# EQUIVALENCE OF NEARBY DIFFERENTIABLE ACTIONS OF A COMPACT GROUP

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In this note we will be concerned with the proof and consequences of the following fact: if $\phi_0$ is a differentiable action of a compact Lie group on a compact differentiable manifold $M$, then any differentiable action of $G$ on $M$ sufficiently close to $\phi_0$ in the $C^1$-topology is equivalent to $\phi_0$.

1. **Notation.** In what follows differentiable means class $C^\infty$. If $M$ and $V$ are differentiable manifolds, $\mathfrak{M}(M, V)$ is the space of differentiable maps of $M$ into $V$ in the $C^K$-topology where $K$ is a positive integer or $\infty$ fixed throughout. We denote by Diff $(M)$ the group of automorphisms of $M$ topologized as a subspace of $\mathfrak{M}(M, M)$. As such it is a topological group. $\mathfrak{D}(M)$ is the subgroup of Diff $(M)$ consisting of diffeomorphisms which are the identity outside of some compact set and $\mathfrak{D}_0(M)$ is the arc component of $i_M$, the identity map of $M$, in $\mathfrak{D}(M)$. If $M$ is compact $\mathfrak{D}(M)$ is locally arcwise connected and $\mathfrak{D}_0(M)$ is open in $\mathfrak{D}(M)$ and in fact in $\mathfrak{M}(M, M)$. For a definition of the $C^K$-topology and a proof of the statements made above, see [6]. If $G$ is a Lie group we denote by $\mathfrak{A}(G, M)$ the space of differentiable actions of $G$ on $M$, i.e. continuous homomorphisms of $G$ into Diff $(M)$, topologized with the compact-open topology. If $\phi: g \to g^\phi$ is an element of $\mathfrak{A}(G, M)$ then by a theorem of D. Montgomery [2] $\tilde{\phi}: (g, m) \to g^\phi m$ is an element of $\mathfrak{M}(G \times M, M)$. Given $\phi \in \mathfrak{A}(G, M)$ and $f \in$ Diff $(M)$ then $\phi$ composed with the inner automorphism of Diff $(M)$ defined by $f$ is another element $f\phi$ of $\mathfrak{A}(G, M)(g^{f\phi} = fg^\phi f^{-1})$. Clearly $(f, \phi) \to f\phi$ is jointly continuous[2] and defines an action of Diff $(M)$ on $\mathfrak{A}(G, M)$. We henceforth consider $\mathfrak{A}(G, M)$ as a Diff $(M)$-space and, *a fortiori* as a $\mathfrak{D}(M)$ and $\mathfrak{D}_0(M)$-space. Note that the orbit space $\mathfrak{A}(G, M)/$Diff $(M)$ is just the set of equivalence classes of actions of $G$ on $M$.

2. **Statement of main theorem and consequences.** The following theorem will be proved in §3.

THEOREM A. *If $M$ is a compact differentiable manifold and $G$ is a compact Lie group then the $\mathfrak{D}_0(M)$-space $\mathfrak{A}(G, M)$ admits local cross sections; i.e. given $\phi_0 \in \mathfrak{A}(G, M)$ there is a neighborhood $U$ of $\phi_0$ in*

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[2] This follows from the proposition in [6, §1].

*$\mathfrak{A}(G, M)$ and a continuous map $\chi$: $U \to \mathfrak{D}_0(M)$ such that $\chi(\phi_0) = i_M$ and $\chi(\phi)\phi_0 = \phi$.*

COROLLARY 1. *If $\phi_t$ is a continuous arc in $\mathfrak{A}(G, M)$ then there is a continuous arc $f_t$ in $\mathfrak{D}_0(M)$ such that $f_0 = i_M$ and $\phi_t = f_t\phi_0$.*

REMARKS. Corollary 1 was proved in [7] by the author and T. E. Stewart under the added hypothesis that $(g, m, t) \to \tilde{\phi}_t(g, m)$ was jointly differentiable in all three variables. It was shown there by counter-example that Corollary 1 is invalid if we consider continuous rather than differentiable actions or if we drop either of the conditions that $G$ or $M$ be compact. It follows that all these conditions are also necessary for the validity of Theorem A.

Using that $\mathfrak{D}_0(M)$ is locally arcwise connected:

COROLLARY 2. *$\mathfrak{A}(G, M)$ is locally arcwise connected. If $\phi_0 \in \mathfrak{A}(G, M)$ then its orbit under $\mathfrak{D}_0(M)$ is its arc component in $\mathfrak{A}(G, M)$ hence an open set, and its orbit under $\mathfrak{D}(M)$ (i.e. the class of actions equivalent to $\phi_0$) is also open and so a union of arc components. Moreover if $\Delta = \{f \in \mathfrak{D}(M) \mid f\phi_0 = \phi_0\}$ is the group of automorphisms of the differentiable G-space $(M, \phi_0)$ then $f\Delta \to f\phi_0$ is a homeomorphism of $\mathfrak{D}(M)/\Delta$ onto $\mathfrak{D}(M)\phi_0$.*

Since $\mathfrak{A}(G, M)$ is separable metric and each equivalence class is open:

COROLLARY 3. *There are at most countably many inequivalent differentiable actions of $G$ on $M$.*

REMARKS. It seems likely that by modifying a construction of R. Bing [1] one could construct uncountably many continuous actions of $Z_2$ on $S^3$ with fixed point sets pairwise inequivalently embedded 2-spheres. These actions would of course all be inequivalent.

