HOMEOMORPHIC CONJUGACY OF AUTOMORPHISMS ON THE TORUS[1]

R. L. ADLER AND R. PALAIS

Introduction. Let γ be a continuous map of the n-dimensional torus $T^n=R^n/Z^n$ into itself where R^n is an n-dimensional real Euclidean space and Z^n is the subgroup of R^n with integral coordinates. Let $\pi: R^n\to T^n$ denote the universal covering map. There is a unique $c=(c_1, \cdots, c_n)\in R^n$ with $0\leqq c_i<1$, $i=1, \cdots, n$, such that $\pi(c)=\gamma(0)$ and a unique continuous map $F: R^n\to R^n$ with $F(0)=c$ which is a "lifting" of γ, i.e., which satisfies $\pi F=\gamma\pi$. If we put $G(x)=F(x)-c$, $x\in R^n$, then $G|Z^n$ is a homomorphism of Z^n into itself and therefore extends uniquely to a linear map $L: R^n\to R^n$. In fact making the canonical identification of Z^n with the fundamental group $\pi_1(T^n)$ of T^n, $G|Z^n$ is just the homomorphism of $\pi_1(T^n)$ induced by γ. It follows that if γ is a homeomorphism then $G|Z^n$ is an automorphism of Z^n hence $L\in\mathrm{SL}(n, Z)$, the group of linear automorphisms of R^n whose matrices are unimodular, i.e., have determinant ± 1 and integer entries.

We next note that

$$F(x) = L(x) + P(x) + c \tag{1}$$

where $P: R^n\to R^n$ is a continuous periodic map (i.e., $P(x+\nu)=P(x)$, $x\in R^n$, $\nu\in Z^n$) satisfying $P(0)=0$. This fact is established by considering $P(x)=F(x)-L(x)-c$. Clearly $P(0)=0$. In view of the fact that $L\nu\in Z^n$ whenever $\nu\in Z^n$ we have $\pi(P(x+\nu)-P(x))=\pi(F(x+\nu)-F(x)-L\nu)=\gamma(\pi x+\pi\nu)-\gamma(\pi x)-\pi L\nu=\gamma(\pi x)-\gamma(\pi x)=0$; consequently $P(x+\nu)-P(x)$ is always in Z^n. Since R^n is connected and Z^n is discrete $P(x+\nu)-P(x)$ is (for ν fixed) independent of x. Thus $P(x+\nu)-P(x)=P(0+\nu)-P(0)=P(\nu)=F(\nu)-L\nu-c=G(\nu)-L\nu$ which is zero by definition $L|Z^n=G|Z^n$. We shall call L the linear part, P the periodic part, and c the constant part of the lifting F and when necessary place subscripts on these symbols to indicate the mapping on T^n from which they came.

The linear, periodic, and constant part of a lifting are unique; for let L', P', c', be other ones. $F(x)=L(x)+P(x)+c=L'(x)+P'(x)+c'$, yields $L(x)-L'(x)=P(x)-P'(x)+c-c'$. The right-hand side of the

Received by the editors November 16, 1964.

[1] It has come to the attention of the authors that the following overlaps the present work: D. Z. Arov, *Topological similitude of automorphisms and translations of compact commutative groups*, Uspehi Mat. Nauk **18** (1963), no. 5 (113), 133–138. (Russian)

last relation is linear while the left is periodic. This can only occur if $L-L'=0$. Then since $P(0)=P'(0)=0$, it follows that $c=c'$ and $P=P'$.

If γ is a continuous automorphism of T^n, then F_γ is linear; thus its periodic and constant parts vanish so that $F_\gamma=L_\gamma$. Also every $L\in \mathrm{SL}(n, Z)$ is the lifting of a continuous automorphism on T^n. The correspondence $\gamma\to L_\gamma$ is an isomorphism of the group $\mathrm{Aut}(T^n)$ with $\mathrm{SL}(n, Z)$. More generally the mapping $\gamma\to L_\gamma$ is a homomorphism of the group $\mathrm{Homeo}(T^n)$ onto $\mathrm{SL}(n, Z)$. We shall prove this by showing that $L_{\alpha\beta}=L_\alpha L_\beta$ for any two homeomorphisms α and β of T^n onto itself. By uniqueness of lifting

$$F_{\alpha\beta} = F_\alpha F_\beta. \tag{2}$$

On one hand,

$$F_{\alpha\beta}(x) = L_{\alpha\beta}(x) + P_{\alpha\beta}(x) + c_{\alpha\beta}; \tag{3}$$

on the other hand, using (1) and adding and subtracting $P_\alpha(c_\beta)$,

$$\begin{aligned} F_\alpha F_\beta(x) = L_\alpha L_\beta(x) &+ [L_\alpha P_\beta(x) + P_\alpha(L_\beta(x) + P_\beta(x) + c_\beta) - P_\alpha(c_\beta)] \\ &+ L_\alpha(c_\beta) + P_\alpha(c_\beta) + c_\alpha. \end{aligned} \tag{4}$$

Because $L_\beta Z^n\subseteq Z^n$ the term in the brackets is periodic. This term vanishes when $x=0$ so that it is a periodic part of the lifting $F_{\alpha\beta}$. From the uniqueness of the various parts of a lifting

$$L_{\alpha\beta} = L_\alpha L_\beta, \tag{5}$$

$$P_{\alpha\beta}(x) = L_\alpha P_\beta(x) + P_\alpha(L_\beta(x) + P_\beta(x) + c_\beta) - P_\alpha(c_\beta), \tag{6}$$

$$c_{\alpha\beta} = L_\alpha(c_\beta) + P_\alpha(c_\beta) + c_\alpha. \tag{7}$$

Finally if γ is a continuous automorphism of T^n, it preserves Haar measure on T^n and to such transformations we can apply the notions of ergodic theory [1].

