
C1 Actions of Compact Lie Groups on Compact Manifolds are C^1 -Equivalent to C^∞ Actions

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C¹ ACTIONS OF COMPACT LIE GROUPS ON COMPACT MANIFOLDS ARE C¹-EQUIVALENT TO C[∞] ACTIONS.

By RICHARD S. PALAIS.*

Introduction. Let C_1 and C_2 be two categories and let $F: C_1 \rightsquigarrow C_2$ be a weakening of structure (“forgetful”) functor, i.e. for each pair of objects X_1 and X_2 of C_1 the map of $\text{Morph}(X_1, X_2)$ to $\text{Morph}(F(X_1), F(X_2))$ is injective. Let \hat{C}_1 and \hat{C}_2 denote the equivalence classes of objects of C_1 and C_2 defined by the equivalence relations of isomorphisms in the respective categories. Then F induces a map $\hat{F}: \hat{C}_1 \rightarrow \hat{C}_2$ and the question of whether \hat{F} is injective, surjective, or bijective is often of interest. For example for $0 \leq k \leq \infty$ let Man_k denote the category of C^k paracompact manifolds and C^k maps, and for $k > l$ let $\hat{F}_{lk}: \hat{\text{Man}}_k \rightsquigarrow \hat{\text{Man}}_l$ be the obvious forgetful functor. Then for $l = 0$ $F_{0k}: \hat{\text{Man}}_k \rightarrow \hat{\text{Man}}_0$ is neither surjective nor injective, i.e. there exist topological manifolds which admit no C^k structure [3] and there exist topological manifolds with non-isomorphic C^k structures [5]. However if $l > 0$ then \hat{F}_{lk} is always bijective, i.e. every C^l manifold admits a compatible C^k structure [10] and if two C^k manifolds are C^l diffeomorphic they are C^k diffeomorphic (the latter is trivial from standard approximation theorems).

Below we shall prove the analogous statements for the categories $\text{Man}_k(G)$ of compact C^k G -manifolds where G is a compact Lie group (see §1 for precise definitions). Our main results are summarized in the following theorems.

THEOREM A. *Let G be a compact Lie group and let M_1 and M_2 be closed C^k G -manifolds ($2 \leq k \leq \infty$). If M_1 and M_2 are C^l equivariantly diffeomorphic ($1 \leq l < k$) then they are C^k equivariantly diffeomorphic. In fact any C^l equivariant map $f: M_1 \rightarrow M_2$ can be approximated arbitrarily well in the C^l topology by a C^k equivariant map.*

THEOREM B. *Let G be a compact Lie group and M_0 a closed C^k G -mani-*

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fold $1 \leq k \leq \infty$. Then there is a closed C^∞ G -manifold M_1 which is C^k equivariantly diffeomorphic to M_0 .

The following is a more precise form of Theorem B.

THEOREM C. Let M be a closed C^∞ manifold, G a compact Lie group, and for $1 \leq k \leq \infty$ let $\mathcal{A}^k(G, M)$ denote the space of C^k actions $\alpha: G \times M \rightarrow M$ of G on M with the C^k topology and let $\text{Diff}^k(M)$ denote the group of C^k diffeomorphisms of M with the C^k topology. Given $\alpha_0 \in \mathcal{A}^k(G, M)$ we can find $\alpha \in \mathcal{A}^\infty(G, M)$ arbitrarily near α_0 in $\mathcal{A}^k(G, M)$ and f arbitrarily near the identity in $\text{Diff}^k(M)$ such that $\alpha^f = \alpha_0$; where $\alpha^f \in \mathcal{A}^k(G, M)$ is defined by $\alpha^f(g, x) = f^{-1}\alpha(g, fx)$.

As an immediate consequence of Theorems A and B we have the following

THEOREM D. For $0 \leq k \leq \infty$ let $\text{Man}_k(G)$ denote the category of closed C^k G -manifolds and C^k equivariant maps, and for $l < k$ let $F_{lk}: \text{Man}_k(G) \rightsquigarrow \text{Man}_l(G)$ denote the obvious weakening of structure functor. Then if $l \geq 1$ $\hat{F}_{lk}: \widehat{\text{Man}}_k(G) \rightarrow \widehat{\text{Man}}_l(G)$ is bijective.

It is worth remarking that for the case $l=0$ the situation is again quite the opposite and \hat{F}_{0k} is in general neither injective nor surjective. For example there exist C^∞ actions of Z_2 on a sphere which are C^0 equivalent to the linear reflexion in a line but are not C^1 equivalent to a linear action [4]. Also whereas there are always at most countably many equivalence classes of smooth actions of a compact Lie group G on a compact smooth manifold M [8], so that in fact $\widehat{\text{Man}}_k(G)$ is always countable for $k \geq 1$, Z_2 or S^1 can act in uncountably many topologically inequivalent ways on S^5 , and more generally any non-trivial compact Lie group G can act uncountably many topologically inequivalent ways on some S^n [9], so that $\widehat{\text{Man}}_0(G)$ is always uncountable and hence \hat{F}_{0k} is never surjective for $G \neq e$.

