DEFORMATIONS OF COMPACT DIFFERENTIABLE TRANSFORMATION GROUPS.*

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Let G be a Lie group and M a differentiable $(=C^r, r\geqq 1)$ manifold. We recall that a differentiable action of G on M is a differentiable map $\varphi\colon G\times M\rightarrow M$ such that $\varphi(g_1,\varphi(g_2,p))\equiv\varphi(g_1g_2,p)$, and if e is the identity of G, then $\varphi(e,p)\equiv p$. By a *deformation* of a differentiable action φ of G on M we mean a one-parameter family φ_t $(t\in I=[0,1])$ of differentiable actions of G on M such that $\varphi_0=\varphi$ and the map $(g,p,t)\rightarrow\varphi_t(g,p)$ of $G\times M\times I$ into M is continuous. If the latter map is differentiable, we say that the deformation φ_t is differentiable. Recall that a deformation of M is a one-parameter family ψ_t $(t\in .I)$ of diffeomorphisms of M such that ψ_0 is the identity and $(p,t)\rightarrow\psi_t(p)$ is a continuous map of $M\times I$ into M. If the latter map is differentiable, then ψ_t is called a differentiable deformation of M. Given a differentiable action φ of G on M and a [differentiable] deformation ψ_t of M we define a [differentiable] deformation φ_t of φ by $\varphi_t(g,p)=\psi_t(\varphi(g,\psi_t^{-1}(p)))$. We call a [differentiable] deformation of φ [*differentiably*] *trivial* if it can be expressed in this form. Recall that two differentiable actions φ_0 and φ_1 of G on M are called equivalent if there exists a diffeomorphism ψ of M such that $\varphi_1(g,p)\equiv\psi(\varphi(g,\psi^{-1}(p)))$. Thus if a deformation φ_t of an action φ is trivial, then φ is equivalent to φ_t for all $t\in I$. Now it is an easy consequence of a theorem proved by one of the authors [1, Theorem 1] that every differentiable action of a Lie group on a Euclidean space with at least one stationary point can be differentiably deformed into a linear action. On the other hand, there is an example of a differentiable (in fact real analytic) action of the circle group on five dimensional Euclidean space which is known not to be equivalent to a linear action. Also, it is a trivial observation that all actions of the line on the torus defined by left translating by a one parameter subgroup are differentiably deformable into each other. However, they clearly are not all equivalent, some being periodic and others not. Thus from the hypothesis that G is compact or that M is

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compact we cannot conclude that all differentiable deformations of an action of G on M are trivial. However, we will show that if both G and M are compact, then every differentiable deformation of a differentiable action of G on M is in fact differentiably trivial. First, however, we must recall a fairly well-known relation that exists between differentiable deformations of manifolds and differentiable time dependent vector fields. A differentiable time dependent vector field on M is a family $X(t)$ $(t \in I)$ of vector fields on M such that $(p,t) \to X(t)_p$ is a differentiable map of $M \times I$ into the tangent bundle of M (equivalently, in local coordinates the components of $X(t)_p$ are jointly differentiable in t and the coordinates of p). A differentiable path $\gamma\colon I \to M$ is called an integral curve of $X(t)$ if its tangent at t is $X(t)_{\gamma(t)}$ for all $t \in I$. This is equivalent to the requirement that $t \to (\gamma(t), t)$ be an integral curve of the vector field X^* on $M \times I$ given by $X^*_{(p,t)} = (X(t)_p, D_t)$. If ψ_t is a deformation of M, then we define a differentiable time dependent vector field $X(t)$ on M by $X(t)_{\psi_t(p)} =$ tangent to $s \to \psi_s(p)$ at $s = t$. The uniqueness theorem for ordinary differential equations implies that ψ_t can be recovered from X, for $t \to \psi_t(p)$ is the integral curve of X starting at p. If M is not compact, then not every differentiable time dependent vector field $X(t)$ arises in this way; in fact, there will not in general be an integral curve of $X(t)$ starting at an arbitrary point p of M and defined for the whole unit interval. However, if M is compact, this pathology cannot arise and every differentiable time dependent vector field on M generates a differentiable deformation of M. We now prove

