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ON THE HOMOTOPY TYPE OF CERTAIN GROUPS OF OPERATORS

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INTRODUCTION

GIVEN a sequence of topological spaces $\{X_n\}$ with X_n a subspace of X_{n+1} we denote by $\varinjlim X_n$ their inductive limit, i.e. the space whose point set is $\bigcup_n X_n$ and whose topology is the finest such that each inclusion $X_m \longrightarrow \bigcup_n X_n$ is continuous.

The point of this paper is that certain infinite dimensional manifolds have the homotopy type of an inductive limit of finite dimensional submanifolds. Our basic abstract theorem in this direction, whose proof is given in §1, is

THEOREM (A). *Suppose on a Banach space E there is a sequence $\{\pi_n\}$ of continuous projection operators onto finite dimensional subspaces $E_n = \pi_n E \subseteq E_{n+1}$ which tend strongly to the identity* (i.e. *$\pi_n x \longrightarrow x$ for each $x \in E$*). *Given $\mathcal{O}$ open in E let $\mathcal{O}_n = \mathcal{O} \cap E_n$ and let $\mathcal{O}_\infty = \varinjlim \mathcal{O}_n$ Then the injection map $j : \mathcal{O}_\infty \longrightarrow \mathcal{O}$ is a homotopy equivalence.*

Next suppose H is a separable real or complex Hilbert space, $\{e_n\}$ an orthonormal basis, and let P_n be the orthogonal projection of H onto the subspace H_n spanned by $\{e_1, \dots, e_n\}$. Let $B(H)$ denote the Banach algebra of bounded operators on H (with $\| \ \|_\infty$ the usual norm, defined by $\|A\|_\infty = \operatorname{Sup}\{\|Ax\| \mid x \in D\}$ where D is the closed unit ball in H), $B(H_n)$ the finite dimensional subalgebra of operators which map H_n into itself and are zero on $H_n^\perp$ and let Q_n denote the projection of $B(H)$ onto $B(H_n)$ given by $Q_n(A) = P_n A P_n$. Let $GL(H)$ denote the group of units of $B(H)$, and let $GL(n)$ denote the subgroup consisting of invertible operators of the form $I + A$ where I is the identity and $A \in B(H_n)$. Finally let $GL(\infty) = \varinjlim GL(n)$ (the homotopy groups of $GL(\infty)$ are given by the Bott periodicity theorems [1]).

DEFINITION. A Banach algebra $\mathscr{A}$ of operators on H with topology at least as fine as that induced from $B(H)$ will be called *approximately tame* if $\bigcup_n B(H_n) \subseteq \mathscr{A}$ and if for each $A \in \mathscr{A}$ the sequence $Q_n(A) \rightarrow A$ in $\mathscr{A}$. In this case we define $G(\mathscr{A}) = \{I + A \in GL(H) \mid A \in \mathscr{A}\}$ topologized by the condition that $I + A \rightarrow A$ shall be a bicontinuous map into $\mathscr{A}$.

The following result, which accounts for the title of this paper, will be proved in §2.

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THEOREM (B). *If $\mathscr{A}$ is an approximately tame algebra of operators on H then the injection $j: GL(\infty) \longrightarrow G(\mathscr{A})$ is a homotopy equivalence.*

Let $\mathscr{K}$ denote the set of all completely continuous (or compact) operators on H, i.e. all $A \in B(H)$ such that $A(D)$ is compact, where D as above is the closed unit ball in H. Then as is well-known and elementary $\mathscr{K}$ is a closed ideal in $B(H)$ which contains all operators with finite dimensional range, so in particular $\bigcup_n B(H_n) \subseteq \mathscr{K}$. We note that every approximately tame algebra $\mathscr{A}$ is included in $\mathscr{K}$. For if $A \in \mathscr{A}$ then $Q_n(A) \in B(H_n) \subseteq \mathscr{K}$ and since $Q_n(A) \longrightarrow A$ in $\mathscr{A}$ it *a fortiori* converges to A in the less fine topology of $B(H)$, hence $A \in \mathscr{K}$ since $\mathscr{K}$ is closed in $B(H)$. In §3 we will prove

THEOREM (C). *The algebra $\mathscr{K}$ of completely continuous operators on H is approximately tame.*†

In §§4 and 5 we discuss the algebra L^1 of operators of trace class on H and more generally the algebras L^p $1 \leq p < \infty$, and prove that they are all approximately tame. The algebra L^2 of Hilbert–Schmidt operators is treated separately in §4 because of its comparative simplicity. While the properties of the L^p algebras are well known, and are stated for example in [2] and proved in [3] and [4], treatments in the literature are for the more general L^p spaces of operators associated with a ring of operators with a 'gauge' or 'trace function.' Since this general theory of non-commutative integration is considerably more sophisticated than our special case (in the same way that the theory of the L^p spaces for a general measure space is more sophisticated than that of the algebras l^p of a countable discrete measure space) we have given an elementary treatment of this special case in §5. I would like to thank E. Nelson and E. Stein for their collaboration on this section.

