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# Geometry and topology of isoparametric submanifolds in euclidean spaces

**(Coxeter group/Dynkin diagram/homology/cohomology/taut)**

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**ABSTRACT** **Restrictions are given on the possible marked Dynkin diagrams of isoparametric submanifolds, and their homology and cohomology are computed by an extension of the techniques used by Bott and Samelson [Bott, R. & Samelson, H. (1958) *Am. J. Math.* 80, 964–1029] and by Borel [Borel, A. (1953) *Ann. Math.* 57 (2), 115–207] for G/T.**

An isoparametric hypersurface in a constant curvature space (i.e., a euclidean, spherical, or hyperbolic riemannian space) is, by definition, a level surface of an isoparametric function—namely, a function $f\colon M^{n+1}(c) \to \mathbf{R}$ satisfying $\Delta f$ and $|\nabla f|^2 \equiv 0 \pmod f$. Geometrically, the level surfaces of a given isoparametric function can also be characterized as a "parallel foliation" of hypersurfaces of constant mean curvatures. Such nice geometric structures were first studied by Segre and Cartan. In a series of papers in the late 1930s, Cartan (1–4) was particularly fascinated by the profound depth of the spherical case and its mysterious connection with the Lie group theory. This subject was somehow forgotten until it was revived by a paper of Nomizu (5) and two remarkable papers of Münzner (6, 7). It is rather straightforward to verify that the principal orbits of an orthogonal transformation group $(G, S^n(1))$ with one-dimensional orbit space (i.e., of cohomogeneity 1) are automatically examples of isoparametric hypersurfaces in $S^n(1)$, (cf. ref. 8). Such orthogonal transformation groups have been classified by Hsiang and Lawson (9) and they are exactly those isotropy representations of symmetric spaces of rank 2. Further examples of isoparametric hypersurfaces—i.e., those of nonorbital (or nonhomogeneous) type—were found by Ozeki and Takeuchi (10) and by Ferus *et al.* (11).

In a recent paper, Terng (12) proposed the following definition of isoparametric submanifolds in a constant curvature space; i.e., a submanifold $N^n$ in $M^{n+k}(x)$ is called isoparametric if its normal bundle is flat and the principal curvatures in the directions of any parallel normal vector field are constant. Typical examples that, in fact, motivate the above generalization of isoparametric hypersurfaces are exactly those principal orbits of isotropy representations of symmetric spaces: e.g., orbits of the type G/T in the Lie algebra of a given compact connected Lie group G. A general theory, as well as some basic theorems on such isoparametric submanifolds in euclidean spaces, especially the existence of an associated Coxeter group structure that generalizes the fundamental structure of the Weyl group action on a chosen Cartan subalgebra, is presented in ref. 12.

## Section 1. Isoparametric Submanifolds and Their Coxeter Groups

The orbit structure of the adjoint action of a given compact connected Lie group, G, on its Lie algebra, $\mathfrak{G}$, plays a central role in the study of Lie groups and their representations. The key to the understanding of the above orbit structure is the maximal torus theorem of Cartan. Let T be a maximal torus of G, $\mathfrak{L}$ be its Lie algebra (called a Cartan subalgebra), and W = N(T)/T be the Weyl group. Then (*i*) the principal orbit type is G/T; (*ii*) $\mathfrak{L}$ intersects every G orbit perpendicularly, and (*iii*) $\mathfrak{G}/G \cong \mathfrak{L}/W$ and $(W, \mathfrak{L})$ is a group generated by reflections. The above result was generalized by Cartan to the setting of the isotropy transformation of a symmetric space M = G/K—(*i*) a normal plane, $\mathfrak{L}$, of an arbitrary principal K orbit in $\mathfrak{P}$ automatically intersects all K orbits perpendicularly and (*ii*) there is an induced Coxeter group $(W, \mathfrak{L})$ such that $\mathfrak{P}/K \cong \mathfrak{L}/W$.