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1. Notation and preliminaries. G will always denote a compact Lie group. A (left) action of G on a space X is a continuous map $\alpha: G \times X \rightarrow X$ such that the map $\tilde{\alpha}: G \rightarrow X^X$ defined by $\tilde{\alpha}(g)(x) = \alpha(g, x)$ is a homomorphism of G onto the group of homeomorphisms of X . We call $\tilde{\alpha}(g)$ the operation of g on X (defined by the action α). A G -space is a completely regular space X together with a fixed action α of G on X and generally we write gx for $\alpha(g, x)$. If X and Y are G -spaces a map $f: X \rightarrow Y$ is called equivariant if $f(gx) = gf(x)$. A Fréchet G -module is a G -space V which is

a Fréchet space (= complete, metrizable, locally convex, topological vector space) such that each operation of G on V is linear. If in addition V is finite dimensional we call it a *linear G -space*. In this case by choosing an arbitrary orthogonal structure for V and “averaging it over the group” (see Theorem 1.1) we can find an orthogonal structure for V with respect to which each operation of G is orthogonal, and with this extra structure we call V a *Euclidean G -space*.

By a C^k G -manifold $0 \leq k \leq \infty$ we shall mean a C^k manifold M , possibly with boundary, which is a G -space in such a way that the action $\alpha: G \times M \rightarrow M$ is a C^k map, so in particular each operation of G on M is a C^k diffeomorphism. If M is compact and without boundary we call it a closed C^k G -manifold. In particular we regard G itself as a closed C^∞ G -manifold with the natural left translation action.

If M is a C^k manifold we denote by $\mathcal{A}^k(G, M)$ the space of C^k actions of G on M and by $\text{Diff}^k(M)$ the group of C^k diffeomorphisms of M , both with the C^k topology. There is a natural (right) action of $\text{Diff}^k(M)$ on $\mathcal{A}^k(G, M)$, $(f, \alpha) \mapsto \alpha^f$, where $\alpha^f(g, x) = f^{-1}\alpha(g, fx)$. This action is easily seen to be continuous. Note that f is equivariant from M with the action α^f to M with the action α . If M_1 and M_2 are C^k G -manifolds we denote by $C^k(M_1, M_2)$ the space of C^k maps of M_1 to M_2 with the C^k topology and by $C_G^k(M_1, M_2)$ the subspace of equivariant maps. We note that $C^k(M_1, M_2)$ is in a natural way a G -space; namely if $f \in C^k(M_1, M_2)$ then $gf \in C^k(M_1, M_2)$ is defined by $(gf)(x) = g(f(g^{-1}x))$, and $C_G^k(M_1, M_2)$ is just the set of $f \in C^k(M_1, M_2)$ left fixed by each operation of G . If M is a C^k G -manifold and W is a linear G -space then $C^k(M, W)$ is a Fréchet G -module. In particular if W is any finite dimensional vector space we can regard it as a linear G -space by letting G act trivially and $C^k(M, W)$ becomes a Fréchet G -module, the action of $g \in G$ on $f \in C^k(M, W)$ being given by $(gf)(x) = f(g^{-1}x)$. In particular $C^k(M) = C^k(M, \mathbf{R})$ is a Fréchet G -module and more particularly still $C^k(G)$ is a Fréchet G -module.

The following is a classical and trivial remark (“averaging over the group”).

1.1. THEOREM. *If V is a Fréchet G -module then $A: V \rightarrow V$ defined by $A(v) = \int gv d\mu(g)$, where μ is normalized Haar measure on G , is a continuous linear projection of V onto the subspace V^G of V consisting of elements left fixed by each operation of G .*

As a corollary we have:

1.2. THEOREM. *If M is a C^k G -manifold and W is a linear G -space then $A: C^k(M, W) \rightarrow C^k(M, W)$ defined by $(Af)(x) = \int g(f(g^{-1}x))d\mu(g)$ is a continuous linear projection onto $C_G^k(M, W)$.*

1.3. COROLLARY. *$C_G^k(M, W)$ is dense in $C_G^l(M, W)$ for $0 \leq l \leq k$.*

Proof. Given $f \in C_G^l(M, W)$ choose a sequence $\{f_n\}$ in $C^k(M, W)$ converging to f in $C^l(M, W)$. Then $\{Af_n\}$ is a sequence in $C_G^k(M, W)$ converging to $Af = f$ in $C_G^l(M, W)$. Q. E. D.

For each Fréchet G -module V we define a continuous bilinear map $(f, v) \mapsto f^*v$ of $C^0(G) \times V$ into V by $(f^*v)(x) = \int f(g)gv d\mu(g)$. It is trivial that for fixed $v \in V$ $f \mapsto f^*v$ is a continuous, linear and equivariant (i.e. $(gf)^*v = g(f^*v)$) map of $C^0(G)$ into V and v is in the closure of its image (in fact if $\{f_n\}$ is a sequence in $C^0(G)$ of positive functions with integral one whose supports shrink to the identity, then $f_n^*v \rightarrow v$). An element of a Fréchet G -module V is called *almost invariant* if its orbit spans a finite dimensional subspace of V . From the equivariance of $f \mapsto f^*v$ it follows that if $f \in C^0(G)$ is almost invariant in $C^0(G)$ then f^*v is almost invariant in V . According to the Peter-Weyl Theorem [1] the almost invariant elements of $C^0(G)$ are dense, so since $f \mapsto f^*v$ is continuous and has v in its closure we can find almost invariant f_n in $C^0(G)$ such that $f_n^*v \rightarrow v$. Since each f_n^*v is almost invariant in V this proves the following classical fact:

1.4. THEOREM. *In any Fréchet G -module V the almost invariant elements are dense.*

2. The C^k equivariant embedding theorem. The following theorem is proved in [6], and [7] for the cases $k = 1$ and $k = \infty$ respectively, While either proof extends easily enough to the case of general k we give here the appropriate generalization of Mostow's proof, which is the easier.