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§1. PROOF OF THEOREM (A)

For $0 \leq t < \infty$ we define π_t by $\pi_t = \pi_n + (t-n)(\pi_{n+1} - \pi_n)$, $n \leq t \leq n+1$ and we define $\pi_\infty =$ identity map of E. We note that $\pi_t E_n \subseteq E_n$ for all t and n and that if $n > t$ then $\pi_t E \subseteq E_n$. Since clearly $\pi_t x \longrightarrow x$ as $t \longrightarrow \infty$ for each $x \in E$, by the principle of uniform boundedness the π_t are uniformly bounded in norm and hence equicontinuous. Since $\pi_t \longrightarrow \pi_\infty$ pointwise it follows from equicontinuity that $\pi_t \longrightarrow \pi_\infty$ uniformly on any compact subset of E. In particular if $x_n \longrightarrow x$ in E then $K = \{x_n\} \cup \{x\}$ is a compact subset of E hence if $t_n \longrightarrow \infty$ then $\pi_{t_n} \longrightarrow \pi_\infty$ uniformly on K, from which it follows that $\pi_{t_n} x_n \longrightarrow x$. This proves

LEMMA (1). *The map $\pi : E \times [0, \infty] \longrightarrow E$ defined by $\pi(x, t) = \pi_t x$ is continuous.*

Now let $\mathscr{O}$ be open in E and define a non-negative real valued function f on $\mathscr{O}$ by $f(x) = \operatorname{Sup}\{t \mid t \geq 0 \text{ and } \pi(x, t) \notin \mathscr{O}\}$.

† ADDED IN PROOF: A. S. SVARC has announced [*Dokl. Akad. Nauk SSSR* **154**, pp. 61–63, Translated in *Soviet Mathematics* **5**, pp. 57–59] that $J: GL(\infty) \rightarrow G(\mathscr{K})$ is a weak homotopy equivalence.

LEMMA (2). *f is upper semi-continuous, i.e. if $f(x_0) < M$ then there is a neighborhood U of x_0 such that $f(x) < M$ for all $x \in U$.*

Proof. If not there would exist a sequence $x_n \longrightarrow x_0$ and $t_n \geq M$ such that $\pi(x_n, t_n) \notin \mathcal{O}$ Since $\pi(x_n, t) \in \mathcal{O}$ for n and t sufficiently large by Lemma (1), the t_n are bounded, hence by passing to a subsequence we can assume $t_n \longrightarrow t \geq M$. Then $\pi(x_n, t_n) \longrightarrow \pi(x_0, t)$ and since the complement of $\mathcal{O}$ is closed $\pi(x_0, t) \notin \mathcal{O}$. But then by definition of f, $f(x_0) \geq t \geq M$ a contradiction. Q.E.D.

Now on a paracompact space, hence in particular on a metric space, an upper semi-continuous real valued function is dominated by a continuous real valued function. Applying this fact to f

LEMMA (3). *There is a continuous positive real valued function g on $\mathcal{O}$ such that $\pi(x, t) \in \mathcal{O}$ if $t \geq g(x)$.*

DEFINITION. *We define $h : \mathcal{O} \times I \longrightarrow \mathcal{O}$ by $h(x, t) = \pi(x, g(x)/t)$ where g is as in Lemma (3). For $t \in I$ we define $h_t : \mathcal{O} \longrightarrow \mathcal{O}$ by $h_t(x) = h(x, t)$.*