Let $M^n$ be a given isoparametric submanifold in $\mathbf{R}^{n+k}$ (which cannot be included in any proper affine subspace) and $\nu_x$ be the normal plane of $M^n$ at the point $x \in M^n$. Let $x_0$ be an arbitrary but fixed base point of $M^n$ and $\mathbf{R}^k = \nu_{x_0}$. Then $\mathbf{R}^k$ intersects $M^n$ perpendicularly at a finite number of points, and, moreover, to each given pair of points $x_1, x_2 \in S = \mathbf{R}^k \cap M^n$, there is a unique parallel translation $\Pi_{x_1x_2}\colon \mathbf{R}^k \simeq \nu_{x_1} \to \nu_{x_2} \simeq \mathbf{R}^k$, thus generating an induced transformation group, W, on $\mathbf{R}^k$. It has been proven (12) that $(W, \mathbf{R}^k)$ is automatically a group generated by reflection, called the associated Coxeter group of $M^n \subset \mathbf{R}^{n+k}$. Therefore, the ring of W-invariant polynomials on $\mathbf{R}^k$ is a free algebra generated by $k$ homogeneous generators, say $\{u_i;\ 1 \leq i \leq k\}$. Moreover, each $u_i$ extends uniquely to a homogeneous polynomial on $\mathbf{R}^{n+k}$ that is constant on $M^n$ and

$$u = (u_1, \ldots, u_k)\colon \mathbf{R}^{n+k} \to \mathbf{R}^k$$

defines an "isoparametric map" whose "level surfaces" form a foliation of $\mathbf{R}^{n+k}$ by isoparametric submanifolds and their focal varieties, which generalizes the orbital geometry of K orbits in $\mathfrak{P}$.

The tangent bundle of M, TM, has a canonical splitting as the orthogonal direct sum of the subspaces of equiprincipal directions:

$$TM = \bigoplus_{1 \leq j \leq p} E_j, \qquad \dim E_j = m_j.$$

The integral submanifold of $E_j$ passing through $x_0$ is a round sphere of dimension $m_j$ that intersects $\mathbf{R}^k$ at the antipode point of $x_0$, say $x_j$, and moreover, $\Pi_{x_jx_0}$, $1 \leq j \leq p$, are exactly those generating reflections of W. We shall call $m_j$ the marked multiplicity of the reflection hyperplane (in $\mathbf{R}^k$) and represent the above structure by a Dynkin diagram with marked multiplicities.

We announce here the following results on the possibilities of such marked Dynkin diagrams associated to isoparametric submanifolds in $\mathbf{R}^{n+k}$.

THEOREM 1. *Let* $M^n$ *be an irreducible isoparametric submanifold in* $\mathbf{R}^{n+k}$ *of codimension* $k \geq 3$. *Then its associated Coxeter group* $(W, \mathbf{R}^k)$ *is crystallographic (hence a Weyl*

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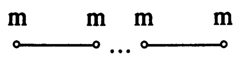
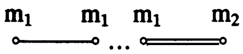
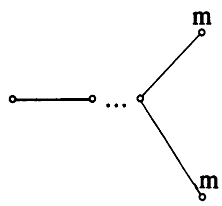
*group) and its marked Dynkin diagram must be one of the following list:*

$A_k$: 

m = 1, 2, *or* 4,

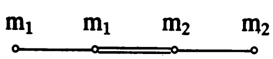
$B_k$: 


$$\begin{cases} m = 1 \text{ and } m_2 \text{ arbitrary} \\ m_1 = 2 \text{ and } m_2 = 2 \text{ or } 2m + 1 \\ m_1 = 4 \text{ and } m_2 = 5 \text{ or } 4m + 3 \\ k = 3,\ m_1 = 8, \text{ and } m_2 = 1, 3, 7, 11, \\ \qquad 16m - 1,\ 16m + 7, \end{cases}$$

$D_k$, $k \geq 4$: 

m = 1, 2, *or* 4,

$E_k$, k = 6, 7, 8: *with uniform multiplicity* m = 1, 2, *or* 4,

$F_4$: 

$m_1 = 1$ *and* $m_2 = 1, 2, 4$ *or* 8,

$m_1 = m_2 = 2$.