2.1. THEOREM. *If M is a compact C^k G -manifold with $1 \leq k \leq \infty$ then there exists an equivariant C^k embedding of M in some Euclidean G -space.*

Proof. Let W be a Euclidean space in which M admits a C^k embedding (e.g. \mathbf{R}^{2n+1} , $n = \dim M$), regarding as a Euclidean G -space by letting G act trivially. Since the space of embeddings of M in W is open in the Fréchet G -module $C^k(M, W)$ we can by Theorem 1.4 find a C^k embedding $f: M \rightarrow W$ with f almost invariant in $C^k(M, W)$. Let U be the linear span of the orbit of f in $C^k(M, W)$, a finite dimensional invariant linear subspace of $C^k(M, W)$ and hence a linear G -space. Choose a G -invariant positive definite product

for U , making U a Euclidean G -space. By a change of scale we can assume $f = f_1$ is a unit vector and we extend f_1 to an orthonormal basis for U ; f_1, \dots, f_m . For $g \in G$ $gf_i = \sum_j a_{ij}(g)f_j$ where, since G acts orthogonally on U and the f_i are orthonormal, $a_{ij}(g^{-1}) = a_{ji}(g)$. Now $U \otimes W$ is a Euclidean G -space (the action of G being defined by $g(u \otimes w) = (gu) \otimes w$) and we will define a C^k equivariant embedding $F: M \rightarrow U \otimes W$.

Every element of $U \otimes W$ can be written uniquely as

$$f_1 \otimes w_1 + \dots + f_m \otimes w_m$$

for w_1, \dots, w_m in W , i.e. the choice of basis f_1, \dots, f_m for U identifies $U \otimes W$ with the direct sum of m copies of W . Define $F: M \rightarrow U \otimes W$ by $F(x) = f_1 \otimes f_1(x) + \dots + f_m \otimes f_m(x)$. Since the first component of F is the C^k embedding $f: M \rightarrow W$, a fortiori F is a C^k embedding of M in $U \otimes W$ so it remains only to show that F is equivariant. But

$$\begin{aligned} F(gx) &= \sum_i f_i \otimes f_i(gx) = \sum_i f_i \otimes (g^{-1}f_i)(x) \\ &= \sum_i f_i \otimes \sum_j a_{ij}(g^{-1})f_j(x) \\ &= \sum_j \left(\sum_i a_{ji}(g)f_i \right) \otimes f_j(x) \\ &= \sum_j (gf_j) \otimes f_j(x) = g \sum_j f_j \otimes f_j(x) = gF(x). \end{aligned}$$

Q. E. D.

3. The tubular neighborhood theorem. Let $\pi: \xi \rightarrow M$ be a C^k vector bundle. If both ξ and η are C^k G -manifolds, π is equivariant, and each operation of G on ξ is a bundle map (i.e. g maps ξ_x linearly onto ξ_{gx}) then we call ξ a C^k -vector bundle. Given a linear G -space V the C^∞ manifold $G_m(V)$ (the Grassmannian of m dimensional linear subspaces of V) is in an obvious and natural way a C^∞ G -space. There is moreover a natural vector bundle ξ_m^V over $G_m(V)$, $\xi_m^V = \{(v, W) \in V \times G_m(V) \mid v \in W\}$, the projection being of course $(v, W) \mapsto W$. If we define $g(v, W) = (gv, gW)$ then ξ_m^V becomes a C^∞ G -vector bundle.

If $\pi: \xi \rightarrow M$ is a C^k G -vector bundle, N a C^k G -manifold, and $f: N \rightarrow M$ is a C^k equivariant map then the induced vector bundle $f^*\xi$ over N is clearly a C^k G -vector bundle (recall that $f^*\xi = \{(x, v) \in N \times \xi \mid fx = \pi v\}$ the action of G on $f^*\xi$ is just the restriction of the "product" action on $N \times \xi$).

If M is a C^k G -manifold ($k \geq 1$) and $f: M \rightarrow V$ is a C^k equivariant immersion of M in a linear G -space V then a C^l G -normal bundle to f ($l \leq k$)

is a C^l G -vector bundle ν over M of the form $\nu = h^* \xi_m^V$ ($m = \dim V - \dim M$) where $h: M \rightarrow G_m(V)$ is a C^l equivariant map such that for each $x \in M$ $h(x) = \nu_x$ is a linear complement to $\text{im}(Df_x)$. We note that such an f always has a C^{k-1} G -normal bundle: namely give V a G -invariant positive definite inner product (i.e. the structure of an orthogonal G -space) and define $h: M \rightarrow G_m(V)$ by $h(x) = (\text{im}(Df_x))^\perp$. We shall see shortly that indeed f has a C^k G -normal bundle.

