C^1 ACTIONS OF COMPACT LIE GROUPS ON COMPACT MANIFOLDS ARE C^1-EQUIVALENT TO C^∞ ACTIONS.

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Introduction. Let C_1 and C_2 be two categories and let $F\colon C_1 \rightsquigarrow C_2$ be a weakening of structure ("forgetful") functor, i.e. for each pair of objects X_1 and X_2 of C_1 the map of Morph (X_1, X_2) to Morph$(F(X_1), F(X_2))$ is injective. Let $\hat{C}_1$ and $\hat{C}_2$ denote the equivalence classes of objects of C_1 and C_2 defined by the equivalence relations of isomorphisms in the respective categories. Then F induces a map $\hat{F}\colon \hat{C}_1 \to \hat{C}_2$ and the question of whether $\hat{F}$ is injective, surjective, or bijective is often of interest. For example for $0 \leqq k \leqq \infty$ let Man_k denote the category of C^k paracompact manifolds and C^k maps, and for $k > l$ let $\hat{F}_{lk}\colon \mathrm{Man}_k \rightsquigarrow \mathrm{Man}_l$ be the obvious forgetful functor. Then for $l = 0$ $F_{0k}\colon \widehat{\mathrm{Man}}_k \to \widehat{\mathrm{Man}}_0$ is neither surjective nor injective, i.e. there exist topological manifolds which admit no C^k structure [3] and there exist topological manifolds with non-isomorphic C^k structures [5]. However if $l > 0$ then $\hat{F}_{lk}$ is always bijective, i.e. every C^l manifold admits a compatible C^k structure [10] and if two C^k manifolds are C^l diffeomorphic they are C^k diffeomorphic (the latter is trivial from standard approximation theorems).

Below we shall prove the analogous statements for the categories $Man_k(G)$ of compact C^k G-manifolds where G is a compact Lie group (see § 1 for precise definitions). Our main results are summarized in the following theorems.

THEOREM A. *Let G be a compact Lie group and let M_1 and M_2 be closed C^k G-manifolds ($2 \leqq k \leqq \infty$). If M_1 and M_2 are C^l equivariantly diffeomorphic ($1 \leqq l < k$) then they are C^k equivariantly diffeomorphic. In fact any C^l equivariant map $f\colon M_1 \to M_2$ can be approximated arbitrarily well in the C^l topology by a C^k equivariant map.*

THEOREM B. *Let G be a compact Lie group and M_0 a closed C^k G-mani-*

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fold $1 \leqq k \leqq \infty$. *Then there is a closed* C^∞ *G-manifold* M_1 *which is* C^k *equivariantly diffeomorphic to* M_0.

The following is a more precise form of Theorem B.

THEOREM C. *Let* M *be a closed* C^∞ *manifold,* G *a compact Lie group, and for* $1 \leqq k \leqq \infty$ *let* $\mathcal{A}^k(G,M)$ *denote the space of* C^k *actions* $\alpha\colon G \times M \to M$ *of* G *on* M *with the* C^k *topology and let* $\mathrm{Diff}^k(M)$ *denote the group of* C^k *diffeomorphisms of* M *with the* C^k *topology. Given* $\alpha_0 \in \mathcal{A}^k(G,M)$ *we can find* $\alpha \in \mathcal{A}^\infty(G,M)$ *arbitrarily near* α_0 *in* $\mathcal{A}^k(G,M)$ *and* f *arbitrarily near the identity in* $\mathrm{Diff}^k(M)$ *such that* $\alpha^f = \alpha_0$; *where* $\alpha^f \in \mathcal{A}^k(G,M)$ *is defined by* $\alpha^f(g,x) = f^{-1}\alpha(g,fx)$.

As an immediate consequence of Theorems A and B we have the following

THEOREM D. *For* $0 \leqq k \leqq \infty$ *let* $\mathrm{Man}_k(G)$ *denote the category of closed* C^k *G-manifolds and* C^k *equivariant maps, and for* $l < k$ *let* $F_{lk}\colon \mathrm{Man}_k(G) \rightsquigarrow \mathrm{Man}_l(G)$ *denote the obvious weakening of structure functor. Then if* $l \geqq 1$ $\hat{F}_{lk}\colon \widehat{\mathrm{Man}}_k(G) \to \widehat{\mathrm{Man}}_l(G)$ *is bijective.*

It is worth remarking that for the case $l = 0$ the situation is again quite the opposite and $\hat{F}_{0k}$ is in general neither injective nor surjective. For example there exist C^∞ actions of Z_2 on a sphere which are C^0 equivalent to the linear reflexion in a line but are not C^1 equivalent to a linear action [4]. Also whereas there are always at most countably many equivalence classes of smooth actions of a compact Lie group G on a compact smooth manifold M [8], so that in fact $\widehat{\mathrm{Man}}_k(G)$ is always countable for $k \geqq 1$, Z_2 or S^1 can act in uncountably many topologically inequivalent ways on S^5, and more generally any non-trivial compact Lie group G can act uncountably many topologically inequivalent ways on some S^n [9], so that $\widehat{\mathrm{Man}}_0(G)$ is always uncountable and hence $\hat{F}_{0k}$ is never surjective for $G \neq e$.

