f \subseteq M \subseteq U$ so $f \in \tilde{G}_0$, i.e. $\tilde{G}_0$ is a neighborhood of $C_0^0(M', M)$. We define $\tilde{A}_0: \tilde{G}_0 \to C_0^0(M', M)$ by $\tilde{A}_0(f) = \tilde{\pi}_0(Af)$. Since $A$ restricts to a continuous linear projection $A_r: C^r(M, V) \to C_0^r(M', V)$ for $0 \leq r \leq k$ by Theorem 1.2 and since $\tilde{\pi}: U \to M$ is an equivariant $C^k$ retraction the proposition follows.

Q. E. D.

3.2. Corollary. For $0 \leq r \leq k$ $C_0^k(M', M)$ is dense in $C_0^r(M', M)$.

Proof. Given $f \in C_0^r(M', M)$ choose a sequence $\{f_n\}$ in $C_0^k(M', M)$ converging to $f$ in $C_0^r(M', M)$. Since $\tilde{G}_r$ is a neighborhood of $f$ in $C^r(M', M)$ we can suppose all $f_n$ are in $\tilde{G}_r$. Then $\tilde{A}_r f_n \in C_0^k(M', M)$ and
3.3. Proposition. If \( M \) is a \( C^\infty \) G-manifold without boundary then 
\[ M = \bigcup_{n=1}^{\infty} M_n \]
where \( M_n \) is an open invariant submanifold of \( M \), \( M_n \subset M_{n+1} \),
and each \( \bar{M}_n \) is a compact \( C^\infty \) G-manifold with boundary.

Proof. Let \( f: M \to R \) be a \( C^\infty \) proper map of \( M \) onto the positive reals
and let \( \bar{f}(x) = \int f'(dx) \), so \( \bar{f} : M \to R \) is an invariant \( C^\infty \) map. Given
\( c > 0 \) \( Gf^{-1}([0,c]) = X \) is compact and if \( x \not\in X \) then clearly \( \bar{f}(x) > c \), so
\( \bar{f}^{-1}([0,c]) \subseteq X \) and so \( \bar{f} \) is a proper map. Let \( M_n = \bar{f}^{-1}([0,c_n]) \)
where \( \{c_n\} \)
is a sequence of regular values of \( \bar{f} \) with \( c_n \to \infty \) monotonically. Q. E. D.

3.4. Theorem. If \( M \) is a \( C^\infty \) G-manifold without boundary then for
any compact \( C^k \) G-manifold \( M' \) \( C_G^k(M',M) \)
is dense in \( C_G^r(M',M) \), \( 0 \leq r \leq k \).

Proof. Represent \( M \) as \( \bigcup_{n=1}^{\infty} M_n \) as in 3.3. If \( f \in C^r(M',M) \) then
\( f(M') \subset M_n \) for some \( n \) since \( M' \) is compact, i.e. \( C_G^r(M',M) = \bigcup_{n=1}^{\infty} C_G^r(M',M_n) \),
so it will suffice to show that \( C_G^k(M',M_n) \)
is dense in \( C_G^r(M',M_n) \) for all \( n \), or equivalently we can assume that \( M \) is the interior of a compact \( C^\infty \) G-manifold \( \bar{M} \), possibly with boundary. But then by Theorem 2.1 \( M \)
adopts a \( C^\infty \) equivariant embedding \( \bar{f} : M \to V \) in a Euclidean G-space \( V \)
(Indeed \( \bar{M} \) does) and \( x \mapsto \text{im}(Df_x)^\perp \) is a \( C^\infty \) equivariant map \( g : M \to G_m(V) \)
where \( m = \text{dim } V - \text{dim } M \), so \( g^*(\xi_m V) \) is a \( C^\infty \) G-normal bundle to \( f \) and the
Theorem follows from 3.2. Q. E. D.

3.5. Tubular Neighborhood Theorem. If \( M \) is a compact \( C^k \) G-manifold, \( k \geq 1 \), and \( f : M \to V \)
is a \( C^k \) equivariant embedding on a linear G-space \( V \) then \( f \) admits a \( C^k \) G-normal bundle and hence if \( \partial M = \emptyset \) then
\( f(M) \) admits a \( C^k \) G-tubular neighborhood in \( V \) \( \bar{f} : U \to f(M) \).

