

## HOMEOMORPHIC CONJUGACY OF AUTOMORPHISMS ON THE TORUS<sup>1</sup>

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**Introduction.** Let  $\gamma$  be a continuous map of the  $n$ -dimensional torus  $T^n = R^n/Z^n$  into itself where  $R^n$  is an  $n$ -dimensional real Euclidean space and  $Z^n$  is the subgroup of  $R^n$  with integral coordinates. Let  $\pi: R^n \rightarrow T^n$  denote the universal covering map. There is a unique  $c = (c_1, \dots, c_n) \in R^n$  with  $0 \leq c_i < 1$ ,  $i = 1, \dots, n$ , such that  $\pi(c) = \gamma(0)$  and a unique continuous map  $F: R^n \rightarrow R^n$  with  $F(0) = c$  which is a "lifting" of  $\gamma$ , i.e., which satisfies  $\pi F = \gamma \pi$ . If we put  $G(x) = F(x) - c$ ,  $x \in R^n$ , then  $G|Z^n$  is a homomorphism of  $Z^n$  into itself and therefore extends uniquely to a linear map  $L: R^n \rightarrow R^n$ . In fact making the canonical identification of  $Z^n$  with the fundamental group  $\pi_1(T^n)$  of  $T^n$ ,  $G|Z^n$  is just the homomorphism of  $\pi_1(T^n)$  induced by  $\gamma$ . It follows that if  $\gamma$  is a homeomorphism then  $G|Z^n$  is an automorphism of  $Z^n$  hence  $L \in \text{SL}(n, Z)$ , the group of linear automorphisms of  $R^n$  whose matrices are unimodular, i.e., have determinant  $\pm 1$  and integer entries.

We next note that

$$(1) \quad F(x) = L(x) + P(x) + c$$

where  $P: R^n \rightarrow R^n$  is a continuous periodic map (i.e.,  $P(x+\nu) = P(x)$ ,  $x \in R^n$ ,  $\nu \in Z^n$ ) satisfying  $P(0) = 0$ . This fact is established by considering  $P(x) = F(x) - L(x) - c$ . Clearly  $P(0) = 0$ . In view of the fact that  $L\nu \in Z^n$  whenever  $\nu \in Z^n$  we have  $\pi(P(x+\nu) - P(x)) = \pi(F(x+\nu) - F(x) - L\nu) = \gamma(\pi x + \pi\nu) - \gamma(\pi x) - \pi L\nu = \gamma(\pi x) - \gamma(\pi x) = 0$ ; consequently  $P(x+\nu) - P(x)$  is always in  $Z^n$ . Since  $R^n$  is connected and  $Z^n$  is discrete  $P(x+\nu) - P(x)$  is (for  $\nu$  fixed) independent of  $x$ . Thus  $P(x + \nu) - P(x) = P(0 + \nu) - P(0) = P(\nu) = F(\nu) - L\nu - c = G(\nu) - L\nu$  which is zero by definition  $L|Z^n = G|Z^n$ . We shall call  $L$  the linear part,  $P$  the periodic part, and  $c$  the constant part of the lifting  $F$  and when necessary place subscripts on these symbols to indicate the mapping on  $T^n$  from which they came.

The linear, periodic, and constant part of a lifting are unique; for let  $L', P', c'$ , be other ones.  $F(x) = L(x) + P(x) + c = L'(x) + P'(x) + c'$ , yields  $L(x) - L'(x) = P(x) - P'(x) + c - c'$ . The right-hand side of the

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last relation is linear while the left is periodic. This can only occur if  $L - L' = 0$ . Then since  $P(0) = P'(0) = 0$ , it follows that  $c = c'$  and  $P = P'$ .

If  $\gamma$  is a continuous automorphism of  $T^n$ , then  $F_\gamma$  is linear; thus its periodic and constant parts vanish so that  $F_\gamma = L_\gamma$ . Also every  $L \in \text{SL}(n, Z)$  is the lifting of a continuous automorphism on  $T^n$ . The correspondence  $\gamma \rightarrow L_\gamma$  is an isomorphism of the group  $\text{Aut}(T^n)$  with  $\text{SL}(n, Z)$ . More generally the mapping  $\gamma \rightarrow L_\gamma$  is a homomorphism of the group  $\text{Homeo}(T^n)$  onto  $\text{SL}(n, Z)$ . We shall prove this by showing that  $L_{\alpha\beta} = L_\alpha L_\beta$  for any two homeomorphisms  $\alpha$  and  $\beta$  of  $T^n$  onto itself. By uniqueness of lifting

$$(2) \quad F_{\alpha\beta} = F_\alpha F_\beta.$$

On one hand,

$$(3) \quad F_{\alpha\beta}(x) = L_{\alpha\beta}(x) + P_{\alpha\beta}(x) + c_{\alpha\beta};$$

on the other hand, using (1) and adding and subtracting  $P_\alpha(c_\beta)$ ,

$$(4) \quad F_\alpha F_\beta(x) = L_\alpha L_\beta(x) + [L_\alpha P_\beta(x) + P_\alpha(L_\beta(x) + P_\beta(x) + c_\beta) - P_\alpha(c_\beta)] \\ + L_\alpha(c_\beta) + P_\alpha(c_\beta) + c_\alpha.$$

Because  $L_\beta Z^n \subseteq Z^n$  the term in the brackets is periodic. This term vanishes when  $x = 0$  so that it is a periodic part of the lifting  $F_{\alpha\beta}$ . From the uniqueness of the various parts of a lifting

$$(5) \quad L_{\alpha\beta} = L_\alpha L_\beta,$$

$$(6) \quad P_{\alpha\beta}(x) = L_\alpha P_\beta(x) + P_\alpha(L_\beta(x) + P_\beta(x) + c_\beta) - P_\alpha(c_\beta),$$

$$(7) \quad c_{\alpha\beta} = L_\alpha(c_\beta) + P_\alpha(c_\beta) + c_\alpha.$$

Finally if  $\gamma$  is a continuous automorphism of  $T^n$ , it preserves Haar measure on  $T^n$  and to such transformations we can apply the notions of ergodic theory [1].

