even, these arcs comprise of $\frac{1}{2}q$ *circles. If* arg A *is not a rational multiple of* 2π, *the* $\{z_n\}$ *lie on a spiral through* z_1 *and the fixed points.*

(c) *If* $A \neq 1$ *is a root of unity,* $\{z_n\}$ *is a finite set on the generalized circle through* z_1 *orthogonal to the pencil of circles through the fixed points.*

(d) *In the remaining case, if* $A \neq 1$, $\{z_n\}$ *is a set densely covering the circle mentioned in* (c).

By the remarks preceding Theorem 2 and 3 we have:

THEOREM 5. (a) *If* (2) *has one fixed point,* $\lim_{n\to\infty} z_n$ *exists and equals the fixed point. The* $\{z_n\}$ *lie in general on two different generalized circles tangent at the fixed point, one through* z_1 *and the other through* z_2. *The* z_n *with even affix lie on one circle and the others on the second.*

(b) *If* (2) *has two fixed points,* $\lim_{n\to\infty} z_n$ *exists and equals one of the fixed points. The* $\{z_n\}$ *lie on two different generalized circles through the fixed points. The* $\{z_n\}$ *with even affix lie on one circle and the others on the second.*

(c) *If* (2) *is noninvolutory and has no fixed points the* $\{z_n\}$ *lie in general on two generalized circles orthogonal to the pencil of circles through the semi-fixed points. For odd* n *the circle passes through* z_1 *and for even through* z_2. *The density of the* $\{z_n\}$ *on the circles depends in an obvious manner on* $h/\bar{h}$.

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MATHEMATICAL NOTES

THE CLASSIFICATION OF REAL DIVISION ALGEBRAS

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Let D be a finite dimensional division algebra over the field $\mathbf{R}$ of real numbers. One way of stating the fundamental theorem of algebra is to say that if D is commutative (i.e. a field) then D is isomorphic over $\mathbf{R}$ to either $\mathbf{R}$ or the field $\mathbf{C}$ of complex numbers. A famous theorem of Frobenius asserts that if we allow D to be noncommutative then there is only one new possibility: D can be isomorphic over $\mathbf{R}$ to the quaternion algebra of Hamilton. This is an algebra $\mathbf{H}$ of dimension four generated as a vector space by basis elements 1, i, j, k which satisfy the multiplication table

$$i^2 = j^2 = k^2 = -1; \quad ij = -ji = k; \quad jk = -kj = i; \quad ki = -ik = j.$$

The proofs of Frobenius' theorem in the literature seem to be of two types.

Either they are elementary, but rather computational, e.g. [2], or else they deduce the theorem from sophisticated general results about division algebras, e.g. [1]. We wish to give here a short, self-contained proof which seems both elementary and conceptual. Besides the inevitable use of the fundamental theorem of algebra we use only the simplest facts about the eigenvalues of linear transformations.

Before starting the proof we note that the two-dimensional subspace of $\mathbf{H}$ generated by 1 and i is isomorphic to the complex numbers. If we denote it by $\mathbf{C}$ then $\mathbf{H}$ becomes a vector space over $\mathbf{C}$ (using left multiplication for the scalar operations). Moreover $\mathbf{C}$ is clearly $\{x\in D \mid ix=xi\}$, while the complementary two-dimensional space spanned by j and k is just $\{x\in D \mid ix=-xi\}$. It is this observation which motivates the proof.

Let 1 denote the unit of D. As usual we can think of $\mathbf{R}$ as embedded in D via the map $x\to x\cdot 1$. We may assume $D\neq\mathbf{R}$. Let d be any element of D not in $\mathbf{R}$ and let $\mathbf{R}\langle d\rangle$ denote the two-dimensional subspace $\mathbf{R}+\mathbf{R}d$ spanned by 1 and d. We claim:

(1) $\mathbf{R}\langle d\rangle$ *is a maximal commutative subset of* D, *consisting of all the elements of* D *which commute with* d. *Moreover it is a field isomorphic to* $\mathbf{C}$.

Proof. Choose a subspace F of D of maximal dimension which includes $\mathbf{R}\langle d\rangle$ and is commutative. If $x\in D$ commutes with everything in F then $F+\mathbf{R}x$ is commutative and so must equal F, so $x\in F$ proving that F is a maximal commutative subset of D. In particular, if $x\neq 0$ is in F then x^{-1} commutes with everything in F (because $xy=yx\Rightarrow yx^{-1}=x^{-1}y$) so $x^{-1}\in F$ and F is a field. By the fundamental theorem of algebra F is isomorphic over $\mathbf{R}$ to $\mathbf{C}$. In particular, F has dimension two so that $F=\mathbf{R}\langle d\rangle$. Finally, if $x\in D$ commutes with d it commutes with everything in $\mathbf{R}\langle d\rangle=F$, hence belongs to F.