Proof. We can assume \( V \) is a Euclidean G-space and we define \( f : M \to G_m(V) \), \( m = \text{dim } V - \text{dim } M \), by \( g(x) = \text{im}(Df_x)^\perp \), so
\( g \in C_G^{k-1}(M,G_m(V)) \).

Since \( G_m(V) \) is a \( C^\infty \) G-manifold, by 3.4 we can approximate \( g \) arbitrarily well
in \( C_G^{k-1}(M,G_m(V)) \) by \( h \in C_G^k(M,V) \). But clearly if \( h \) is sufficiently close
to \( g \) in the \( C^\infty \)-topology then \( h(x) \) like \( g(x) \) will for each \( x \in M \) be a linear
complement to \( \text{im}(Df_x) \) in \( V \). Then \( h^*(\xi_m V) \) is a \( C^k \) G-normal bundle to \( f \).
Q. E. D.
4. Fiberwise transversality.

4.1. Definition. Let \( \pi_1: E_1 \rightarrow M_1 \) and \( \pi_2: E_2 \rightarrow M_2 \) be two \( C^k \) fiber bundles, \( k \geq 1 \), and let \( G_2 \) be a \( C^k \) sub-bundle of \( E_2 \). A \( C^k \) map \( f: E_1 \rightarrow E_2 \) will be called fiber-wise transversal to \( G_2 \) if for each \( x \in M_1 \) \( f \mid (E_1)_x \) is transversal to \( G_2 \).

4.2. Proposition. Given \( \pi_1: E_1 \rightarrow M_1 \), \( \pi_2: E_2 \rightarrow M_2 \) and \( G_2 \) as in 4.1 with \( E_1 \) compact, the set of \( C^k \) maps \( f: E_1 \rightarrow E_2 \) which are fiber-wise transversal to \( G_2 \) is open in the \( C^1 \) topology.

Proof. Obvious.

4.3. Lemma. Let \( \pi: E \rightarrow M \) be a \( C^k \) fiber bundle, \( k \geq 1 \) and let \( G \) be a compact \( C^k \) submanifold of \( E \) with \( \partial G = \emptyset \). A necessary and sufficient condition for \( G \) to be a \( C^k \) sub-bundle of \( E \) (i.e. for \( \tilde{\pi} = \pi \mid G: G \rightarrow M \) to be a \( C^k \) fiber bundle) is that for each \( x \in M \) \( G_x = G \cap E_x \) be a submanifold of \( G \) of codimension equal to the dimension of \( M \).

Proof. Necessity is trivial. To prove sufficiency it is enough by a well-known theorem of Ehresmann [2] to check that when the condition is satisfied \( \tilde{\pi}: G \rightarrow M \) is a submersion, i.e. that for each \( y \in G \) \( (D\tilde{\pi})_y: TG_y \rightarrow TM_y \) has rank equal to \( \dim M \). Now \( \text{rank}(D\tilde{\pi})_y = \dim G = \dim (\ker(D\tilde{\pi})_y) \). Since \( \tilde{\pi} \) is a fibering \( \ker(D\tilde{\pi})_y = T(E_x)_y \) and since \( (D\tilde{\pi})_y = (D\pi)_y \mid (TG)_y \),

\[
\ker(D\pi)_y = \ker(D\tilde{\pi})_y \cap (TG)_y = T(E_x)_y \cap (TG)_y = T(E_x \cap G)_y = T(G_x)_y,
\]

so \( \text{rank}(D\tilde{\pi})_y = \dim G - \dim G_x = \text{codim} G_x \) in \( G = \dim M \). Q. E. D.

4.4. Theorem. Let \( \pi_1: E_1 \rightarrow M_1 \), \( \pi_2: E_2 \rightarrow M_2 \) be two \( C^k \) fiber bundles, \( k \geq 1 \), and let \( G_2 \) be a \( C^k \) sub-bundle of \( E_2 \). Assume \( E_1 \) is compact, \( \partial M_1 = \emptyset \), and \( \partial G_2 = \emptyset \). Let \( f: E_1 \rightarrow E_2 \) be a \( C^k \) map which is fiber-wise transversal to \( G_2 \) with \( f(\partial E_1) \cap G_2 = \emptyset \). Then \( G_1 = f^{-1}(G_2) \) is a \( C^k \) sub-bundle of \( E_1 \), \( \partial G_1 = \emptyset \), and the fiber codimension of \( G_1 \) in \( E_1 \) equals the fiber codimension of \( G_2 \) in \( E_2 \).