*Remark*: Except for the case of uniform multiplicity $m = 4$ in $E_k$ and $D_k$ ($k \geq 4$) and the case of $B_3$ with $m_1 = 8$ and $m_2 > 1$, all of the above possibilities of marked Dynkin diagrams with $k \geq 3$ are realized by homogeneous isoparametric examples. We conjecture that except perhaps for $B_3$ none of these missing marked diagrams do actually occur in the inhomogeneous case either.

**Section 2. Topology of Isoparametric Submanifolds and Focal Varieties**

Recall that in the typical case of G orbits in $\mathfrak{G}$, one has the following results of Borel (13) and of Bott and Samelson (14), respectively. (*i*) All G orbits in $\mathfrak{G}$ are of the type G/H with H connected and of maximal rank and one has the following theorem on the rational cohomologies of such homogeneous spaces:

$$H^*(G/H; Q) \cong H^*(B_T; Q)^{W(H)}/\langle H^+(B_T; Q)^{W(G)}\rangle,$$

where $H^*(B_T; Q)^{W(H)} \cong H^*(B_H; Q)$ and $\langle \ldots \rangle$ denotes the ideal generated by homogeneous W(G)-invariant elements of positive degrees. (*ii*) Morse theory has been applied to the computation of the integral cohomology of G/T in terms of the geometry of (W, $\mathfrak{L}$).

Roughly speaking, isoparametric submanifolds are exactly a kind of "geometric abstraction" of the above situation of G/T in $\mathfrak{G}$. Therefore, it is natural to investigate how much of those "Lie theoretical" results on the topology of G/T can indeed be generalized in the purely geometric setting of isoparametric submanifolds. We now announce the following initial results along this direction:

THEOREM 2. *Let* $M^n \subseteq \mathbf{R}^{n+k}$ *be an isoparametric submanifold with* $(W, \mathbf{R}^k)$ *and* $\{m_i: 1 \leq i \leq p\}$ *as its associated Coxeter group and marked multiplicities* (*cf.* Section 1). *Then, we have the following.*

(i) *If all* $m_i \geq 2$, *then M is simply connected, and* $H^*(M;\mathbf{Z})$ *is torsion free and its rank is equal to* $|W|$, *the order of W. Moreover, the rank of* $H_i(M; \mathbf{Z})$ *can easily be computed in terms of the geometry of* $(W, \mathbf{R}^k)$ *and the multiplicities* $m_i$ *exactly as in ref. 14.* (ii) *If some of* $m_i = 1$, *then dim* $H^*(M; \mathbf{Z}_2) = |W|$ *and dim* $H_i(M; \mathbf{Z}_2)$ *can be computed in terms of* $(W, \mathbf{R}^k)$ *and* $m_i$ *as in* i.

COROLLARY 1. *Every isoparametric submanifold* $M^n \subset R^{n+k}$ *is taut.*

*Remark*: For $k = 2$, *Corollary 1* has been proven by Cecil and Ryan (15).

THEOREM 3. *Let* $M^n \subseteq \mathbf{R}^{n+k}$ *be an isoparametric submanifold and* $(W, \mathbf{R}^k)$, $m_i$, $1 \leq i \leq p$, *be its associated Coxeter group and multiplicities. If all* $m_i$ *are even, then*

$$H^*(M^n; \mathbf{Q}) \cong \mathbf{Q}[x_1, \ldots, x_k]/\langle \mathbf{Q}^+[x_1, \ldots, x_k]^W\rangle$$

*where* $\mathbf{Q}[x_1, \ldots, x_k]$ *is the ring of weighted polynomials generated by a chosen system of simple roots of W,* $\Pi = \{x_1, \ldots, x_k\}$, *with degree*$(x_j)$ = *multi*$(x_j)$, and $\langle .,. \rangle$ *denotes the ideal generated by homogeneous W-invariant elements of positive degree.*

COROLLARY 2. *Let* $V \subset R^{n+k}$ *be a focal variety and W(V) be the isotropy Coxeter subgroup of V. Then,*

$$H^*(V, \mathbf{Q}) = \mathbf{Q}[x_1, \ldots, x_k]^{W(V)}/\langle \mathbf{Q}^+[x_1, \ldots, x_k]^W\rangle.$$

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