THEOREM. *If α and β are continuous automorphisms of T^n such that*

$$\gamma\alpha\gamma^{-1} = \beta \tag{8}$$

where γ is a homeomorphism of T^n onto itself then

(i) $L_\gamma L_\alpha L_\gamma^{-1}=L_\beta$ (*α and β are conjugate elements in the group of measure preserving transformations on T^n*).

(ii) *c_γ is a fixed point of L_β ($\gamma(0)$ is a fixed point of β).*

(iii) *If α is ergodic then $P_\gamma=0$ (γ is a continuous automorphism of T^n composed with a rotation. The rotation is by a fixed point of β and the continuous automorphism satisfies the conjugacy relation between α and β).*

PROOF. Relation (i) follows immediately from (5).

From (7)

$$c_{\gamma\alpha} = L_\gamma(c_\alpha) + P_\gamma(c_\alpha) + c_\gamma$$

and

$$c_{\beta\gamma} = L_\beta(c_\gamma) + P_\beta(c_\gamma) + c_\beta.$$

From (8) $\gamma\alpha = \beta\gamma$. Since the constant part of the lifting for this mapping is unique, $c_{\gamma\alpha} = c_{\beta\gamma}$.

Therefore because $P_\beta = 0$ and $c_\alpha = c_\beta = 0$, we have statement (ii) that $L_\beta(c_\gamma) = c_\gamma$.

It is convenient to prove (iii) in the following steps.

Step I. Define the function Q by $Q(x) = L_\gamma^{-1}P_\gamma(x)$. Then $QL_\alpha^m = L_\alpha^m Q$ for all $m \in Z$. To prove this we first derive from (6) that $P_{\gamma\alpha}(x) = P_\gamma L_\alpha(x)$ and $P_{\beta\gamma}(x) = L_\beta P_\gamma(x)$. By hypothesis (8) and uniqueness of periodic part $P_\gamma L_\alpha = L_\beta P_\gamma$. Substituting (i) in this expression $P_\gamma L_\alpha = L_\gamma L_\alpha L_\gamma^{-1} P_\gamma$ so that $QL_\alpha = L_\alpha Q$. Thereupon Step I is obtained by induction on m.

Step II. We next recall that if $L: R^n \to R^n$ is an invertible linear operator and if $\{L^m x: m \in Z\}$ is bounded then either $x = 0$ or $\{L^m x: m \in Z\}$ is bounded away from zero. One way to see this is to express x in a basis that puts L in Jordan form. It can be verified that the hypothesis $\{L^m x: m \in Z\}$ is bounded implies that x is a linear combination of characteristic vectors of L belonging to characteristic values of absolute value one.

Step III. If $\{L_\alpha^m Q(x): m \in Z\}$ is not bounded away from zero then $Q(x) = 0$. This follows from Step II. The set $\{L_\alpha^m Q(x) = QL_\alpha^m(x): m \in Z\}$ is bounded because Q being continuous and periodic is bounded.

Step IV. If $\{L_\alpha^m x: m \in Z\}$ is not bounded away from Z^n then $Q(x) = 0$. Since Q is periodic, $L_\alpha^m Q(x) = L_\alpha^m(x) = Q(L_\alpha^m(x) - \nu)$ for $\nu \in Z^n$. By hypothesis there exists a subsequence $\{m_i: i = 1, 2, \cdots\}$ of Z and a subset $\{\nu_i: i = 1, 2, \cdots\}$ of Z^n such that $L_\alpha^{m_i} x - \nu_i \to 0$, $i \to \infty$. Since Q is continuous and $Q(0) = 0$, $L_\alpha^{m_i} Q(x) = Q(L_\alpha^{m_i}(x) - \nu_i) \to 0$. By Step III $Q(x) = 0$.

Now α is ergodic and so almost all orbits of α are dense in T^n. In particular the zero element in T^n is a limit point for almost all orbits or in other words $\{L_\alpha^m x: m \in Z\}$ is not bounded away from Z^n for almost all $x \in R^n$. From Step IV $Q(x) = 0$ almost everywhere. By continuity $Q = 0$ and since L_γ^{-1} is nonsingular $P_\gamma = 0$.

Remarks. Questions arise whether the theorem holds under weaker hypotheses. In (iii) one cannot merely drop the assumption of ergodicity, for choosing α to be the identity transformation removes all restrictions on the homeomorphism γ. Another question is whether the theorem holds if γ is a measure preserving transformation instead of a homeomorphism. A positive answer would be significant in ergodic theory, for then an example could be constructed of two Kolmogoroff transformations [2] with the same entropy but which are not conjugate.

REFERENCES

1. P. R. Halmos, *Lectures in ergodic theory*, Publ. Math. Soc. Japan No. 3, The Mathematical Society of Japan, Tokyo, 1956.

2. Ja. G. Sinaĭ, *Probabilistic ideas in ergodic theory*, Amer. Math. Soc. Transl. (2) **31** (1963), 62–84.

INTERNATIONAL BUSINESS MACHINES CORPORATION, YORKTOWN HEIGHTS AND BRANDEIS UNIVERSITY