The following extension theorem generalizes Theorem A. On the other hand it is an easy consequence of Theorem A above and Theorem B of [6].

THEOREM B. *Let $H$ be a Lie group, $W$ a differentiable manifold (neither necessarily compact), $G$ a compact subgroup of $H$, and $M$ a compact submanifold of $W$. Let $\psi_0 \in \mathfrak{A}(H, W)$ such that $M$ is invariant under $\psi_0 | G$ and let $\phi_0 \in \mathfrak{A}(G, M)$ be the induced action of $G$ on $M$. Then given any neighborhood $\mathfrak{O}$ of $M$ in $W$ there exists a neighborhood $U$ of $\phi_0$ in $\mathfrak{A}(G, M)$ and a map $\psi$: $U \to \mathfrak{A}(H, W)$ such that $\psi(\phi_0) = \psi_0$, $\psi(\phi) | G$ leaves $M$ invariant and induces $\phi$ on $M$, and $\psi(\phi)$ agrees with $\psi_0$ outside $\mathfrak{O}$. In fact there is a continuous map $\chi$: $U \to \mathfrak{D}_0(W)$ such that $\chi(\phi)$ is the identity outside $\mathfrak{O}$ and such that $\psi(\phi) = \chi(\phi)\psi_0$ satisfies the above conditions.*

3. **Proof of Theorem A.** By a theorem proved independently by the author [5] and G. D. Mostow [4] there exists an orthogonal representation $g \to g^{\psi}$ of $G$ in a Euclidean vector space $V$ and a differentiable $\phi_0$-equivariant embedding $f_0: M \to V$. Let $\mathcal{O}$ be a tubular neighborhood of $f_0(M)$ in $V$ with respect to the Euclidean metric. Then $\mathcal{O}$ is invariant under the representation $\psi$ and the map $\pi: \mathcal{O} \to f_0(M)$ carrying a point of $\mathcal{O}$ into the unique nearest point of $f_0(M)$ is a differentiable equivariant retraction of $\mathcal{O}$ onto $f_0(M)$. Given $\phi \in \mathcal{A}(G, M)$ define $f_\phi: M \to V$ by $f_\phi(m) = \int g^{-1\psi} f_0(g^\phi m) dg$ where the integral is with respect to Haar measure on $G$. Then (cf. [4, p. 434]) $f_\phi$ is $\phi$-equivariant and clearly $f_{\phi_0} = f_0$. The map $F_\phi \in \mathfrak{M}(G \times M, V)$ defined by $F_\phi(g, m) = \tilde{\psi}(g^{-1}, f_0 \circ \tilde{\phi}(g, m))$ is easily seen[2] to depend continuously on $\phi \in \mathcal{A}(G, M)$ and since $f_\phi = \int F_\phi(g, m) dg$ it follows that $\phi \to f_\phi$ is a continuous map of $\mathcal{A}(G, M)$ into $\mathfrak{M}(M, V)$. Then for $\phi$ in a neighborhood $U'$ of $\phi_0$ in $\mathcal{A}(G, M)$ $f_\phi(M) \subseteq \mathcal{O}$ so $\sigma(\phi) = f_{\phi_0}^{-1} \circ \pi \circ f_\phi \in \mathfrak{M}(M, M)$. Now $\sigma: U' \to \mathfrak{M}(M, M)$ is continuous[2] and clearly $\sigma(\phi_0) = i_M$. Since $\mathfrak{D}_0(M)$ is open in $\mathfrak{M}(M, M)$, for some smaller neighborhood $U$ of $\phi_0$ in $\mathcal{A}(G, M)$ $\sigma: U \to \mathfrak{D}_0(M)$. Since $f_\phi$, $\pi$, at $f_{\phi_0}$ are respectively $\phi$-, $\pi$- and $\phi_0$-equivariant maps into $(V, \psi)$ it follows that $\sigma(\phi) g^\phi = g^{\phi_0} \sigma(\phi)$ or putting $\chi(\phi) = \sigma(\phi)^{-1}$, $\chi(\phi)\phi_0 = \phi$. Q.E.D.

4. **Conjugacy of neighboring compact subgroups of** Diff$(M)$. It is suggested by Theorem A that an analogue of the Montgomery and Zippin conjugacy theorem for neighboring compact subgroups of a Lie group [3] might hold for Diff $(M)$, i.e. that given a compact subgroup $G$ of Diff $(M)$ every compact subgroup of Diff $(M)$ sufficiently close to $G$ is conjugate in Diff $(M)$ to a subgroup of $G$. This in fact is the case and was the basis of an earlier more complicated proof of Theorem A. A proof will appear elsewhere.

## References


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