Then it is immediate from Lemmas (1) and (3) that

LEMMA (4). *h_t is a homotopy of h_1 with the identity map of $\mathcal{O}$.*

Given $x_0 \in \mathcal{O}$ let n be an integer greater than $g(x_0)$. Then $h_1(x_0) = \pi(x_0, g(x_0)) \in \mathcal{O} \cap E_n = \mathcal{O}_n$. This shows that h_1 maps $\mathcal{O}$ into $\bigcup_n \mathcal{O}_n = \mathcal{O}_\infty$. Let us denote by h_1^* the function h_1 considered as a function from $\mathcal{O}$ into the topological space $\mathcal{O}_\infty$ and show that h_1^* is continuous. Let U be a neighborhood of x_0 such that $n > g(x)$ for $x \in U$. Then it will suffice to show that $h_1^*|U$ is continuous. But $h_1(x) \in \mathcal{O}_n$ for $x \in U$ and since we know $h_1|U : U \longrightarrow \mathcal{O}_n$ is continuous and since $\mathcal{O}_n$ is a subspace of $\mathcal{O}_\infty$ it follows that $h_1^*|U$ is continuous. Now if j is the injection of $\mathcal{O}_\infty$ into $\mathcal{O}$ then clearly $jh_1 = h_1$ so by Lemma (4) jh_1^* is homotopic to the identity map of $\mathcal{O}$. On the other hand consider the map $h_1^* j : \mathcal{O}_\infty \longrightarrow \mathcal{O}_\infty$. Since $\pi_t E_n \subseteq E_n$ it follows that h_t maps each $\mathcal{O}_n$ into itself and defines a homotopy of $h_1|\mathcal{O}_n$ with the identity map of $\mathcal{O}_n$. It follows that $h_t j$ defines a homotopy of $h_1 j$ with the identity map of $\mathcal{O}_\infty$. Thus h_1^* is a homotopy inverse for j which proves Theorem (A).

§2. PROOF OF THEOREM (B)

First we note that since $B(H_n)$ is finite dimensional it is a topological subspace of $\mathscr{A}$, from which it follows that $GL(n)$ is a topological subspace of $G(\mathscr{A})$ and hence that the injection $j : GL(\infty) \longrightarrow G(\mathscr{A})$ is continuous.

Let F denote the map $I + A \longrightarrow A$ of $G(\mathscr{A})$ into $\mathscr{A}$, which by definition is a homeomorphism onto the subspace $\mathcal{O}$ of $\mathscr{A}$ defined by $\mathcal{O} = \{A \in \mathscr{A} | I + A \in GL(H)\}$. Since $GL(H)$ is open in $B(H)$, and the injection of $\mathscr{A}$ into $B(H)$ is by assumption continuous, it follows that $\mathcal{O}$ is open in $\mathscr{A}$. Since $P_n \in B(H_n) \subseteq \mathscr{A}$ and since left and right multiplication operators are continuous in a Banach algebra, it follows that the projection Q_n of $\mathscr{A}$ onto the finite dimensional subspace $B(H_n)$ is continuous as an operator on $\mathscr{A}$, and by definition of an approximately tame algebra, Q_n tends strongly to the identity map of $\mathscr{A}$. By Theorem

(A) the injection map of $\varinjlim(\mathcal{O} \cap B(H_n)) \longrightarrow \mathcal{O}$ is a homotopy equivalence. On the other hand F clearly maps $\overrightarrow{GL(n)}$ homeomorphically onto $\mathcal{O} \cap B(H_n)$ and hence maps $GL(\infty)$ homeomorphically onto $\varinjlim(\mathcal{O} \cap B(H_n))$. Since F maps $G(\mathscr{A})$ homeomorphically onto $\mathcal{O}$ the theorem follows.

§3. PROOF OF THEOREM (C)

Since we already know that $\mathscr{K}$ is a closed subalgebra (even an ideal) in $B(H)$ and $\bigcup_n B(H_n) \subseteq \mathscr{K}$ it remains only to show that $Q_n(A) = P_nAP_n$ converges in $B(H)$ to A for each $A \in \mathscr{K}$. Now P_n clearly converges strongly to the identity map of H so as remarked at the beginning of section 1 P_n converges uniformly to the identity on each compact subset of H. In particular if D is the closed unit ball in H then $A(D)$ is compact so P_n converges uniformly to the identity on $A(D)$, or equivalently P_nA converges uniformly to A on D. But this means

$$\|A - P_nA\|_\infty \longrightarrow 0,$$

and since A^* is also in $\mathscr{K}$,

$$\|A^* - P_nA^*\|_\infty \longrightarrow 0$$

hence

$$\|A - AP_n\|_\infty = \|(A^* - P_nA^*)^*\|_\infty \longrightarrow 0.$$

Since $\| P_n\| = 1$,

$$\|P_nA - P_nAP_n\|_\infty = \|P_n(A - AP_n)\|_\infty \longrightarrow 0$$

hence

$$\|A - Q_n(A)\|_\infty = \|A - P_nA + P_nA - P_nAP_n\|_\infty \longrightarrow 0.$$

Q.E.D.