THEOREM. *If φ is a differentiable action of a compact Lie group G on the compact differentiable manifold M, then any differentiable deformation of φ is differentiably trivial.*

Proof. If φ_t is a differentiable deformation of φ, then $\Phi\colon G \times M \times I \to M \times I$ defined by $\Phi(g,(p,t)) = (\varphi_t(g,p),t)$ is clearly a differentiable action of G on $M \times I$. If we write Φ_g for the diffeomorphism $(p,t) \to \Phi(g,(p,t))$ of $M \times I$ and π for the projection of $M \times I$ on I, then $\pi\Phi_g = \pi$ for all $g \in G$. Let Δ denote the vector field $(0,D)$ on $M \times I$. Since G is compact, we can "average" Δ over G, i.e. form the vector field Δ^* defined by $\Delta^*_{(p,t)} = \int \delta\Phi_g(\Delta_{\Phi_g^{-1}(p,t)})\,d\mu(g)$ where μ is Haar measure on G. Since Φ is differentiable, Δ^* is a differentiable vector field. By invariance of Haar measure Δ^* is invariant, in the sense that $\delta\Phi_g(\Delta^*_{(p,t)}) = \Delta^*_{\Phi_g(p,t)}$. Because $\delta\pi\delta\Phi_g = \delta\pi$ for all $g \in G$ and $\delta\pi(\Delta_{\Phi_g^{-1}(p,t)}) = D_t$, it follows that $\delta\pi(\Delta^*_{(p,t)}) = D_t$, and therefore $\Delta^*_{(p,t)} = (X(t)_p, D_t)$ for some differentiable time dependent vector field $X(t)$ on M. If ψ_t is the corresponding differentiable deformation of M, then $(\psi_t(p), t)$ is the integral curve of Δ^* starting at $(p, 0)$. Since Δ^* is invariant, it follows that $\Phi_g(\psi_t(p), t)$ is the integral curve of Δ^* starting at $\Phi_g(p, 0) = (\varphi_0(g, p), 0) = (\varphi(g, p), 0)$. But $\Phi_g(\psi_t(p), t) = (\varphi_t(g, \psi_t(p)), t)$ and the integral curve of Δ^* starting at $(\varphi(g, p), 0)$ is $(\psi_t(\varphi(g, p)), t)$, so $\varphi_t(g, \psi_t(p)) = \psi_t(\varphi(g, p))$. Replacing p by $\psi_t^{-1}(p)$ we get $\varphi_t(g, p) = \psi_t(\varphi(g, \psi_t^{-1}(p)))$. q. e. d.

The notion of a deformation of a general group action on a space can be defined, of course, exactly as above replacing all hypothesis of differentiability by continuity, diffeomorphism by homeomorphism, etc. However, the above theorem is false then even under the most stringent demands short of the assumptions of the theorem (indeed "almost all" actions of compact groups on a sphere can be deformed to linear actions). In a sense this is fortunate since we know that the conclusion of the theorem, i. e. two actions that can be deformed into one another are equivalent by conjugation, is too demanding. Saying that two actions φ, φ' are D-equivalent if there is a deformation φ_t such that $\varphi_0 = \varphi$, $\varphi_1 = \varphi'$ provides us with a partition of all action of a compact group G on M which is conceivably more convenient for many of the theorems of transformation groups. Assuming that the group G is compact and the space M on which it acts is a compact, generalized manifold the following problems seem reasonable:

(1) If φ and φ' are D-equivalent actions of G on M, are the orbit spaces of the same homotopy type?

(2) If φ and φ' are D-equivalent, what is the relation of their orbit structures? In particular, are the fixed point sets of the same homotopy type?

It is not even clear in (2) that if φ has fixed points, then the same must be true of φ'.

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REFERENCES.

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