Assume now that $1 \leq l \leq k$ and that $\pi: \nu \rightarrow M$ is a C^l G -normal bundle to the C^k G -equivariant embedding $f: M \rightarrow V$. Identify M with its image under f and also with the zero section of ν and define $E: \nu \rightarrow V$ by $E(w) = \pi(w) + w$. Then E is clearly an equivariant C^l map and it is classical that if $\partial M = \emptyset$ then E maps a neighborhood N of M in ν diffeomorphically onto a "tubular neighborhood" U of M in V . If we define $\tilde{\pi}: U \rightarrow M$ by $\tilde{\pi} = \pi \circ (E|N)^{-1}$ then $\tilde{\pi}$ is a C^l equivariant retraction of U onto M and we call $\tilde{\pi}: U \rightarrow M$ the C^l G -tubular neighborhood of M defined by ν .

3.1. PROPOSITION. *Let M be a C^k G -manifold without boundary ($k \geq 1$) which admits a C^k equivariant embedding in a linear G -space V with a C^k G -normal bundle ν . If M' is a compact C^k G -manifold then there is a neighborhood $\tilde{\mathcal{O}}_0$ of $C_G^0(M', M)$ in $C^0(M', M)$ and a continuous retraction $\tilde{A}_0: \tilde{\mathcal{O}}_0 \rightarrow C_G^0(M', M)$ such that for $0 \leq r \leq k$ \tilde{A}_0 restricts to a continuous retraction $\tilde{A}_r: \tilde{\mathcal{O}}_r \rightarrow C_G^r(M', M)$ where $\tilde{\mathcal{O}}_r = \tilde{\mathcal{O}}_0 \cap C^r(M', M)$.*

Proof. Let $\mathcal{O}_0 = \{f \in C^0(M', V) \mid \text{im } f \subseteq U\}$ where $\tilde{\pi}: U \rightarrow M$ is a C^k G -tubular neighborhood of M defined by ν as above. Note that $C^0(M', M) \subseteq C^0(M', V)$ so we can define $\tilde{\mathcal{O}}_0 = \{f \in C^0(M', M) \mid Af \in \mathcal{O}_0\}$ where

$$A: C^0(M', V) \rightarrow C_G^0(M', V)$$

is the continuous linear projection of Theorem 1.2. Since \mathcal{O}_0 is open in $C^0(M', V)$, $\tilde{\mathcal{O}}_0$ is open in $C^0(M', M)$. Moreover if $f \in C_G^0(M', M)$ then $Af = f$ and so $\text{im}(Af) = \text{im } f \subseteq M \subseteq U$ so $f \in \tilde{\mathcal{O}}_0$, i.e. $\tilde{\mathcal{O}}_0$ is a neighborhood of $C_G^0(M', M)$. We define $\tilde{A}_0: \tilde{\mathcal{O}}_0 \rightarrow C_G^0(M', M)$ by $\tilde{A}_0(f) = \tilde{\pi}_0(Af)$. Since A restricts to a continuous linear projection $A_r: C^r(M, V) \rightarrow C_G^r(M, V)$ for $0 \leq r \leq k$ by Theorem 1.2 and since $\tilde{\pi}: U \rightarrow M$ is an equivariant C^k retraction the proposition follows. Q. E. D.

3.2. COROLLARY. *For $0 \leq r \leq k$ $C_G^k(M', M)$ is dense in $C_G^r(M', M)$.*

Proof. Given $f \in C_G^r(M', M)$ choose a sequence $\{f_n\}$ in $C^k(M', M)$ converging to f in $C^r(M', M)$. Since $\tilde{\mathcal{O}}_r$ is a neighborhood of f in $C^r(M', M)$ we can suppose all f_n are in $\tilde{\mathcal{O}}_r$. Then $\tilde{A}_r f_n \in C_G^k(M', M)$ and

$$\tilde{A}_{kf_n} = \tilde{A}_r f_n \rightarrow \tilde{A}_r f = f.$$

Q. E. D.

3.3. PROPOSITION. *If M is a C^∞ G -manifold without boundary then $M = \bigcup_{n=1}^\infty M_n$ where M_n is an open invariant submanifold of M , $M_n \subset M_{n+1}$, and each \bar{M}_n is a compact C^∞ G -manifold with boundary.*

Proof. Let $f: M \rightarrow \mathbf{R}$ be a C^∞ proper map of M onto the positive reals and let $\bar{f}(x) = \int f(gx) dg$, so $\bar{f}: M \rightarrow \mathbf{R}$ is an invariant C^∞ map. Given $c > 0$ $Gf^{-1}([0, c]) = X$ is compact and if $x \notin X$ then clearly $\bar{f}(x) > c$, so $\bar{f}^{-1}([0, c]) \subseteq X$ and so \bar{f} is a proper map. Let $M_n = \bar{f}^{-1}([0, c_n])$ where $\{c_n\}$ is a sequence of regular values of \bar{f} with $c_n \rightarrow \infty$ monotonically. Q. E. D.