**THEOREM.** *If  $\alpha$  and  $\beta$  are continuous automorphisms of  $T^n$  such that*

$$(8) \quad \gamma\alpha\gamma^{-1} = \beta$$

*where  $\gamma$  is a homeomorphism of  $T^n$  onto itself then*

(i)  $L_\gamma L_\alpha L_\gamma^{-1} = L_\beta$  ( $\alpha$  and  $\beta$  are conjugate elements in the group of measure preserving transformations on  $T^n$ ).

(ii)  $c_\gamma$  is a fixed point of  $L_\beta$  ( $\gamma(0)$  is a fixed point of  $\beta$ ).

(iii) If  $\alpha$  is ergodic then  $P_\gamma = 0$  ( $\gamma$  is a continuous automorphism of  $T^n$  composed with a rotation. The rotation is by a fixed point of  $\beta$  and the continuous automorphism satisfies the conjugacy relation between  $\alpha$  and  $\beta$ ).

PROOF. Relation (i) follows immediately from (5).

From (7)

$$c_{\gamma\alpha} = L_\gamma(c_\alpha) + P_\gamma(c_\alpha) + c_\gamma$$

and

$$c_{\beta\gamma} = L_\beta(c_\gamma) + P_\beta(c_\gamma) + c_\beta.$$

From (8)  $\gamma\alpha = \beta\gamma$ . Since the constant part of the lifting for this mapping is unique,  $c_{\gamma\alpha} = c_{\beta\gamma}$ .

Therefore because  $P_\beta = 0$  and  $c_\alpha = c_\beta = 0$ , we have statement (ii) that  $L_\beta(c_\gamma) = c_\gamma$ .

It is convenient to prove (iii) in the following steps.

*Step I.* Define the function  $Q$  by  $Q(x) = L_\gamma^{-1}P_\gamma(x)$ . Then  $QL_\alpha^m = L_\alpha^mQ$  for all  $m \in Z$ . To prove this we first derive from (6) that  $P_{\gamma\alpha}(x) = P_\gamma L_\alpha(x)$  and  $P_{\beta\gamma}(x) = L_\beta P_\gamma(x)$ . By hypothesis (8) and uniqueness of periodic part  $P_\gamma L_\alpha = L_\beta P_\gamma$ . Substituting (i) in this expression  $P_\gamma L_\alpha = L_\gamma L_\alpha L_\gamma^{-1} P_\gamma$  so that  $QL_\alpha = L_\alpha Q$ . Thereupon Step I is obtained by induction on  $m$ .

*Step II.* We next recall that if  $L: R^n \rightarrow R^n$  is an invertible linear operator and if  $\{L^m x: m \in Z\}$  is bounded then either  $x = 0$  or  $\{L^m x: m \in Z\}$  is bounded away from zero. One way to see this is to express  $x$  in a basis that puts  $L$  in Jordan form. It can be verified that the hypothesis  $\{L^m x: m \in Z\}$  is bounded implies that  $x$  is a linear combination of characteristic vectors of  $L$  belonging to characteristic values of absolute value one.

*Step III.* If  $\{L_\alpha^m Q(x): m \in Z\}$  is not bounded away from zero then  $Q(x) = 0$ . This follows from Step II. The set  $\{L_\alpha^m Q(x) = QL_\alpha^m(x): m \in Z\}$  is bounded because  $Q$  being continuous and periodic is bounded.

*Step IV.* If  $\{L_\alpha^m x: m \in Z\}$  is not bounded away from  $Z^n$  then  $Q(x) = 0$ . Since  $Q$  is periodic,  $L_\alpha^m Q(x) = L_\alpha^m(x) = Q(L_\alpha^m(x) - \nu)$  for  $\nu \in Z^n$ . By hypothesis there exists a subsequence  $\{m_i: i = 1, 2, \dots\}$  of  $Z$  and a subset  $\{\nu_i: i = 1, 2, \dots\}$  of  $Z^n$  such that  $L_\alpha^{m_i} x - \nu_i \rightarrow 0, i \rightarrow \infty$ . Since  $Q$  is continuous and  $Q(0) = 0, L_\alpha^{m_i} Q(x) = Q(L_\alpha^{m_i}(x) - \nu_i) \rightarrow 0$ . By Step III  $Q(x) = 0$ .

Now  $\alpha$  is ergodic and so almost all orbits of  $\alpha$  are dense in  $T^n$ . In particular the zero element in  $T^n$  is a limit point for almost all orbits or in other words  $\{L_\alpha^m x: m \in Z\}$  is not bounded away from  $Z^n$  for almost all  $x \in R^n$ . From Step IV  $Q(x) = 0$  almost everywhere. By continuity  $Q = 0$  and since  $L_\gamma^{-1}$  is nonsingular  $P_\gamma = 0$ .

**Remarks.** Questions arise whether the theorem holds under weaker hypotheses. In (iii) one cannot merely drop the assumption of ergodicity, for choosing  $\alpha$  to be the identity transformation removes all

restrictions on the homeomorphism  $\gamma$ . Another question is whether the theorem holds if  $\gamma$  is a measure preserving transformation instead of a homeomorphism. A positive answer would be significant in ergodic theory, for then an example could be constructed of two Kolmogoroff transformations [2] with the same entropy but which are not conjugate.

#### REFERENCES

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