§4. THE ALGEBRA OF HILBERT-SCHMIDT OPERATORS

We begin by recalling a few elementary facts about the Hilbert–Schmidt norm $\| \;\|_2$ and the set of Hilbert–Schmidt operators. The proofs will be found in [5].

If $A \in B(H)$ then the positive real number or ∞ $\|A\|_2$, defined by

$$\|A\|_2^2 = \operatorname{tr} A^*A = \sum_{m,n=1}^{\infty} |(A\phi_m, \psi_n)|^2 = \sum_{n=1}^{\infty} (A^*A\phi_n, \phi_n),$$

where $\{\phi_n\}$ and $\{\psi_n\}$ are orthonormal bases, is independent of the choice of these bases. Elements $A \in B(H)$ such that $\|A\|_2 < \infty$ are called Hilbert–Schmidt operators and the set L^2 of all Hilbert–Schmidt operators is a Hilbert space in the norm $\| \;\|_2$, the corresponding inner product being

$$\langle A, B\rangle = \operatorname{tr} B^*A = \sum_{m,n=1}^{\infty} (A\phi_m, \psi_n)(\psi_n, B\phi_m) = \sum_{n=1}^{\infty} (B^*A\phi_n, \phi_n),$$

these sums being absolutely convergent and independent of the orthonormal bases $\{\phi_n\}$ and $\{\psi_n\}$. If $x \in H$ with $\|x\| = 1$, say $x = \sum_{m=1}^{\infty} x_m\phi_m$ then

$$\|Ax\|^2 = \sum_n |(Ax, \phi_n)|^2 = \sum_n |\sum_m x_m(A\phi_m, \phi_n)|^2$$

so by Schwarz's inequality

$$\|Ax\|^2 \leq \sum_n \|x\|^2 \sum_m |(A\phi_m, \phi_n)|^2 = \|A\|_2^2.$$

Since $\|A\|_\infty = \mathrm{Sup}\{\|Ax\| \,|\, \|x\| = 1\}$ it follows that $\|A\|_\infty \leq \|A\|_2$. This proves that the topology of L^2 is at leas as fine as that induced from $B(H)$. If $A, B \in B(H)$ then

$$\|AB\|_2^2 = \sum_{m,n} |(AB\phi_m, \phi_n)|^2 = \sum_{m,n} |\sum_k (A\phi_k, \phi_n)(B\phi_m, \phi_k)|^2.$$

If $x \in H$ then

$$\|A\|_\infty^2 \|x\|^2 \geq \|Ax\|^2 = \sum_n |\sum_k (A\phi_k, \phi_n)(x, \phi_k)|^2.$$

Taking $x = B\phi_m$ in the latter gives

$$\|A\|_\infty^2 \sum_n |(B\phi_m, \phi_n)|^2 \geq \sum_n | \sum_k (A\phi_k, \phi_n)(B\phi_m, \phi_k)|^2$$

and summation over m gives

$$\|A\|_\infty \|B\|_2 \geq \|AB\|_2.$$

An entirely similar argument proves

$$\|A\|_2 \|B\|_\infty \geq \|AB\|_2.$$

It follows that L^2 is a (non-closed) ideal in $B(H)$, hence in particular an algebra. Moreover if $A, B \in L^2$ then $\|AB\|_2 \leq \|A\|_\infty \|B\|_2 \leq \|A\|_2 \|B\|_2$ so L^2 is a Banach algebra. Clearly each of the projections P_n on H_n is in L^2, in fact

$$\|P_n\|_2^2 = \sum_m |(P_n e_m, e_m)|^2 = n,$$

hence

$$P_n B(H) P_n = B(H_n) \subseteq L^2,$$

since L^2 is an ideal in $B(H)$. Moreover if $A \in L^2$ then

$$\|A - Q_n(A)\|_2^2 = \|A - P_n A P_n\|_2^2 = \sum_{m,k>n} |(Ae_k, e_m)|^2,$$

and since $\sum_{m,k} |(Ae_k, e_m)|^2 = \|A\|_2^2 < \infty$,

$$\|A - Q_n(A)\|_2 \longrightarrow 0, \text{ i.e. } Q_n(A) \longrightarrow A$$

in L^2. This completes the proof of

THEOREM (D). *The algebra L^2 of Hilbert–Schmidt operators is approximately tame.*