3.4. THEOREM. *If M is a C^∞ G -manifold without boundary then for any compact C^k G -manifold M' $C_G^k(M', M)$ is dense in $C_G^r(M', M)$, $0 \leq r \leq k$.*

Proof. Represent M as $\bigcup_{n=1}^\infty M_n$ as in 3.3. If $f \in C^r(M', M)$ then $f(M') \subset M_n$ for some n since M' is compact, i. e. $C_G^r(M', M) = \bigcup_{n=1}^\infty C_G^r(M', M_n)$, so it will suffice to show that $C_G^k(M', M_n)$ is dense in $C_G^r(M', M_n)$ for all n , or equivalently we can assume that M is the interior of a compact C^∞ G -manifold \bar{M} , possibly with boundary. But then by Theorem 2.1 M admits a C^∞ equivariant embedding $f: M \rightarrow V$ in a Euclidean G -space V (indeed \bar{M} does) and $x \mapsto \text{im}(Df_x)^\perp$ is a C^∞ equivariant map $g: M \rightarrow G_m(V)$ where $m = \dim V - \dim M$, so $g^*(\xi_m^V)$ is a C^∞ G -normal bundle to f and the Theorem follows from 3.2. Q. E. D.

3.5. TUBULAR NEIGHBORHOOD THEOREM. *If M is a compact C^k G -manifold, $k \geq 1$, and $f: M \rightarrow V$ is a C^k equivariant embedding on a linear G -space V then f admits a C^k G -normal bundle and hence if $\partial M = \emptyset$ then $f(M)$ admits a C^k G -tubular neighborhood in V $\tilde{\pi}: U \rightarrow f(M)$.*

Proof. We can assume V is a Euclidean G -space and we define $f: M \rightarrow G_m(V)$, $m = \dim V - \dim M$, by $g(x) = \text{im}(Df)_x^\perp$, so

$$g \in C_G^{k-1}(M, G_m(V)).$$

Since $G_m(V)$ is a C^∞ G -manifold, by 3.4 we can approximate g arbitrarily well in $C_G^{k-1}(M, G_m(V))$ by $h \in C_G^k(M, V)$. But clearly if h is sufficiently close to g in the C^0 -topology then $h(x)$ like $g(x)$ will for each $x \in M$ be a linear complement to $\text{im}(Df_x)$ in V . Then $h^*(\xi_m^V)$ is a C^k G -normal bundle to f .

Q. E. D.

4. Fiberwise transversality.

4.1. *Definition.* Let $\pi_1: E_1 \rightarrow M_1$ and $\pi_2: E_2 \rightarrow M_2$ be two C^k fiber bundles, $k \geq 1$, and let G_2 be a C^k sub-bundle of E_2 . A C^k map $f: E_1 \rightarrow E_2$ will be called fiber-wise transversal to G_2 if for each $x \in M_1$ $f|_{(E_1)_x}$ is transversal to G_2 .

4.2. *PROPOSITION.* Given $\pi_1: E_1 \rightarrow M_1$, $\pi_2: E_2 \rightarrow M_2$ and G_2 as in 4.1 with E_1 compact, the set of C^k maps $f: E_1 \rightarrow E_2$ which are fiber-wise transversal to G_2 is open in the C^1 topology.

Proof. Obvious.

4.3. *LEMMA.* Let $\pi: E \rightarrow M$ be a C^k fiber bundle, $k \geq 1$ and let G be a compact C^k submanifold of E with $\partial G = \emptyset$. A necessary and sufficient condition for G to be a C^k sub-bundle of E (i. e. for $\tilde{\pi} = \pi|_G: G \rightarrow M$ to be a C^k fiber bundle) is that for each $x \in M$ $G_x = G \cap E_x$ be a submanifold of G of codimension equal to the dimension of M .

Proof. Necessity is trivial. To prove sufficiency it is enough by a well-known theorem of Ehressman [2] to check that when the condition is satisfied $\tilde{\pi}: G \rightarrow M$ is a submersion, i. e. that for each $y \in G$ $(D\tilde{\pi})_y: TG_y \rightarrow TM_x$ has rank equal to $\dim M$. Now $\text{rank}(D\tilde{\pi})_y = \dim G - \dim(\ker(D\tilde{\pi})_y)$. Since π is a fibering $\ker(D\pi)_y = T(E_x)_y$ and since $(D\tilde{\pi})_y = (D\pi)_y|_{(TG)_y}$,

$$\ker(D\pi)_y = \ker(D\tilde{\pi})_y \cap (TG)_y = T(E_x)_y \cap (TG)_y = T(E_x \cap G)_y = T(G_x)_y,$$

so $\text{rank}(D\tilde{\pi})_y = \dim G - \dim G_x = \text{codim } G_x \text{ in } G = \dim M$. Q. E. D.

4.4. *THEOREM.* Let $\pi_1: E_1 \rightarrow M_1$, $\pi_2: E_2 \rightarrow M_2$ be two C^k fiber bundles, $k \geq 1$, and let G_2 be a C^k sub-bundle of E_2 . Assume E_1 is compact, $\partial M_1 = \emptyset$, and $\partial G_2 = \emptyset$. Let $f: E_1 \rightarrow E_2$ be a C^k map which is fiber-wise transversal to G_2 with $f(\partial E_1) \cap G_2 = \emptyset$. Then $G_1 = f^{-1}(G_2)$ is a C^k sub-bundle of E_1 , $\partial G_1 = \emptyset$, and the fiber codimension of G_1 in E_1 equals the fiber codimension of G_2 in E_2 .