Proof. Since \( f \) is fiber-wise transversal to \( G_2 \) it is a fortiori transversal to \( G_2 \) and hence \( G_1 = f^{-1}(G_2) \) is a compact \( C^k \) submanifold of \( E_1 \) of codimension equal to the codimension of \( G_2 \) in \( E_2 \), and \( \partial G_1 = (f \mid \partial E_1)^{-1}G_2 = \emptyset \). Putting \( (G_1)_x = (f \mid (E_1)_x)^{-1}G_2 = G_1 \cap (E_1)_x \), \( (G_1)_x \) is a submanifold of \( (E_1)_x \) of codimension equal to the codimension of \( G_2 \) in \( E_2 \) which, since \( G_2 \) is a sub-bundle of \( E_2 \) is the same as the fiber codimension of \( G_2 \) in \( E_2 \). Now since \( \dim E - \dim G = \dim (E_1)_x - \dim (G_1)_x = \dim E_1 - \dim \) it follows
that \( \dim G = \dim (G_1)_x = \dim E = \dim (E_1)_x = \dim M \), and Lemma 4.3 completes the proof.

Q. E. D.

4.5. **Theorem.** Let \( \pi_1 : E_1 \to M_1 \) and \( \pi_2 : E_2 \to M_2 \) and \( G_2 \) be as in Theorem 4.4 and let \( f_t : E_1 \to E_2 \) be a homotopy of \( C^k \) maps with each \( f_t \) fiberwise transversal to \( G_2 \) and all \( f_t(\partial E_1) \cap G_2 \) empty. Then \( f_{t}^{-1}(G_2) \) and \( f_{1}^{-1}(G_2) \) are \( C^k \) equivalent sub-bundles of \( E_1 \) and in particular for each \( x \in M_1 \) their fibers at \( x \) are \( C^k \) diffeomorphic.

**Proof.** Let \( S^1 \) denote \([0,1]\) with 0 and 1 identified. It will suffice to construct a \( C^k \) fiber bundle over \( S^1 \times M_1 \) which restricted to \( \{0\} \times M_1 \) is \( f_0^{-1}(G_2) \) and restricted to \( \{\frac{1}{2}\} \times M_1 \) is \( f^{-1}(G_2) \) (because bundles over \([0,\frac{1}{2}] \times M_1 \) are always of the form \([0,\frac{1}{2}] \times B \) for some bundle over \( M_1 \)). By 4.2 we can approximate \( f_t \) so that \((t,x) \mapsto f_t(x) \) is \( C^k \) without changing \( f_0 \) or \( f_1 \) and maintaining the other given conditions. By a standard argument we can also suppose that \( f_t = f_0 \) for \( 0 \leq t < \epsilon \) and \( f_t = f_1 \) for \( 1 - \epsilon \leq t \leq 1 \). Define \( h_t = f_{2t} \) for \( 0 \leq t \leq \frac{1}{2} \) and \( h_t = f_{2t-2\epsilon} \) for \( \frac{1}{2} \leq t \leq 1 \). Then \((t,x) \mapsto h_t(x) \) is a \( C^k \) map of \( S^1 \times E_1 \to E_2 \) and if we regard \( S^1 \times E_1 \) and \( S^1 \times E_2 \) as \( C^k \) fiber bundles over \( S^1 \times M_1 \) and \( S^1 \times M_2 \) respectively then \( H : S^1 \times E_1 \to S^1 \times E_2 \) defined by \((t,x) \mapsto (t,h_t(x)) \) is clearly fiber-wise transversal to \( S^1 \times G_2 \) and \( H(\partial(S^1 \times E_1)) \cap S^1 \times G_2 = \emptyset \) so by 4.4 \( H^{-1}(S^1 \times G_2) \) is a \( C^k \) sub-bundle of \( S^1 \times E_1 \). Clearly \( H^{-1}(S^1 \times G_2) \cap \{t\} \times M_1 \) is just \( h_t^{-1}(G_2) \) so taking \( t = 0, \frac{1}{2} \) respectively we get \( f_0^{-1}(G_2) \) and \( f_1^{-1}(G_2) \) respectively as desired.