Note that as a corollary of Theorem (D) we recover the well-known result that if $A \in L^2$ then A is completely continuous. *A fortiori* A^*A is completely continuous and since it is self adjoint there is an orthonormal basis $\{\phi_n\}$ such that $A^*A\phi_n = \lambda_n \phi_n$. Then

$$\|A\|_2^2 = \sum_n (A^*A\phi_n, \phi_n) = \sum_n \lambda_n,$$

i.e. the square of the Hilbert–Schmidt norm of A is the sum of the eigenvalues of A^*A (with multiplicity). In particular if A is a self adjoint Hilbert–Schmidt operator then $\|A\|_2^2$ is the sum of the squares of the eigenvalues of A.

§5. THE ALGEBRAS L^p

If $A \in B(H)$ is a non-negative operator we define $\mathrm{tr}\, A = \|A^{1/2}\|_2^2$, where $A^{1/2}$ is the non-negative square root of A. If $A \in B(H)$ we write its canonical polar decomposition as

$A = U(A)|A|$, so $|A| = (A^*A)^{1/2}$ and $U(A)$ is a partial isometry with initial space $\overline{A^*(H)}$ and final space $\overline{A(H)}$, and we define $\|A\|_1 = \operatorname{tr}|A|$, and more generally for $1 \le p < \infty$ we define $\|A\|_p$ by $\|A\|_p^p = \operatorname{tr}|A|^p$. The set L^1 of operators of trace class is defined by $L^1 = \{A \in B(H) \| A\|_1 < \infty\} = \{A \in B(H) \| A|^{1/2} \in L^2\}$. Since L^2 is an ideal, if $A \in L^1$ then not only $|A|^{1/2}$ but also $|A|^{1/2}U(A)^*$ is Hilbert–Schmidt, so $\operatorname{tr} A = \langle |A|^{1/2}, |A|^{1/2}U(A)^*\rangle$ is well-defined and for any orthonormal basis $\{\phi_n\}$, $\operatorname{tr} A$ is given by the absolutely convergent series $\sum_n (A\phi_n, \phi_n)$.

More generally if $1 \le p < \infty$ we define $L^p = \{A \in B(H) \| \|A\|_p < \infty\}$ and we define q by $q = \infty$ if $p = 1$ and by $1/p + 1/q = 1$ otherwise. Also we make the conventions that $L^\infty = B(H)$ and that $(\sum_n |a_n|^q)^{1/q}$ means $\operatorname{Sup}_n |a_n|$ if $q = \infty$. If $A \in L^p$ then $|A|^p \in L^1$ so $|A|^{p/2} \in L^2$ and hence is completely continuous. Since $|A|^{p/2}$ is self adjoint there is an orthonormal basis $\{\theta_n\}$ of eigenvectors of $|A|^{p/2}$, hence of $|A|$, say $|A|\theta_n = \lambda_n\theta_n$. Then if $1 \le p \le r < \infty$,

$$\|A\|_r^r = \sum_n (|A|^r\theta_n, \theta_n) = \sum_n \lambda_n^r = \sum_n (\lambda_n^p)^{r/p} \le (\sum_n \lambda_n^p)^{r/p} = \|A\|_p^r,$$

i.e.

$$\|A\|_r \le \|A\|_p.$$

Also

$$\|A\|_\infty^2 = \|A^*A\|_\infty = \operatorname{Sup}_n \lambda_n^2$$

so

$$\|A\|_\infty^p = \operatorname{Sup}_n \lambda_n^p \le \sum_n \lambda_n^p = \|A\|_p^p$$

which is the same inequality for $r = \infty$. These inequalities are trivial with $A \notin L^p$, since then $\|A\|_p = \infty$. We have proved

LEMMA (1). *If $1 \le p \le r \le \infty$ then $\|A\|_r \le \|A\|_p$ for all $A \in B(H)$, hence $L^p \subseteq L^r$. In particular $L^1 \subseteq L^p$.*

Again let $A \in L^p$ and maintain the above notation. Let $\{\phi_n\}$ and $\{\psi_n\}$ be any orthonormal sequences of vectors in H and let $B \in B(H)$. Then by Schwarz's inequality

(1) $\sum_m |(B\psi_n, \theta_m)| \cdot |(\theta_m, \phi_n)| \le \|B\psi_n\| \cdot \|\phi_n\| \le \|B\|_\infty$

and

(2) $\sum_n |(B\psi_n, \theta_m)| \cdot |(\theta_m, \phi_n)| \le \|B^*\theta_m\| \cdot \|\theta_m\| \le \|B\|_\infty$.

