A Covering Homotopy Theorem and the Classification of G-Spaces
Author(s): Richard S. Palais
Reviewed work(s):
Source: *Proceedings of the National Academy of Sciences of the United States of America*, Vol. 45, No. 6 (Jun. 15, 1959), pp. 857-859
Published by: National Academy of Sciences
Stable URL: http://www.jstor.org/stable/89904
Accessed: 12/11/2012 14:42

This also holds if f is an L_1-limit of our analytic functions and $b = \int f d\sigma$.

THEOREM 4. *Let $w(x)$ be non-negative and summable. It has a representation*

$$w = |b + \chi|^2, \qquad b \neq 0$$

where χ is in the L_2-closure of Φ if and only if

$$\int \log w \, d\sigma > -\infty.$$

THEOREM 5. *Let $f \in L_1$ and*

$$f = a + \chi, \qquad a \neq 0,$$

where χ is in the L_1-closure of Φ. There exist functions $g = b + \chi_1$, $h = c + \chi_2$ with χ_1, χ_2 in the L_2-closure of Φ such that $f = g \cdot h$.

THEOREM 6 (Beurling). *Let H be the family*

$$a + \chi$$

with χ in the L_2 closure of Φ. For $f \in H$ let C_f be the smallest closed linear manifold containing

$$(b + \varphi)f$$

for all constants b and all $\varphi \in \Phi$, then $C_f = H$ if and only if

$$\int \log |f| d\sigma = \log \left| \int f d\sigma \right| > -\infty.$$

* This research was supported by the United States Air Force through the Office of Scientific Research of the Air Force Research and Development Command.

[1] Bochner, S., "Additive Set Functions on Groups," *Annals of Math.*, **40**, (1939) 769–799 esp. pp. 791–793.

[2] Dunford, N., and J. T. Schwartz, *Linear Operators I* (Interscience Publishers, 1958), 525.

[3] Helson, H., and D. Lowdenslager, "Prediction Theory and Fourier Series in Several Variables," *Acta Mathematica*, **99**, 165–202, (1958).

A COVERING HOMOTOPY THEOREM AND THE CLASSIFICATION OF G-SPACES

BY RICHARD S. PALAIS

THE INSTITUTE FOR ADVANCED STUDY

Communicated by Deane Montgomery, April 3, 1959

In this note we shall describe a classification theory for compact Lie transformation groups with a finite number of orbit types. The classification is analogous to the classification of principal bundles in terms of universal bundles and their classifying spaces, the latter in fact being the special case in which there is only one orbit type which is equivalent to the group acting on itself by translation. It is expected that details and proofs will appear in a Memoir of the American Mathematical Society.

Let G be a compact Lie group with identity e. We write $H \subseteq G$ to mean that H is a closed subgroup of G, and we write (H) for the set $\{gHg^{-1} | g \in G\}$ of subgroups of G conjugate to H. Such a set we call a *G-orbit type*. A *G-space* is a locally compact, second countable, Hausdorff space with a fixed action of G on X. We write Gx for the orbit of $x \in X$ under G and G_x for the isotropy group at x. The set of all isotropy groups on a single orbit is an orbit type which we call the type of the orbit. We write $X_{(H)}$ for the union of all orbits in X of type (H) and $\tilde{X}_{(H)}$ for the set orbits of X of type (H). If X/G is the orbit space of X (a locally compact, second countable Hausdorff space) and Π_X the natural map of $X \to X/G$ then $\tilde{X}_{(H)} = \Pi_X(X_{(H)})$. A map f of a G-space X into a G-space Y is called *isovariant* if it is equivariant (i. e., $f(gx) \equiv gf(x)$) and one-to-one on each orbit. For such a map we have $G_{f(x)} \equiv G_x$ so that $f(X_{(H)}) \subseteq Y_{(H)}$ and $\tilde{f}(\tilde{X}_{(H)}) \subseteq \tilde{Y}_{(H)}$ where $\tilde{f}:X/G \to Y/G$ is the induced map which satisfies $\tilde{f}\Pi_X = \Pi_Y f$. We make the product $X \times I$ of a G-space X with the unit interval into a G-space by $g(x, t) = (gx, t)$ and a homotopy $f:X \times I \to Y$ of an isovariant map $f_0:X \to Y$ is called an *isovariant* homotopy of f_0 if it is isovariant in this G-space structure. With a natural identification $(X \times I)/G = X/G \times I$, and the induced map $\tilde{f}$ of an isovariant homotopy of f_0 (which is clearly a homotopy of $\tilde{f}_0$) clearly satisfies $\tilde{f}(\tilde{X}_{(H)} \times I) \subseteq \tilde{Y}_{(H)}$. The covering homotopy theorem says that this condition is also sufficient for a homotopy $\tilde{f}$ of $\tilde{f}_0$ to be the induced map of an isovariant homotopy of f_0.