Proof. Since f is fiber-wise transversal to G_2 it is a fortiori transversal to G_2 and hence $G_1 = f^{-1}(G_2)$ is a compact C^k submanifold of E_1 of codimension equal to the codimension of G_2 in E_2 , and $\partial G_1 = (f|_{\partial E_1})^{-1}G_2 = \emptyset$. Putting $(G_1)_x = (f|_{(E_1)_x})^{-1}G_2 = G_1 \cap (E_1)_x$, $(G_1)_x$ is a submanifold of $(E_1)_x$ of codimension equal to the codimension of G_2 in E_2 which, since G_2 is a sub-bundle of E_2 is the same as the fiber codimension of G_2 in E_2 . Now since $\dim E - \dim G = \dim(E_1)_x - \dim(G_1)_x = \dim E_2 - \dim G_2$ it follows

that $\dim G - \dim(G_1)_x = \dim E - \dim(E_1)_x = \dim M$, and Lemma 4.3 completes the proof. Q. E. D.

4.5. THEOREM. *Let $\pi_1: E_1 \rightarrow M_1$ and $\pi_2: E_2 \rightarrow M_2$ and G_2 be as in Theorem 4.4 and let $f_t: E_1 \rightarrow E_2$ be a homotopy of C^k maps with each f_t fiberwise transversal to G_2 and all $f_t(\partial E_1) \cap G_2$ empty. Then $f_0^{-1}(G_2)$ and $f_1^{-1}(G_2)$ are C^k equivalent sub-bundles of E_1 and in particular for each $x \in M_1$ their fibers at x are C^k diffeomorphic.*

Proof. Let S^1 denote $[0, 1]$ with 0 and 1 identified. It will suffice to construct a C^k fiber bundle over $S^1 \times M_1$ which restricted to $\{0\} \times M_1$ is $f_0^{-1}(G_2)$ and restricted to $\{\frac{1}{2}\} \times M_1$ is $f^{-1}(G_2)$ (because bundles over $[0, \frac{1}{2}] \times M_1$ are always of the form $[0, \frac{1}{2}] \times B$ for some bundle over M_1). By 4.2 we can approximate f_t so that $(t, x) \mapsto f_t(x)$ is C^k without changing f_0 or f_1 and maintaining the other given conditions. By a standard argument we can also suppose that $f_t = f_0$ for $0 \leq t < \epsilon$ and $f_t = f_1$ for $1 - \epsilon \leq t \leq 1$. Define $h_t = f_{2t}$ for $0 \leq t \leq \frac{1}{2}$ and $h_t = f_{2-2t}$ for $\frac{1}{2} \leq t \leq 1$. Then $(t, x) \mapsto h_t(x)$ is a C^k map of $S^1 \times E_1 \rightarrow E_2$ and if we regard $S^1 \times E_1$ and $S^1 \times E_2$ as C^k fiber bundles over $S^1 \times M_1$ and $S^1 \times M_2$ respectively then $H: S^1 \times E_1 \rightarrow S^1 \times E_2$ defined by $(t, x) \mapsto (t, h_t(x))$ is clearly fiber-wise transversal to $S^1 \times G_2$ and $H(\partial(S^1 \times E_1)) \cap S^1 \times G_2 = \emptyset$ so by 4.4 $H^{-1}(S^1 \times G_2)$ is a C^k sub-bundle of $S^1 \times E_1$. Clearly $H^{-1}(S^1 \times G_2) | \{t\} \times M_1$ is just $h_t^{-1}(G_2)$ so taking $t = 0, \frac{1}{2}$ respectively we get $f_0^{-1}(G_2)$ and $f_1^{-1}(G_2)$ respectively as desired.

Q. E. D.

4.6. COROLLARY. *Let $\pi_1: E_1 \rightarrow M_1$, $\pi_2: E_2 \rightarrow M_2$, G and f be as in Theorem 4.4 and in addition assume that E_1 is compact. Then there is a neighborhood U of f in $C^k(E_1, E_2)$, which is in fact open in the C^1 topology, such that for all $h \in U$, $h^{-1}(G_2)$ is a C^k sub-bundle of E_1 which is C^k equivalent to $f^{-1}(G_2)$.*

Proof. There is a C^0 neighborhood of f in $C^k(E_1, E_2)$ say U_0 such that for all $h \in U_0$ $h(\partial E_1) \cap G_2 = \emptyset$. There is a C^1 neighborhood U_1 of f by 4.2 such that h is fiberwise transversal to G_2 for all $h \in U_1$. Now $C^k(E_1, E_2)$ is locally contractible in the C^1 topology so we can take for U any contractible (or even pathwise connected) C^1 -neighborhood of f in $U_0 \cap U_1$. Q. E. D.