Q. E. D.

4.6. **Corollary.** Let \( \pi_1 : E_1 \to M_1 \), \( \pi_2 : E_2 \to M_2 \), \( G \) and \( f \) be as in Theorem 4.4 and in addition assume that \( E_1 \) is compact. Then there is a neighborhood \( U \) of \( f \) in \( C^k(E_1,E_2) \), which is in fact open in the \( C^1 \) topology, such that for all \( h \in U \), \( h^{-1}(G_2) \) is a \( C^k \) sub-bundle of \( E_1 \) which is \( C^k \) equivalent to \( f^{-1}(G_2) \).

**Proof.** There is a \( C^0 \) neighborhood of \( f \) in \( C^k(E_1,E_2) \) say \( U_0 \) such that for all \( h \in U_0 \) \( h(\partial E_1) \cap G_2 = \emptyset \). There is a \( C^1 \) neighborhood \( U_1 \) of \( f \) by 4.2 such that \( h \) is fiberwise transversal to \( G_2 \) for all \( h \in U_1 \). Now \( C^k(E_1,E_2) \) is locally contractible in the \( C^1 \) topology so we can take for \( U \) any contractible (or even pathwise connected) \( C^2 \)-neighborhood of \( f \) in \( U_0 \cap U_1 \). Q. E. D.

4.7. **Theorem.** Let \( \pi_1 : E_1 \to M_1 \) and \( \pi_2 : E_2 \to M_2 \) be \( C^k \) fiber bundles, \( k \geq 1 \), with compact total spaces and \( \partial M_1 = \partial M_2 = \emptyset \). Let \( f_0 : E_1 \to E_2 \) be a \( C^k \) map which maps each fiber of \( E_1 \) diffeomorphically onto a fiber of \( E_2 \). If \( \sigma \) is any \( C^k \) section of \( E_2 \) with \( \sigma \cap \partial E_2 = \emptyset \) then there is a neighborhood \( U \)
of \( f_0 \) in \( C^k(E_1,E_2) \), which in fact is open in the \( C^1 \) topology, such that for every \( f \in U \), \( f^{-1}(\sigma) \) is a \( C^k \) section of \( E_1 \) disjoint from \( \partial E_2 \). Moreover the map \( f \mapsto f^{-1}(\sigma) \) is continuous from \( U \subseteq C^k(E_1,E_2) \) into the space \( C^k(E_1) \) of \( C^k \) sections of \( E_1 \) with the \( C^k \) topology.

Proof. In 4.6 take \( G_2 = \sigma \). Since \( f_0 \) maps each fiber of \( E_1 \) diffeomorphically onto a fiber of \( E_2 \) it is clear that \( f_0 \) is fiberwise transversal to \( \sigma \), and hence \( \sigma_0 = f_0^{-1}(\sigma) \) is a \( C^k \) sub-bundle of \( E_1 \) of fiber dimension zero, and in fact since \( f_0(E_1)_x \) meets \( \sigma \) in exactly one point, \( \sigma_0 \) is a section of \( E_1 \). It now follows from 4.6 that \( f^{-1}(\sigma) \) is also a \( C^k \) sub-bundle of \( E_1 \) with fiber a point, i.e. a \( C^k \) section of \( E_1 \), if \( f \in C^k(E_1,E_2) \) is sufficiently near \( f_0 \) in the \( C^1 \) topology. It remains only to show the continuity of \( f \mapsto f^{-1}(\sigma) \) as a map of \( U \) into \( C^k(E_1) \). Since \( M_1 \) is compact it will suffice to show that if \( f_n \to f \) then \( f_n^{-1}(\sigma) \to \sigma \in U \) in the \( C^k \) topology in some neighborhood \( W \) of each point of \( x \in M_1 \). We can choose coordinates in \( E_2 \) near \( \sigma(x) \) so that \( E_2 \) is identified with \( \mathbb{R}^k \times \mathbb{R}^m \), \( \pi_2 \) with the projection \( \mathbb{R}^k \times \mathbb{R}^m \to \mathbb{R}^m \) and \( \sigma \) with \( \mathbb{R}^k \times \{0\} \). Then choosing coordinates near \( \sigma_0(x) \) in \( E_1 \) so that \( E_1 \) is locally \( \mathbb{R}^n \times \mathbb{R}^m \), \( \pi_1 \) is the projection \( \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n \) and \( \sigma_0 \) is \( \mathbb{R}^n \times \{0\} \), \( f_0 \) is given locally by a map \( (x,y) \mapsto (g_0(x),h_0(x,y)) \) of \( \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^k \times \mathbb{R}^m \) with \( g_0(0) = 0 \), \( h_0(0,0) = 0 \) and \( y \mapsto h_0(0,y) \) a local \( C^k \) diffeomorphism of one neighborhood of \( 0 \) in \( \mathbb{R}^m \) onto another. Given \( f: \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^k \times \mathbb{R}^m \) near \( f_0 \), say \( f(x,y) = (g(x,y),h(x,y)) \), \( f^{-1}(\sigma) \) is represented as the unique \( C^k \) map \( \sigma_f: \mathbb{R}^n \to \mathbb{R}^m \) near zero which solves the equation \( h(x,\sigma_f(x)) = 0 \). By the implicit function theorem \( \sigma_f \) and its derivatives through order \( k \) depend continuous on \( h \) and its derivatives through order \( k \), which in turn of course depend continuously on \( f \) in the \( C^k \) topology.