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1. Notation and preliminaries. G will always denote a compact Lie group. A (left) *action* of G on a space X is a continuous map $\alpha\colon G \times X \to X$ such that the map $\tilde{\alpha}\colon G \to X^X$ defined by $\tilde{\alpha}(g)(x) = \alpha(g,x)$ is a homomorphism of G onto the group of homeomorphisms of X. We call $\tilde{\alpha}(g)$ the *operation* of g on X (defined by the action α). A *G-space* is a completely regular space X together with a fixed action α of G on X and generally we write gx for $\alpha(g,x)$. If X and Y are G-spaces a map $f\colon X \to Y$ is called equivariant if $f(gx) = gf(x)$. A *Fréchet G-module* is a G-space V which is a Fréchet space (= complete, metrizeable, locally convex, topological vector space) such that each operation of G on V is linear. If in addition V is finite dimensional we call it a *linear G-space.* In this case by choosing an arbitrary orthogonal structure for V and "averaging it over the group" (see Theorem 1.1) we can find an orthogonal structure for V with respect to which each operation of G is orthogonal, and with this extra structure we call V a *Euclidean G-space.*

By a C^k *G-manifold* $0 \leqq k \leqq \infty$ we shall mean a C^k manifold M, possibly with boundary, which is a G-space in such a way that the action $\alpha: G \times M \to M$ is a C^k map, so in particular each operation of G on M is a C^k diffeomorphism. If M is compact and without boundary we call it a closed C^k G-manifold. In particular we regard G itself as a closed C^∞ G-manifold with the natural left translation action.

If M is a C^k manifold we denote by $\mathfrak{A}^k(G,M)$ the space of C^k actions of G on M and by $\mathrm{Diff}^k(M)$ the group of C^k diffeomorphisms of M, both with the C^k topology. There is a natural (right) action of $\mathrm{Diff}^k(M)$ on $\mathfrak{A}^k(G,M)$, $(f,\alpha) \mapsto \alpha^f$, where $\alpha^f(g,x) = f^{-1}\alpha(g,fx)$. This action is easily seen to be continuous. Note that f is equivariant from M with the action α^f to M with the action α. If M_1 and M_2 are C^k G-manifolds we denote by $C^k(M_1,M_2)$ the space of C^k maps of M_1 to M_2 with the C^k topology and by $C_G{}^k(M_1,M_2)$ the subspace of equivariant maps. We note that $C^k(M_1,M_2)$ is in a natural way a G-space; namely if $f \in C^k(M_1,M_2)$ then $gf \in C^k(M_1,M_2)$ is defined by $(gf)(x) = g(f(g^{-1}x))$, and $C_G{}^k(M_1,M_2)$ is just the set of $f \in C^k(M_1,M_2)$ left fixed by each operation of G. If M is a C^k G-manifold and W is a linear G-space then $C^k(M,W)$ is a Fréchet G-module. In particular if W is any finite dimensional vector space we can regard it as a linear G-space by letting G act trivially and $C^k(M,W)$ becomes a Fréchet G-module, the action of $g \in G$ on $f \in C^k(M,W)$ being given by $(gf)(x) = f(g^{-1}x)$. In particular $C^k(M) = C^k(M,\boldsymbol{R})$ is a Fréchet G-module and more particularly still $C^k(G)$ is a Fréchet G-module.

The following is a classical and trivial remark ("averaging over the group").

1.1. THEOREM. *If V is a Fréchet G-module then $A: V \mapsto V$ defined by $A(v) = \int gv\,d\mu(g)$, where μ is normalized Haar measure on G, is a continuous linear projection of V onto the subspace V^G of V consisting of elements left fixed by each operation of G.*

As a corollary we have:

1.2. THEOREM. *If M is a C^k G-manifold and W is a linear G-space then $A: C^k(M, W) \to C^k(M, W)$ defined by $(Af)(x) = \int g(f(g^{-1}x))\,d\mu(g)$ is a continuous linear projection onto $C_G{}^k(M, W)$.*

1.3. COROLLARY. *$C_G{}^k(M, W)$ is dense in $C_G{}^l(M, W)$ for $0 \leqq l \leqq k$.*

Proof. Given $f \in C_G{}^l(M, W)$ choose a sequence $\{f_n\}$ in $C^k(M, W)$ converging to f in $C^l(M, W)$. Then $\{Af_n\}$ is a sequence in $C_G{}^k(M, W)$ converging to $Af = f$ in $C_G{}^l(M, W)$. Q.E.D.

For each Fréchet G-module V we define a continuous bilinear map $(f, v) \mapsto f^*v$ of $C^0(G) \times V$ into V by $(f^*v)(x) = \int f(g)gv\,d\mu(g)$. It is trivial that for fixed $v \in V$ $f \mapsto f^*v$ is a continuous, linear and equivariant (i.e. $(gf)^*v = g(f^*v)$) map of $C^0(G)$ into V and v is in the closure of its image (in fact if $\{f_n\}$ is a sequence in $C^0(G)$ of positive functions with integral one whose supports shrink to the identity, then $f_n{}^*v \to v$). An element of a Fréchet G-module V is called *almost invariant* if its orbit spans a finite dimensional subspace of V. From the equivariance of $f \mapsto f^*v$ it follows that if $f \in C^0(G)$ is almost invariant in $C^0(G)$ then f^*v is almost invariant in V. According to the Peter-Weyl Theorem [1] the almost invariant elements of $C^0(G)$ are dense, so since $f \mapsto f^*v$ is continuous and has v in its closure we can find almost invariant f_n in $C^0(G)$ such that $f_n{}^*v \to v$. Since each $f_n{}^*v$ is almost invariant in V this proves the following classical fact:

1.4. THEOREM. *In any Fréchet G-module V the almost invariant elements are dense.*

2. The C^k equivariant embedding theorem.

The following theorem is proved in [6], and [7] for the cases $k = 1$ and $k = \infty$ respectively, While either proof extends easily enough to the case of general k we give here the appropriate generalization of Mostow's proof, which is the easier.