4.7. THEOREM. *Let $\pi_1: E_1 \rightarrow M_1$ and $\pi_2: E_2 \rightarrow M_2$ be C^k fiber bundles, $k \geq 1$, with compact total spaces and $\partial M_1 = \partial M_2 = \emptyset$. Let $f_0: E_1 \rightarrow E_2$ be a C^k map which maps each fiber of E_1 diffeomorphically onto a fiber of E_2 . If σ is any C^k section of E_2 with $\sigma \cap \partial E_2 = \emptyset$ then there is a neighborhood U*

of f_0 in $C^k(E_1, E_2)$, which in fact is open in the C^1 topology, such that for every $f \in U$, $f^{-1}(\sigma)$ is a C^k section of E_1 disjoint from ∂E_1 . Moreover the map $f \mapsto f^{-1}(\sigma)$ is continuous from $U \subseteq C^k(E_1, E_2)$ into the space $C^k(E_1)$ of C^k sections of E_1 with the C^k topology.

Proof. In 4.6 take $G_2 = \sigma$. Since f_0 maps each fiber of E_1 diffeomorphically onto a fiber of E_2 it is clear that f_0 is fiberwise transversal to σ , and hence $\sigma_0 = f_0^{-1}(\sigma)$ is a C^k sub-bundle of E_1 of fiber dimension zero, and in fact since $f_0(E_1)_x$ meets σ in exactly one point, σ_0 is a section of E_1 . It now follows from 4.6 that $f^{-1}(\sigma)$ is also a C^k sub-bundle of E_1 with fiber a point, i. e. a C^k section of E_1 , if $f \in C^k(E_1, E_2)$ is sufficiently near f_0 in the C^1 topology. It remains only to show the continuity of $f \mapsto f^{-1}(\sigma)$ as a map of U into $C^k(E_1)$. Since M_1 is compact it will suffice to show that if $f_n \rightarrow f$ then $f_n^{-1}(\sigma) \mid W \rightarrow f^{-1}(\sigma) \mid W$ in the C^k topology in some neighborhood W of each point of $x \in M_1$. We can choose coordinates in E_2 near $\sigma(x)$ so that E_2 is identified with $\mathbf{R}^k \times \mathbf{R}^m$, π_2 with the projection $\mathbf{R}^k \times \mathbf{R}^m \rightarrow \mathbf{R}^m$ and σ with $\mathbf{R}^k \times \{0\}$. Then choosing coordinates near $\sigma_0(x)$ in E_1 so that E_1 is locally $\mathbf{R}^n \times \mathbf{R}^m$, π_1 is the projection $\mathbf{R}^n \times \mathbf{R}^m \rightarrow \mathbf{R}^n$ and σ_0 is $\mathbf{R}^n \times \{0\}$, f_0 is given locally by a map $(x, y) \mapsto (g_0(x), h_0(x, y))$ of $\mathbf{R}^n \times \mathbf{R}^m \rightarrow \mathbf{R}^k \times \mathbf{R}^m$ with $g_0(0) = 0$, $h_0(0, 0) = 0$ and $y \mapsto h_0(0, y)$ a local C^k diffeomorphism of one neighborhood of 0 in \mathbf{R}^m onto another. Given $f: \mathbf{R}^n \times \mathbf{R}^m \rightarrow \mathbf{R}^k \times \mathbf{R}^m$ near f_0 , say $f(x, y) = (g(x, y), h(x, y))$, $f^{-1}(\sigma)$ is represented as the unique C^k map $\sigma_f: \mathbf{R}^n \rightarrow \mathbf{R}^m$ near zero which solves the equation $h(x, \sigma_f(x)) = 0$. By the implicit function theorem σ_f and its derivatives through order k depend continuously on h and its derivatives through order k , which in turn of course depend continuously on f in the C^k topology. Q. E. D.

5. Proofs of Theorems A, B, and C. The second conclusion of Theorem A is just Theorem 3.4. The first conclusion is an immediate consequence of the second since the space $\text{Diff}^l(M_1, M_2)$ of C^l diffeomorphisms of M_1 with M_2 is open in $C^l(M_1, M_2)$, and hence if we approximate a C^l equivariant diffeomorphism $f: M_1 \approx M_2$ well enough (in $C^l(M_1, M_2)$) by $g \in C_G^k(M_1, M_2)$ then g will be a C^k equivariant diffeomorphism of M_1 with M_2 . This completes the proof of Theorem A.

Now let M_0 be a closed C^k manifold and let $\lambda: M_0 \rightarrow V$ be a C^k embedding of M_0 in some Euclidean space. Let $\pi: \nu \rightarrow M_0$ be C^k normal bundle to λ and ν_ϵ the ϵ -disc sub-bundle of ν . As usual we identify M_0 with $\lambda(M_0)$ and with the zero section of ν , and for ϵ small enough we identify ν_ϵ with the ϵ -tubular neighborhood of M_0 it defines, namely its image under the map

$v \mapsto \pi(v) + v$. Let $k = \dim V - \dim M$ and let $\xi_k^V(\epsilon)$ denote the ϵ -disc bundle of the vector bundle $\xi_k^V \rightarrow G_k(V)$. Define $f_0: \nu_\epsilon \rightarrow \xi_k^V(\epsilon)$ by $f_0(v) = (v, \nu_{\pi(v)})$ (the Thom-Pontryagin map) so that clearly for each $x \in M_0$, f_0 maps $\nu_x(\epsilon)$ diffeomorphically onto the fiber of $\xi_k^V(\epsilon)$ over ν_x . If σ is the zero section ξ_k^V then $f_0^{-1}(\sigma) = M_0$. Now by 4.7 if $f_1: \nu(\epsilon) \rightarrow \xi_k^V(\epsilon)$ is a C^k map sufficiently close to f_0 in the C^1 topology then $M_1 = f_1^{-1}(\sigma)$ is a C^k section of ν and hence $\pi|_{M_1}: M_1 \rightarrow M_0$ is a C^k diffeomorphism. In particular if we approximate f_0 by a C^∞ f_1 , relative to the natural C^∞ structure for ξ_k^V and that for $\nu(\epsilon)$ inherited from V , then M_1 will be a C^∞ submanifold of V mapped C^k diffeomorphically onto M_0 by π .