COVERING HOMOTOPY THEOREM. *Let X and Y be G-spaces and $f_0:X \to Y$ an isovariant map. If $\tilde{f}:X/G \times I \to Y/G$ is a homotopy of the induced map $\tilde{f}_0$ of f_0 which satisfies $\tilde{f}(\tilde{X}_{(H)} \times I) \subseteq \tilde{Y}_{(H)}$ for all G-orbit types (H) then $\tilde{f}$ is the induced map of an isovariant homotopy of f_0.*

Now let Σ be a finite collection of G-orbit types. We assume henceforth that all G-spaces X satisfy $X = \cup\{X_{(H)} | (H) \in \Sigma\}$, and hence are Σ-spaces, as are their orbit spaces, in the following sense. A *Σ-space* is a locally compact, second countable space Z with a fixed partition $\{Z_{(H)}\}$ into subsets indexed by Σ. We make $Z \times I$ into a Σ-space by $(Z \times I)_{(H)} = Z_{(H)} \times I$. A map f of a Σ-space Z into a Σ-space X is called a *Σ-map* if $f(Z_{(H)}) \subseteq X_{(H)}$ for all $(H) \in \Sigma$, and a Σ-equivalence if, in addition, it is a homeomorphism of Z onto X. If f_0 and f_1 are two Σ-maps of Z into X they are called *strongly Σ-homotopic* if there is a Σ-map $f:Z \times I \to X$ which is a homotopy of f_0 with f_1. If f_0 is strongly Σ-homotopic to $f_1 \circ h$ where h is some Σ-equivalence of Z with itself then f_0 and f_1 are called *weakly Σ-homotopic*. A *G-space over the Σ-space* Z is a triple (Y, Z, h) where Y is a G-space and h is a Σ-equivalence of Y/G with Z. It is called *weakly equivalent* to a second G-space over Z (Y', Z, h') if there is an equivariant homeomorphism f of Y onto Y', and *strongly equivalent* (to Y', Z, h') if, in addition, f can be so chosen that $h' \circ f \circ h^{-1}$ is the identity map of Z.

If Z is a Σ-space, X a G-space and $f^*:Z \to X/G$ a Σ-map we define a G-space (Y, Z, h) over Z denoted by $f^{*-1}(X)$ and called the *G-space over Z induced by f^** as follows:

$$
\begin{aligned}
Y &= \{(x, z) \in X \times Z | \pi_X(x) = f^*(z)\} \\
g(x, z) &= (gx, z) \\
h \circ \pi_X(x, z) &= Z.
\end{aligned}
$$

It follows from the Covering Homotopy Theorem that

THEOREM. f^{*-1} (X) *depends to within weak (strong) equivalence only on the weak (strong) Σ-homotopy class of f^*.*

If n is a positive integer then a G-space X is called (Σ, n)-*universal* if given any isovariant map f of a closed invariant subspace of a G-space Y into X there exists an extension of f to an isovariant map of Y into X, provided dim $(Y/G) \leq n$.

CLASSIFICATION THEOREM. *Let X be a* (Σ, $n + 1$)*-universal G-space and let Z be a Σ-space of dimension $\leq n$. Then the map which takes the strong (weak) Σ-homotopy class of $f^* \rightarrow$ strong (weak) equivalence class of $f^{*-1}(X)$ is a one-to-one correspondence between the strong (weak) Σ-homotopy classes of Σ-maps of Z into X/G and the strong (weak) equivalence classes of G-spaces over Z.*

To give content to the above theorem we must be able to construct (Σ, n)-universal G-spaces. This is done as follows. Given G-spaces $X_1, \ldots, X_n$ let $X_1 \circ \ldots \circ X_n$ denote their join (see J. Milnor, The construction of universal bundles II, *Annals of Math.*, **63,** no. 3, May, 1956) made into a G-space by $g(t_1x_1, \ldots, t_nx_n) = (t_1gx_1, \ldots, t_ngx_n)$. We define the *reduced join* $X_1 * \ldots * X_n$ of $X_1, \ldots X_n$ to be the invariant subset of their join consisting of those points $(t_1x_1, \ldots, t_nx_n)$ for which the set of isotropy groups G_{x_i} for which $t_i \neq 0$ has a smallest element under inclusion. We denote the k-fold join (reduced join) of a space X by $X^{(ok)}(X^{(*k)})$. Let $\Sigma = ((H_1), \ldots, (H_m))$ and let n be any positive integer. It follows from Lemma 2.3 (Milnor, *loc. cit.*) that we can find integers $k_1, \ldots, k_m$ such that $(N(H_i)/(H_i)^{(ok_i)}$ is n-connected, where $N(H_i)$ is the normalizer of H_i in G. Then

THEOREM. $(G/H_1)^{(*k_1)} * \ldots * (G/H_m)^{(*k_m)}$ *is* (Σ, n)*-universal.*

ORTHOGONAL LATIN SQUARES

BY E. T. PARKER

REMINGTON RAND UNIVAC, SPERRY RAND CORPORATION, ST. PAUL, MINNESOTA

Communicated by A. A. Albert, April 20, 1959

1. *Introduction.*—The principal purpose of this note is to display the first pair of orthogonal latin squares of order 10. A *latin square* of order n is an $n \times n$ matrix with exactly n distinct symbols, each symbol in each row (necessarily only once) and each symbol in each column (once). Two latin squares of order n are called *orthogonal* if each ordered pair of symbols occurs (once) in some cell, the first symbol in the first latin square and the second symbol in the second latin square. More generally, several latin squares of order n are a *mutually* (or *pairwise*) *orthogonal* set if each pair of latin squares is orthogonal. Unlike the somewhat similar magic squares, orthogonal latin squares are today no isolated combinatorial curiosity. For a set of $n - 1$ mutually orthogonal latin squares of order n is equivalent to an affine plane of order n. In turn, an affine plane determines a projective plane of the same order; and a projective plane is the completion of at least one affine plane of like order. The nonspecialist is referred to expository papers[1-3] and their bibliographies.

Eulre[4] introduced the concept of pair of orthogonal latin squares (also called Greco-Latin squares). He obtained several results, and conjectured that no pair exists of any order $n \equiv 2 \pmod 4$. Tarry[5] demonstrated by a lengthy case-by-case argument the truth of Euler's conjecture for order 6; his result made the conjecture