2.1. THEOREM. *If M is a compact C^k G-manifold with $1 \leqq k \leqq \infty$ then there exists an equivariant C^k embedding of M in some Euclidean G-space.*

Proof. Let W be a Euclidean space in which M admits a C^k embedding (e.g. $\boldsymbol{R}^{2n+1}$, $n = \dim M$), regarding as a Euclidean G-space by letting G act trivially. Since the space of embeddings of M in W is open in the Fréchet G-module $C^k(M, W)$ we can by Theorem 1.4 find a C^k embedding $f: M \to W$ with f almost invariant in $C^k(M, W)$. Let U be the linear span of the orbit of f in $C^k(M, W)$, a finite dimensional invariant linear subspace of $C^k(M, W)$ and hence a linear G-space. Choose a G-invariant positive definite product for U, making U a Euclidean G-space. By a change of scale we can assume $f = f_1$ is a unit vector and we extend f_1 to an orthonormal basis for U; $f_1, \cdots, f_m$. For $g \in G$ $gf_i = \sum_j a_{ij}(g) f_j$ where, since G acts orthogonally on U and the f_i are orthonormal, $a_{ij}(g^{-1}) = a_{ji}(g)$. Now $U \otimes W$ is a Euclidean G-space (the action of G being defined by $g(u \otimes w) = (gu) \otimes w)$ and we will define a C^k equivariant embedding $F\colon M \to U \otimes W$.

Every element of $U \otimes W$ can be written uniquely as

$$f_1 \otimes w_1 + \cdots + f_m \otimes w_m$$

for $w_1, \cdots, w_m$ in W, i.e. the choice of basis $f_1, \cdots, f_m$ for U identifies $U \otimes W$ with the direct sum of m copies of W. Define $F\colon M \to U \otimes W$ by $F(x) = f_1 \otimes f_1(x) + \cdots + f_m \otimes f_m(x)$. Since the first component of F is the C^k embedding $f\colon M \to W$, a fortiori F is a C^k embedding of M in $U \otimes W$ so it remains only to show that F is equivariant. But

$$\begin{aligned} F(gx) &= \sum_i f_i \otimes f_i(gx) = \sum_i f_i \otimes (g^{-1} f_i)(\mathrm{x}) \\ &= \sum_i f_i \otimes \sum_j a_{ij}(g^{-1}) f_j(x) \\ &= \sum_j (\sum_i a_{ji}(g) f_i) \otimes f_j(x) \\ &= \sum_j (gf_j) \otimes f_j(x) = g \sum_j f_j \otimes f_j(x) = gF(x). \end{aligned}$$

Q. E. D.

3. The tubular neighborhood theorem. Let $\pi\colon \xi \to M$ be a C^k vector bundle. If both ξ and η are C^k G-manifolds, π is equivariant, and each operation of G on ξ is a bundle map (i.e. g maps ξ_x linearly onto ξ_{gx}) then we call ξ a C^k-vector bundle. Given a linear G-space V the C^∞ manifold $G_m(V)$ (the Grassmannian of m dimensional linear subspaces of V) is in an obvious and natural way a C^∞ G-space. There is moreover a natural vector bundle $\xi_m{}^V$ over $G_m(V)$, $\xi_m{}^V = \{(v, W) \in V \times G_m(V) \mid v \in W\}$, the projection being of course $(v, W) \mapsto W$. If we define $g(v, W) = (gv, gW)$ then $\xi_m{}^V$ becomes a C^∞ G-vector bundle.

If $\pi\colon \xi \to M$ is a C^k G-vector bundle, N a C^k G-manifold, and $f\colon N \to M$ is a C^k equivariant map then the induced vector bundle $f^*\xi$ over N is clearly a C^k G-vector bundle (recall that $f^*\xi = \{(x, v) \in N \times \xi \mid fx = \pi v\}$ the action of G on $f^*\xi$ is just the restriction of the "product" action on $N \times \xi$).

If M is a C^k G-manifold $(k \geqq 1)$ and $f\colon M \to V$ is a C^k equivariant immersion of M in a linear G-space V then a C^l *G-normal bundle to f* $(l \leqq k)$

is a C^l G-vector bundle ν over M of the form $\nu = h^*\xi_m{}^V$ ($m = \dim V - \dim M$) where $h: M \to G_m(V)$ is a C^l equivariant map such that for each $x \in M$ $h(x) = \nu_x$ is a linear complement to $\operatorname{im}(Df_x)$. We note that such an f always has a C^{k-1} G-normal bundle: namely give V a G-invariant positive definite inner product (i. e. the structure of an orthogonal G-space) and define $h: M \to G_m(V)$ by $h(x) = (\operatorname{im}(Df_x))^{\perp}$. We shall see shortly that indeed f has a C^k G-normal bundle.