Q. E. D.

5. Proofs of Theorems A, B, and C. The second conclusion of Theorem A is just Theorem 3.4. The first conclusion is an immediate consequence of the second since the space \( \text{Diff}^1(M_1,M_2) \) of \( C^1 \) diffeomorphisms of \( M_1 \) with \( M_2 \) is open in \( C^1(M_1,M_2) \), and hence if we approximate a \( C^1 \) equivariant diffeomorphism \( f: M_1 \simeq M_2 \) well enough (in \( C^1(M_1,M_2) \)) by \( g \in C^k_0(M_1,M_2) \) then \( g \) will be a \( C^k \) equivariant diffeomorphism of \( M_1 \) with \( M_2 \). This completes the proof of Theorem A.

Now let \( M_0 \) be a closed \( C^k \) manifold and let \( \lambda: M_0 \to V \) be a \( C^k \) embedding of \( M_0 \) in some Euclidean space. Let \( \pi: V \to M_0 \) be \( C^k \) normal bundle to \( \lambda \) and \( v_\epsilon \) the \( \epsilon \)-disc sub-bundle of \( v \). As usual we identify \( M_0 \) with \( \lambda(M_0) \) and with the zero section of \( v \), and for \( \epsilon \) small enough we identify \( v_\epsilon \) with the \( \epsilon \)-tubular neighborhood of \( M_0 \) it defines, namely its image under the map
v \mapsto \pi(v) + v$. Let $k = \dim V - \dim M$ and let $\xi_k(V)$ denote the $\epsilon$-disc bundle of the vector bundle $\xi_k(V) \to G_k(V)$. Define $f_0 : v_\epsilon \to \xi_k(V)$ by $f_0(v) = (v, v_{\pi(v)})$ (the Thom-Pontryagin map) so that clearly for each $x \in M_0$ $f_0$ maps $v_\epsilon(x)$ diffeomorphically onto the fiber of $\xi_k(V)$ over $v_\epsilon$. If $\sigma$ is the zero section $\xi_k(V)$ then $f_0^{-1}(\sigma) = M_0$. Now by 4.7 if $f_1 : v(\epsilon) \to \xi_k(V(\epsilon)$ is a $C^k$ map sufficiently close to $f_0$ in the $C^1$ topology then $M = f_1^{-1}(\sigma)$ is a $C^k$ section of $v$ and hence $\pi | M_1 : M_1 \to M_0$ is a $C^k$ diffeomorphism. In particular if we approximate $f_0$ by a $C^\infty$ $f_1$, relative to the natural $C^\infty$ structure for $\xi_k(V)$ and that for $v(\epsilon)$ inherited from $V$, then $M_1$ will be a $C^\infty$ submanifold of $V$ mapped $C^k$ diffeomorphically onto $M_0$ by $\pi$.

Now suppose $M_0$ is a $C^k$ $G$-manifold. Then by Theorem 2.1 we can suppose $\lambda$ is an equivariant $C^k$ embedding of $M_0$ in a Euclidean $G$-space $V$ and by 3.5 we can suppose $\pi_0 : v \to M_0$ is a $C^k$ $G$-normal bundle to $\lambda$ in $V$ so that $v(\epsilon)$ is a $C^k$ $G$-tubular neighborhood. Finally since $f_0$ is clearly equivariant we can by Theorem 3.4 assume that our approximating $f_1 : v(\epsilon) \to \xi_k(V(\epsilon)$ is equivariant. Then $M_1$ is clearly an invariant $C^\infty$ submanifold of $V$, hence a $C^\infty$ $G$-manifold, and $\pi | M_1$ is of course equivariant. This completes the proof of Theorem B and we pass now to the proof of Theorem C.