According to (1) we can select an element $i\in D$ such that $i^2=-1$ and we may identify $\mathbf{R}\langle i\rangle$ with $\mathbf{C}$. We can now view D not merely as a vector space over $\mathbf{R}$, but also as a vector space (of half the dimension) over $\mathbf{C}$ as well, the scalar operations of $\mathbf{C}$ on D being given by multiplication on the left. On the other hand multiplication on the *right* by i can then be interpreted as a (complex) linear transformation T on the (complex) vector space D; i.e., we define

(2) $Tx\equiv xi$.

Since $T^2=-(\text{identity})$, the only possible eigenvalues of T are $+i$ and $-i$; denote by D^+ and D^- the corresponding eigenspaces:

(3) $D^+=\{x\in D \mid xi=ix\}$, $D^-=\{x\in D \mid xi=-ix\}$.

Of course $D^+\cap D^-=\{0\}$. We claim moreover

(4) $D=D^+\oplus D^-$.

This follows immediately from the decomposition $x=\frac{1}{2}(x-ixi)+\frac{1}{2}(x+ixi)$ for all $x\in D$, the two summands being respectively in D^+ and D^- as one checks by (3). We next note that

(5) $D^+=\mathbf{C}$ and $x, y\in D^-\Rightarrow xy\in D^+$.

The first statement is immediate from (1), the second from (3).

If $D^- = 0$ then by (4) and (5) we have $D = \mathbf{C}$, so let us assume $D^- \neq 0$ and show that D must be isomorphic to $\mathbf{H}$. First of all the real dimension of D must be four, i.e., its complex dimension must be two. This follows from (4), (5) and

(6) $\dim \mathbf{C} D^- = 1$.

Proof. Select any nonzero $\alpha \in D^-$. Then right multiplication by α gives a complex linear transformation on D which is nonsingular (its inverse is right multiplication by α^{-1}), and it interchanges D^+ and D^- by (5) so $\dim_{\mathbf{C}} D^- = \dim_{\mathbf{C}} D^+ = 1$. Moreover

(7) $\alpha^2 \in \mathbf{R}$ and $\alpha^2 < 0$.

Proof. Since by (1) $\mathbf{R}\langle\alpha\rangle$ is a field it contains α^2. But also $\alpha^2 \in \mathbf{C}$ by (5) and therefore $\alpha^2 \in \mathbf{C} \cap \mathbf{R}\langle\alpha\rangle = \mathbf{R}$. If $\alpha^2 > 0$ it would have two square roots in $\mathbf{R}$ hence three square roots in the field $\mathbf{R}\langle\alpha\rangle$ which is impossible by field theory (or more concretely here, because $\mathbf{R}\langle\alpha\rangle \simeq \mathbf{C}$).

By (7) a suitable positive multiple of α is an element j of D^- satisfying $j^2 = -1$. Define $k = ij$, so that by (6), j and k form a basis for D^- over $\mathbf{R}$, and hence by (4) the elements $1, i, j, k$ form a basis for D over $\mathbf{R}$. Since $j, k \in D^-$, they anticommute with i. This together with $i^2 = j^2 = -1$ and $k = ij$ show that $1, i, j, k$ satisfy the multiplication table given above for the quaternions.

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APPROXIMATE FUNCTION VALUES AND HYPERPLANES

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By using an appropriate Helly-type theorem [1] it is possible to give a characterization of hyperplanes through a condition involving approximate function values on arbitrary finite point sets. The development gives a result that generalizes a theorem of Karlin and Shapley [2] in that it removes a condition on boundedness from the hypothesis.

1. Preliminaries. General geometric terminology of n-dimensional euclidean space, R^n, is used, and particular concepts are noted for clarity. When a function F, of n variables, is specified to be between two given functions I and S, over a domain D, we say that *approximate values* of the function are given. That is, for all $\mathbf{x} = (x_1, x_2, \cdots, x_n)$ in D, $I(\mathbf{x}) \leqq F(\mathbf{x}) \leqq S(\mathbf{x})$. For convenience, call the closed segment from the point $(\mathbf{x}, I(\mathbf{x}))$ to the point $(\mathbf{x}, S(\mathbf{x}))$, in R^{n+1} an *approximate point*

$$\mathcal{P}(\mathbf{x}) = \{(\mathbf{x}, z) : I(\mathbf{x}) \leqq z \leqq S(\mathbf{x})\}.$$

Also for a specified class $\Phi = \{\phi_1, \phi_2, \cdots, \phi_k\}$ of linearly independent functions, term $L(A, \mathbf{x}) = \sum_{i=1}^{k} a_i\phi_i(\mathbf{x})$ an (linear) *approximating function* from the Φ class established by the parameter set $A = \{a_1, a_2, \cdots, a_k\}$. In case a particular