Assume now that $1 \leqq l \leqq k$ and that $\pi: \nu \to M$ is a C^l G-normal bundle to the C^k G-equivariant embedding $f: M \to V$. Identify M with its image under f and also with the zero section of ν and define $E: \nu \to V$ by $E(w) = \pi(w) + w$. Then E is clearly an equivariant C^l map and it is classical that if $\partial M = \emptyset$ then E maps a neighborhood N of M in ν diffeomorphically onto a "tubular neighborhood" U of M in V. If we define $\tilde{\pi}: U \to M$ by $\tilde{\pi} = \tilde{\pi} \circ (E \mid N)^{-1}$ then $\tilde{\pi}$ is a C^l equivariant retraction of U onto M and we call $\tilde{\pi}: U \to M$ the C^l G-tubular neighborhood of M defined by ν.

3.1. PROPOSITION. *Let M be a C^k G-manifold without boundary ($k \geqq 1$) which admits a C^k equivariant embedding in a linear G-space V with a C^k G-normal bundle ν. If M' is a compact C^k G-manifold then there is a neighborhood $\tilde{\mathcal{O}}_0$ of $C_G{}^0(M', M)$ in $C^0(M', M)$ and a continuous retraction $\tilde{A}_0: \tilde{\mathcal{O}}_0 \to C_G{}^0(M', M)$ such that for $0 \leqq r \leqq k$ $\tilde{A}_0$ restricts to a continuous retraction $\tilde{A}_r: \tilde{\mathcal{O}}_r \to C_G{}^r(M', M)$ where $\tilde{\mathcal{O}}_r = \tilde{\mathcal{O}}_0 \cap C^r(M', M)$.*

Proof. Let $\mathcal{O}_0 = \{f \in C^0(M', V) \mid \operatorname{im} f \subseteq U\}$ where $\tilde{\pi}: U \to M$ is a C^k G-tubular neighborhood of M defined by ν as above. Note that $C^0(M', M) \subseteq C^0(M', V)$ so we can define $\tilde{\mathcal{O}}_0 = \{f \in C^0(M', M) \mid Af \in \mathcal{O}_0\}$ where

$$A: C^0(M', V) \to C_G{}^0(M', V)$$

is the continuous linear projection of Theorem 1.2. Since $\mathcal{O}_0$ is open in $C^0(M', V)$, $\tilde{\mathcal{O}}_0$ is open in $C^0(M', M)$. Moreover if $f \in C_G{}^0(M', M)$ then $Af = f$ and so $\operatorname{im}(Af) = \operatorname{im} f \subseteq M \subseteq U$ so $f \in \tilde{\mathcal{O}}_0$, i. e. $\tilde{\mathcal{O}}_0$ is a neighborhood of $C_G{}^0(M', M)$. We define $\tilde{A}_0: \tilde{\mathcal{O}}_0 \to C_G{}^0(M', M)$ by $\tilde{A}_0(f) = \tilde{\pi}_0(Af)$. Since A restricts to a continuous linear projection $A_r: C^r(M, V) \to C_G{}^r(M', V)$ for $0 \leqq r \leqq k$ by Theorem 1.2 and since $\tilde{\pi}: U \to M$ is an equivariant C^k retraction the proposition follows. Q. E. D.

3.2. COROLLARY. *For $0 \leqq r \leqq k$ $C_G{}^k(M', M)$ is dense in $C_G{}^r(M', M)$.*

Proof. Given $f \in C_G{}^r(M', M)$ choose a sequence $\{f_n\}$ in $C^k(M', M)$ converging to f in $C^r(M', M)$. Since $\tilde{\mathcal{O}}_r$ is a neighborhood of f in $C^r(M', M)$ we can suppose all f_n are in $\tilde{\mathcal{O}}_r$. Then $\tilde{A}_k f_n \in C_G{}^k(M', M)$ and

$$\tilde{A}_k f_n = \tilde{A}_r f_n \to \tilde{A}_r f = f.$$

Q. E. D.

3.3. PROPOSITION. *If M is a C^∞ G-manifold without boundary then $M = \bigcup_{n=1}^{\infty} M_n$ where M_n is an open invariant submanifold of M, $M_n \subset M_{n+1}$, and each $\bar{M}_n$ is a compact C^∞ G-manifold with boundary.*

Proof. Let $f\colon M \to \boldsymbol{R}$ be a C^∞ proper map of M onto the positive reals and let $\bar{f}(x) = \int f(gx)\,dg$, so $\bar{f}\colon M \to \boldsymbol{R}$ is an invariant C^∞ map. Given $c > 0$ $Gf^{-1}([0,c]) = X$ is compact and if $x \notin X$ then clearly $\bar{f}(x) > c$, so $\bar{f}^{-1}([0,c]) \subseteq X$ and so $\bar{f}$ is a proper map. Let $M_n = \bar{f}^{-1}([0,c_n])$ where $\{c_n\}$ is a sequence of regular values of $\bar{f}$ with $c_n \to \infty$ monotonically. Q. E. D.