Let $M$ be any closed $C^\infty$ manifold, $\alpha_0 \in \mathcal{A}(G, M)$ and denote by $M_0$ the resulting $C^k$ $G$-manifold. According to Theorem B we can choose a $C^\infty$ $G$-manifold $M_1$ for which there exists a $C^k$ equivariant diffeomorphism $\gamma : M_1 \to M_0$. If $\alpha_1 \in \mathcal{A}(G, M_1)$ is the given action of $G$ on $M_1$ then of course $\alpha_0 = \alpha_1 \gamma^{-1}$, i.e. $\alpha_0(g, x) = \gamma \alpha_1(g, \gamma^{-1}x)$. Given $f \in \text{Diff}^k(M)$ we have

$$\alpha = \alpha_1(\gamma^{-1})^{-1} \in \mathcal{A}(G, M), \quad \alpha(g, x) = f_\gamma \alpha_1(g, \gamma^{-1} f^{-1} x)$$

and $\alpha' = \alpha_1 \gamma^{-1} = \alpha_0$. Clearly $f \mapsto \alpha_1(\gamma^{-1})^{-1}$ is continuous from $\text{Diff}^k(M)$ to $\mathcal{A}(G, M)$ so in particular if $f \mapsto e$ in $\text{Diff}^k(M)$ then $\alpha_1(\gamma^{-1})^{-1} \to \alpha_0$ in $\mathcal{A}(G, M)$. Thus to complete the proof of Theorem C it will suffice to find $f \in \text{Diff}^k(M)$ arbitrarily close to the identity such that $\alpha_1(\gamma^{-1})^{-1} \in \mathcal{A}(G, M)$. But since $\alpha_1 \in \mathcal{A}(G, M)$ it is clear that if $\gamma : M_1 \to M$ is $C^\infty$, and hence $(\gamma^{-1})^{-1} : M \to M_1$ is $C^\infty$, then $\alpha_1(\gamma^{-1})^{-1} \in \mathcal{A}(G, M)$. Now $\text{Diff}^\infty(M_1, M)$ is dense in $\text{Diff}^k(M_1, M_2)$, so there exists $g \in \text{Diff}^\infty(M_1, M)$ arbitrarily near $\gamma$ in $\text{Diff}^k(M_1, M)$. Then $f = g \gamma^{-1}$ is arbitrarily near the identity in $\text{Diff}^k(M_1, M_2)$ and $f_\gamma = g$ is $C^\infty$.

Q. E. D.

5.1. **Theorem.** If $M$ is a closed $C^k$ $G$-manifold, $k \geq 1$, and $M'$ is any compact $C^k$ $G$-manifold, then $C^k(G, M)$ is dense in $C^r(G', M)$ for $0 \leq r \leq k$.

**Proof.** By Theorem B we can give $M$ a $C^\infty$ structure which makes it a $C^\infty$ $G$-manifold, and the theorem then follows from 3.4.
Definition. A $C^k$ manifold $M$ is called $C^k$-asymmetric if there is no non-trivial $C^k$-action of any compact connected Lie group on $M$, or equivalently if there is no effective $C^k$ action of the circle group $S^1$ on $M$.

5.2. Theorem. Let $M$ be a closed $C^\infty$ manifold and let $1 \leq l < k \leq \infty$. If $M$ is $C^k$-asymmetric then $M$ is $C^1$-asymmetric.

Proof. Immediate from Theorem C.

It would be interesting to know whether we can take $l = 0$ in Theorem 5.2. In particular in a paper to appear Atiyah and Hirzebruch have shown that if $M$ is a closed, orientable $4k$ dimensional manifold with $w_2(M) = 0$, then $M$ is $C^\infty$-asymmetric (hence $C^1$-asymmetric) provided the $A$-genus of $M$ (a topological invariant!) is non-zero. It would be quite remarkable if such $M$ could admit non-trivial $C^0$-circle actions.*

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[8] ———, “Equivalence of nearby differentiable actions of a compact group,”

*Added in Proof: Atiyah has in fact constructed just such a remarkable example.