Now suppose M_0 is a C^k G -manifold. Then by Theorem 2.1 we can suppose λ is an equivariant C^k embedding of M_0 in a Euclidean G -space V and by 3.5 we can suppose $\pi_0: \nu \rightarrow M_0$ is a C^k G -normal bundle to λ in V so that $\nu(\epsilon)$ is a C^k G -tubular neighborhood. Finally since f_0 is clearly equivariant we can by Theorem 3.4 assume that our approximating $f_1: \nu(\epsilon) \rightarrow \xi_k^V(\epsilon)$ is equivariant. Then M_1 is clearly a invariant C^∞ submanifold of V , hence a C^∞ G -manifold, and $\pi|_{M_1}$ is of course equivariant. This completes the proof of Theorem B and we pass now to the proof of Theorem C.

Let M be any closed C^∞ manifold, $\alpha_0 \in \mathcal{A}^k(G, M)$ and denote by M_0 the resulting C^k G -manifold. According to Theorem B we can choose a C^∞ G -manifold M_1 for which there exists a C^k equivariant diffeomorphism $\gamma: M_1 \rightarrow M_0$. If $\alpha_1 \in \mathcal{A}^\infty(G, M_1)$ is the given action of G on M_1 then of course $\alpha_0 = \alpha_1^{\gamma^{-1}}$, i. e. $\alpha_0(g, x) = \gamma\alpha_1(g, \gamma^{-1}x)$. Given $f \in \text{Diff}^k(M)$ we have

$$\alpha = \alpha_1^{(f\gamma)^{-1}} \in \mathcal{A}^k(G, M), \quad \alpha(g, x) = f\gamma\alpha_1(g, \gamma^{-1}f^{-1}x)$$

and $\alpha^f = \alpha_1^{\gamma^{-1}} = \alpha_0$. Clearly $f \mapsto \alpha_1^{(f\gamma)^{-1}}$ is continuous from $\text{Diff}^k(M)$ to $\mathcal{A}^k(G, M)$ so in particular if $f \rightarrow e$ in $\text{Diff}^k(M)$ then $\alpha_1^{(f\gamma)^{-1}} \rightarrow \alpha_0$ in $\mathcal{A}^k(G, M)$. Thus to complete the proof of Theorem C it will suffice to find $f \in \text{Diff}^k(M)$ arbitrarily close to the identity such that $\alpha_1(f\gamma)^{-1} \in \mathcal{A}^\infty(G, M)$. But since $\alpha_1 \in \mathcal{A}^\infty(G, M)$ it is clear that if $f\gamma: M_1 \rightarrow M$ is C^∞ , and hence $(f\gamma)^{-1}: M \rightarrow M_1$ is C^∞ , then $\alpha_1^{(f\gamma)^{-1}} \in \mathcal{A}^\infty(G, M)$. Now $\text{Diff}^\infty(M_1, M)$ is dense in $\text{Diff}^k(M_1, M_2)$, so there exists $g \in \text{Diff}^\infty(M_1, M)$ arbitrarily near γ in $\text{Diff}^k(M_1, M)$. Then $f = g\gamma^{-1}$ is arbitrarily near the identity in $\text{Diff}^k(M_1, M_2)$ and $f\gamma = g$ is C^∞ .

Q. E. D.

5.1. THEOREM. *If M is a closed C^k G -manifold, $k \geq 1$, and M' is any compact C^k G -manifold, then $C_{G^r}^k(M', M)$ is dense in $C_{G^r}(M', M)$ for $0 \leq r \leq k$.*

Proof. By Theorem B we can give M a C^∞ structure which makes it a C^∞ G -manifold, and the theorem then follows from 3.4.

Definition. A C^k manifold M is called C^k -asymmetric if there is no non-trivial C^k -action of any compact connected Lie group on M , or equivalently if there is no effective C^k action of the circle group S^1 on M .

5.2. THEOREM. *Let M be a closed C^∞ manifold and let $1 \leq l < k \leq \infty$. If M is C^k -asymmetric then M is C^l -asymmetric.*

Proof. Immediate from Theorem C.

It would be interesting to know whether we can take $l=0$ in Theorem 5.2. In particular in a paper to appear Atiyah and Hirzebruch have shown that if M is a closed, orientable $4k$ dimensional manifold with $w_2(M) = 0$, then M is C^∞ -asymmetric (hence C^1 -asymmetric) provided the \hat{A} -genus of M (a topological invariant!) is non-zero. It would be quite remarkable if such M could admit non-trivial C^0 -circle actions.*

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* *Added in Proof:* Atiyah has in fact constructed just such a remarkable example.

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