3.4. THEOREM. *If M is a C^∞ G-manifold without boundary then for any compact C^k G-manifold M' $C_G{}^k(M',M)$ is dense in $C_G{}^r(M',M)$, $0 \leqq r \leqq k$.*

Proof. Represent M as $\bigcup_{n=1}^{\infty} M_n$ as in 3.3. If $f \in C^r(M',M)$ then $f(M') \subset M_n$ for some n since M' is compact, i. e. $C_G{}^r(M',M) = \bigcup_{n=1}^{\infty} C_G{}^r(M',M_n)$, so it will suffice to show that $C_G{}^k(M',M_n)$ is dense in $C_G{}^r(M',M_n)$ for all n, or equivalently we can assume that M is the interior of a compact C^∞ G-manifold $\bar{M}$, possibly with boundary. But then by Theorem 2.1 M admits a C^∞ equivariant embedding $f\colon M \to V$ in a Euclidean G-space V (indeed $\bar{M}$ does) and $x \mapsto \operatorname{im}(Df_x)^\perp$ is a C^∞ equivariant map $g\colon M \to G_m(V)$ where $m = \dim V - \dim M$, so $g^*(\xi_m{}^V)$ is a C^∞ G-normal bundle to f and the Theorem follows from 3.2. Q. E. D.

3.5. TUBULAR NEIGHBORHOOD THEOREM. *If M is a compact C^k G-manifold, $k \geqq 1$, and $f\colon M \to V$ is a C^k equivariant embedding on a linear G-space V then f admits a C^k G-normal bundle and hence if $\partial M = \emptyset$ then $f(M)$ admits a C^k G-tubular neighborhood in V $\tilde{\pi}\colon U \to f(M)$.*

Proof. We can assume V is a Euclidean G-space and we define $f\colon M \to G_m(V)$, $m = \dim V - \dim M$, by $g(x) = \operatorname{im}(Df)_x{}^\perp$, so

$$g \in C_G{}^{k-1}(M, G_m(V)).$$

Since $G_m(V)$ is a C^∞ G-manifold, by 3.4 we can approximate g arbitrarily well in $C_G{}^{k-1}(M, G_m(V))$ by $h \in C_G{}^k(M,V)$. But clearly if h is sufficiently close to g in the C^0-topology then $h(x)$ like $g(x)$ will for each $x \in M$ be a linear complement to $\operatorname{im}(Df_x)$ in V. Then $h^*(\xi_m{}^V)$ is a C^k G-normal bundle to f.

Q. E. D.

4. Fiberwise transversality.

4.1. *Definition.* Let $\pi_1\colon E_1 \to M_1$ and $\pi_2\colon E_2 \to M_2$ be two C^k fiber bundles, $k \geqq 1$, and let G_2 be a C^k sub-bundle of E_2. A C^k map $f\colon E_1 \to E_2$ will be called fiber-wise transversal to G_2 if for each $x \in M_1$ $f \mid (E_1)_x$ is transversal to G_2.

4.2. PROPOSITION. *Given $\pi_1\colon E_1 \to M_1$, $\pi_2\colon E_2 \to M_2$ and G_2 as in 4.1 with E_1 compact, the set of C^k maps $f\colon E_1 \to E_2$ which are fiber-wise transversal to G_2 is open in the C^1 topology.*

Proof. Obvious.

4.3. LEMMA. *Let $\pi\colon E \to M$ be a C^k fiber bundle, $k \geqq 1$ and let G be a compact C^k submanifold of E with $\partial G = \emptyset$. A necessary and sufficient condition for G to be a C^k sub-bundle of E (i.e. for $\tilde{\pi} = \pi \mid G\colon G \to M$ to be a C^k fiber bundle) is that for each $x \in M$ $G_x = G \cap E_x$ be a submanifold of G of codimension equal to the dimension of M.*

Proof. Necessity is trivial. To prove sufficiency it is enough by a well-known theorem of Ehressman [2] to check that when the condition is satisfied $\tilde{\pi}\colon G \to M$ is a submersion, i.e. that for each $y \in G$ $(D\tilde{\pi})_y\colon TG_y \to TM_x$ has rank equal to $\dim M$. Now $\operatorname{rank}(D\tilde{\pi})_y = \dim G - \dim(\ker(D\tilde{\pi})_y)$. Since π is a fibering $\ker(D\pi)_y = T(E_x)_y$ and since $(D\tilde{\pi})_y = (D\pi)_y \mid (TG)_y$,

$$\ker(D\pi)_y = \ker(D\tilde{\pi})_y \cap (TG)_y = T(E_x)_y \cap (TG)_y = T(E_x \cap G)_y = T(G_x)_y,$$

so $\operatorname{rank}(D\tilde{\pi})_y = \dim G - \dim G_x = \operatorname{codim} G_x$ in $G = \dim M$. Q.E.D.

4.4. THEOREM. *Let $\pi_1\colon E_1 \to M_1$, $\pi_2\colon E_2 \to M_2$ be two C^k fiber bundles, $k \geqq 1$, and let G_2 be a C^k sub-bundle of E_2. Assume E_1 is compact, $\partial M_1 = \emptyset$, and $\partial G_2 = \emptyset$. Let $f\colon E_1 \to E_2$ be a C^k map which is fiber-wise transversal to G_2 with $f(\partial E_1) \cap G_2 = \emptyset$. Then $G_1 = f^{-1}(G_2)$ is a C^k sub-bundle of E_1, $\partial G_1 = \emptyset$, and the fiber codimension of G_1 in E_1 equals the fiber codimension of G_2 in E_2.*

Proof. Since f is fiber-wise transversal to G_2 it is a fortiori transversal to G_2 and hence $G_1 = f^{-1}(G_2)$ is a compact C^k submanifold of E_1 of codimension equal to the codimension of G_2 in E_2, and $\partial G_1 = (f \mid \partial E_1)^{-1} G_2 = \emptyset$. Putting $(G_1)_x = (f \mid (E_1)_x)^{-1} G_2 = G_1 \cap (E_1)_x$, $(G_1)_x$ is a submanifold of $(E_1)_x$ of codimension equal to the codimension of G_2 in E_2 which, since G_2 is a sub-bundle of E_2 is the same as the fiber codimension of G_2 in E_2. Now since $\dim E - \dim G = \dim(E_1)_x - \dim(G_1)_x = \dim E_2 - \dim G_2$ it follows that $\dim G - \dim(G_1)_x = \dim E - \dim(E_1)_x = \dim M$, and Lemma 4.3 completes the proof. Q.E.D.

4.5. THEOREM. *Let $\pi_1\colon E_1 \to M_1$ and $\pi_2\colon E_2 \to M_2$ and G_2 be as in Theorem 4.4 and let $f_t\colon E_1 \to E_2$ be a homotopy of C^k maps with each f_t fiberwise transversal to G_2 and all $f_t(\partial E_1) \cap G_2$ empty. Then $f_0^{-1}(G_2)$ and $f_1^{-1}(G_2)$ are C^k equivalent sub-bundles of E_1 and in particular for each $x \in M_1$ their fibers at x are C^k diffeomorphic.*

Proof. Let S^1 denote $[0,1]$ with 0 and 1 identified. It will suffice to construct a C^k fiber bundle over $S^1 \times M_1$ which restricted to $\{0\} \times M_1$ is $f_0^{-1}(G_2)$ and restricted to $\{\frac{1}{2}\} \times M_1$ is $f^{-1}(G_2)$ (because bundles over $[0,\frac{1}{2}] \times M_1$ are always of the form $[0,\frac{1}{2}] \times B$ for some bundle over M_1). By 4.2 we can approximate f_t so that $(t,x) \mapsto f_t(x)$ is C^k without changing f_0 or f_1 and maintaining the other given conditions. By a standard argument we can also suppose that $f_t = f_0$ for $0 \leqq t < \epsilon$ and $f_t = f_1$ for $1 - \epsilon \leqq t \leqq 1$. Define $h_t = f_{2t}$ for $0 \leqq t \leqq \frac{1}{2}$ and $h_t = f_{2-2t}$ for $\frac{1}{2} \leqq t \leqq 1$. Then $(t,x) \mapsto h_t(x)$ is a C^k map of $S^1 \times E_1 \to E_2$ and if we regard $S^1 \times E_1$ and $S^1 \times E_2$ as C^k fiber bundles over $S^1 \times M_1$ and $S^1 \times M_2$ respectively then $H\colon S^1 \times E_1 \to S^1 \times E_2$ defined by $(t,x) \mapsto (t, h_t(x))$ is clearly fiber-wise transversal to $S^1 \times G_2$ and $H(\partial(S^1 \times E_1)) \cap S^1 \times G_2 = \emptyset$ so by 4.4 $H^{-1}(S^1 \times G_2)$ is a C^k sub-bundle of $S^1 \times E_1$. Clearly $H^{-1}(S^1 \times G_2) \mid \{t\} \times M_1$ is just $h_t^{-1}(G_2)$ so taking $t = 0, \frac{1}{2}$ respectively we get $f_0^{-1}(G_2)$ and $f_1^{-1}(G_2)$ respectively as desired. Q.E.D.

4.6. COROLLARY. *Let $\pi_1\colon E_1 \to M_1$, $\pi_2\colon E_2 \to M_2$, G and f be as in Theorem 4.4 and in addition assume that E_1 is compact. Then there is a neighborhood U of f in $C^k(E_1, E_2)$, which is in fact open in the C^1 topology, such that for all $h \in U$, $h^{-1}(G_2)$ is a C^k sub-bundle of E_1 which is C^k equivalent to $f^{-1}(G_2)$.*

Proof. There is a C^0 neighborhood of f in $C^k(E_1, E_2)$ say U_0 such that for all $h \in U_0$ $h(\partial E_1) \cap G_2 = \emptyset$. There is a C^1 neighborhood U_1 of f by 4.2 such that h is fiberwise transversal to G_2 for all $h \in U_1$. Now $C^k(E_1, E_2)$ is locally contractible in the C^1 topology so we can take for U any contractible (or even pathwise connected) C^1-neighborhood of f in $U_0 \cap U_1$. Q.E.D.

4.7. THEOREM. *Let $\pi_1\colon E_1 \to M_1$ and $\pi_2\colon E_2 \to M_2$ be C^k fiber bundles, $k \geqq 1$, with compact total spaces and $\partial M_1 = \partial M_2 = \emptyset$. Let $f_0\colon E_1 \to E_2$ be a C^k map which maps each fiber of E_1 diffeomorphically onto a fiber of E_2. If σ is any C^k section of E_2 with $\sigma \cap \partial E_2 = \emptyset$ then there is a neighborhood U*

of f_0 in $C^k(E_1,E_2)$, which in fact is open in the C^1 topology, such that for every $f\in U$, $f^{-1}(\sigma)$ is a C^k section of E_1 disjoint from ∂E_1. Moreover the map $f\mapsto f^{-1}(\sigma)$ is continuous from $U\subseteq C^k(E_1,E_2)$ into the space $C^k(E_1)$ of C^k sections of E_1 with the C^k topology.

Proof. In 4.6 take $G_2=\sigma$. Since f_0 maps each fiber of E_1 diffeomorphically onto a fiber of E_2 it is clear that f_0 is fiberwise transversal to σ, and hence $\sigma_0=f_0^{-1}(\sigma)$ is a C^k sub-bundle of E_1 of fiber dimension zero, and in fact since $f_0(E_1)_x$ meets σ in exactly one point, σ_0 is a section of E_1. It now follows from 4.6 that $f^{-1}(\sigma)$ is also a C^k sub-bundle of E_1 with fiber a point, i.e. a C^k section of E_1, if $f\in C^k(E_1,E_2)$ is sufficiently near f_0 in the C^1 topology. It remains only to show the continuity of $f\mapsto f^{-1}(\sigma)$ as a map of U into $C^k(E_1)$. Since M_1 is compact it will suffice to show that if $f_n\to f$ then $f_n^{-1}(\sigma)\mid W\to f^{-1}(\sigma)\mid W$ in the C^k topology in some neighborhood W of each point of $x\in M_1$. We can choose coordinates in E_2 near $\sigma(x)$ so that E_2 is identified with $\boldsymbol{R}^k\times\boldsymbol{R}^m$, π_2 with the projection $\boldsymbol{R}^k\times\boldsymbol{R}^m\to\boldsymbol{R}^m$ and σ with $\boldsymbol{R}^k\times\{0\}$. Then choosing coordinates near $\sigma_0(x)$ in E_1 so that E_1 is locally $\boldsymbol{R}^n\times\boldsymbol{R}^m$, π_1 is the projection $\boldsymbol{R}^n\times\boldsymbol{R}^m\to\boldsymbol{R}^n$ and σ_0 is $\boldsymbol{R}^n\times\{0\}$, f_0 is given locally by a map $(x,y)\mapsto(g_0(x),h_0(x,y))$ of $\boldsymbol{R}^n\times\boldsymbol{R}^m\to\boldsymbol{R}^k\times\boldsymbol{R}^m$ with $g_0(0)=0$, $h_0(0,0)=0$ and $y\to h_0(0,y)$ a local C^k diffeomorphism of one neighborhood of 0 in $\boldsymbol{R}^m$ onto another. Given $f\colon\boldsymbol{R}^n\times\boldsymbol{R}^m\to\boldsymbol{R}^k\times\boldsymbol{R}^m$ near f_0, say $f(x,y)=(g(x,y),h(x,y))$, $f^{-1}(\sigma)$ is represented as the unique C^k map $\sigma_f\colon\boldsymbol{R}^n\to\boldsymbol{R}^m$ near zero which solves the equation $h(x,\sigma_f(x))=0$. By the implicit function theorem σ_f and its derivatives through order k depend continuous on h and its derivatives through order k, which in turn of course depend continuously on f in the C^k topology. Q.E.D.

5. Proofs of Theorems A, B, and C. The second conclusion of Theorem A is just Theorem 3.4. The first conclusion is an immediate consequence of the second since the space $\mathrm{Diff}^l(M_1,M_2)$ of C^l diffeomorphisms of M_1 with M_2 is open in $C^l(M_1,M_2)$, and hence if we approximate a C^l equivariant diffeomorphism $f\colon M_1\approx M_2$ well enough (in $C^l(M_1,M_2)$) by $g\in C_G{}^k(M_1,M_2)$ then g will be a C^k equivariant diffeomorphism of M_1 with M_2. This completes the proof of Theorem A.

Now let M_0 be a closed C^k manifold and let $\lambda\colon M_0\to V$ be a C^k embedding of M_0 in some Euclidean space. Let $\pi\colon\nu\to M_0$ be C^k normal bundle to λ and ν_ϵ the ϵ-disc sub-bundle of ν. As usual we identify M_0 with $\lambda(M_0)$ and with the zero section of ν, and for ϵ small enough we identify ν_ϵ with the ϵ-tubular neighborhood of M_0 it defines, namely its image under the map

$v \mapsto \pi(v) + v$. Let $k = \dim V - \dim M$ and let $\xi_k{}^V(\epsilon)$ denote the ϵ-disc bundle of the vector bundle $\xi_k{}^V \to G_k(V)$. Define $f_0 \colon \nu_\epsilon \to \xi_k{}^V(\epsilon)$ by $f_0(v) = (v, \nu_{\pi(v)})$ (the Thom-Pontryagin map) so that clearly for each $x \in M_0$ f_0 maps $\nu_x(\epsilon)$ diffeomorphically onto the fiber of $\xi_k{}^V(\epsilon)$ over ν_x. If σ is the zero section $\xi_k{}^V$ *then* $f_0{}^{-1}(\sigma) = M_0$. Now by 4.7 if $f_1 \colon \nu(\epsilon) \to \xi_k{}^V(\epsilon)$ is a C^k map sufficiently close to f_0 in the C^1 topology then $M_1 = f_1{}^{-1}(\sigma)$ is a C^k section of ν and hence $\pi \mid M_1 \colon M_1 \to M_0$ is a C^k diffeomorphism. In particular if we approximate f_0 by a C^∞ f_1, relative to the natural C^∞ structure for $\xi_k{}^V$ and that for $\nu(\epsilon)$ inherited from V, then M_1 will be a C^∞ submanifold of V mapped C^k diffeomorphically onto M_0 by π.

Now suppose M_0 is a C^k G-manifold. Then by Theorem 2.1 we can suppose λ is an equivariant C^k embedding of M_0 in a Euclidean G-space V and by 3.5 we can suppose $\pi_0 \colon \nu \to M_0$ is a C^k G-normal bundle to λ in V so that $\nu(\epsilon)$ is a C^k G-tubular neighborhood. Finally since f_0 is clearly equivariant we can by Theorem 3.4 assume that our approximating $f_1 \colon \nu(\epsilon) \to \xi_k{}^v(\epsilon)$ is equivariant. Then M_1 is clearly a invariant C^∞ submanifold of V, hence a C^∞ G-manifold, and $\pi \mid M_1$ is of course equivariant. This completes the proof of Theorem B and we pass now to the proof of Theorem C.

Let M be any closed C^∞ manifold, $\alpha_0 \in \mathfrak{A}^k(G, M)$ and denote by M_0 the resulting C^k G-manifold. According to Theorem B we can choose a C^∞ G-manifold M_1 for which there exists a C^k equivariant diffeomorphism $\gamma \colon M_1 \to M_0$. If $\alpha_1 \in \mathfrak{A}^\infty(G, M_1)$ is the given action of G on M_1 then of course $\alpha_0 = \alpha_1{}^{\gamma^{-1}}$, i.e. $\alpha_0(g, x) = \gamma\alpha_1(g, \gamma^{-1}x)$. Given $f \in \mathrm{Diff}^k(M)$ we have

$$\alpha = \alpha_1{}^{(f\gamma)^{-1}} \in \mathfrak{A}^k(G, M), \quad \alpha(g, x) = f\gamma\alpha_1(g, \gamma^{-1}f^{-1}x)$$

and $\alpha^f = \alpha_1{}^{\gamma^{-1}} = \alpha_0$. Clearly $f \mapsto \alpha_1{}^{(f\gamma)^{-1}}$ is continuous from $\mathrm{Diff}^k(M)$ to $\mathfrak{A}^k(G, M)$ so in particular if $f \to e$ in $\mathrm{Diff}^k(M)$ then $\alpha_1{}^{(f\gamma)^{-1}} \to \alpha_0$ in $\mathfrak{A}^k(G, M)$. Thus to complete the proof of Theorem C it will suffice to find $f \in \mathrm{Diff}^k(M)$ arbitrarily close to the identity such that $\alpha_1(f\gamma)^{-1} \in \mathfrak{A}^\infty(G, M)$. But since $\alpha_1 \in \mathfrak{A}^\infty(G, M)$ it is clear that if $f\gamma \colon M_1 \to M$ is C^∞, and hence $(f\gamma)^{-1} \colon M \to M_1$ is C^∞, then $\alpha_1{}^{(f\gamma)^{-1}} \in \mathfrak{A}^\infty(G, M)$. Now $\mathrm{Diff}^\infty(M_1, M)$ is dense in $\mathrm{Diff}^k(M_1, M_2)$, so there exists $g \in \mathrm{Diff}^\infty(M_1, M)$ arbitrarily near γ in $\mathrm{Diff}^k(M_1, M)$. Then $f = g\gamma^{-1}$ is arbitrarily near the identity in $\mathrm{Diff}^k(M_1, M_2)$ and $f\gamma = g$ is C^∞.

Q.E.D.

5.1. THEOREM. *If M is a closed C^k G-manifold, $k \geqq 1$, and M' is any compact C^k G-manifold, then $C_G{}^k(M', M)$ is dense in $C_G{}^r(M', M)$ for $0 \leqq r \leqq k$.*

Proof. By Theorem B we can give M a C^∞ structure which makes it a C^∞ G-manifold, and the theorem then follows from 3.4.

Definition. A C^k manifold M is called *C^k-asymmetric* if there is no non-trivial C^k-action of any compact connected Lie group on M, or equivalently if there is no effective C^k action of the circle group S^1 on M.

5.2. THEOREM. *Let M be a closed C^∞ manifold and let $1 \leqq l < k \leqq \infty$. If M is C^k-asymmetric then M is C^l-asymmetric.*

Proof. Immediate from Theorem C.

It would be interesting to know whether we can take $l = 0$ in Theorem 5.2. In particular in a paper to appear Atiyah and Hirzebruch have shown that if M is a closed, orientable $4k$ dimensional manifold with $w_2(M) = 0$, then M is C^∞-asymmetric (hence C^1-asymmetric) provided the $\hat{A}$-genus of M (a topological invariant!) is non-zero. It would be quite remarkable if such M could admit non-trivial C^0-circle actions.*

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* *Added in Proof*: Atiyah has in fact constructed just such a remarkable example.

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