Applying Hölder's inequality in the form

$$\begin{aligned}\sum_m |a_m| \cdot |b_m| &= \sum_m (|a_m| \cdot |b_m|^{1/p})(|b_m|^{1/q}) \\ &\le (\sum_m |a_m|^p |b_m|)^{1/p} (\sum_m |b_m|)^{1/q}\end{aligned}$$

with $a_m = \lambda_m$ and $b_m = (B\psi_n, \theta_m)(\theta_m, \phi_n)$ and taking (1) into account gives

$$(\sum_m \lambda_m |(B\psi_n, \theta_m)|\, |(\theta_m, \phi_n)|)^p \le \|B\|_\infty^{p/q} \sum_m \lambda_m^p |(B\psi_n, \theta_m)|\, |(\theta_m, \phi_n)|$$

and summing over n and taking account of (2) gives

$$\sum_n \Big(\sum_m \lambda_m |(B\psi_n, \theta_m)| \cdot |(\theta_m, \phi_n)|\Big)^p \leq \|B\|_\infty^{p/q+1} \sum_m \lambda_m^p \leq \|B\|_\infty^p \|A\|_\infty^p$$

since $p/q + 1 = p$. Now

$$|A|B\psi_n = |A| \sum_m (B\psi_n, \theta_m)\theta_m = \sum_m \lambda_m (B\psi_n, \theta_m)\theta_m$$

hence

$$|(|A|B\psi_n, \phi_n)| \leq \sum_m \lambda_m |(B\psi_n, \theta_m)|\, |(\theta_m, \phi_n)|$$

so

$$\sum_n (|A|B\psi_n, \phi_n)|^p \leq \|A\|_p^p \|B\|_\infty^p.$$

Since

$$|(BA\phi_n, \psi_n)| = |(|A|(U(A)^* B^*)\psi_n, \phi_n)| \quad \text{and} \quad \|U(A)^* B^*\|_\infty \leq \|B\|_\infty$$

we get

$$\Big(\sum_n |(BA\phi_n, \psi_n)|^p\Big)^{1/p} \leq \|A\|_p \|B\|_\infty.$$

In case $B = I$ we get equality if we take $\phi_n = \theta_n$ and $\psi_n = U(A)\theta_n$, hence

LEMMA (2). $\|A\|_p = \operatorname{Sup}(\sum_n |(A\phi_n, \psi_n)|^p)^{1/p}$ where the Sup is over all pairs $\{\phi_n\}$ and $\{\psi_n\}$ of orthonormal sequences.

Replacing A by BA in Lemma (2) the above inequality gives

LEMMA (3). $\|BA\|_p \leq \|A\|_p \|B\|_\infty$.

LEMMA (4). $\|A^*\|_p = \|A\|_p$ *and* $\|AB\|_p \leq \|A\|_p \|B\|_\infty$.

Proof. Since $|(A\phi_n, \psi_n)| = |(A^*\psi_n, \phi_n)|$ the equality of $\|A^*\|_p$ and $\|A\|_p$ is immediate from Lemma (2). Then by Lemma (3)

$$\|AB\|_p = \|B^* A^*\|_p \leq \|A^*\|_p \|B^*\|_\infty = \|A\|_p \|B\|_\infty.$$

Q.E.D.

THEOREM (E). *If* $1 \leq p < \infty$ *then* L^p *is an ideal in* $B(H)$ *and is a Banach algebra in the norm* $\|\ \|_p$ *with topology finer than that induced from* $B(H)$. *Moreover* L^p *is approximately tame.*

Proof. For any sequences $\{\phi_n\}$ and $\{\psi_n\}$ in H it follows from Minkowski's inequality that $A \longrightarrow (\sum_n |(A\phi_n, \psi_n)|^p)^{1/p}$ is a pseudo norm for $B(H)$, hence by Lemma (2) $\|\ \|_p$ is a norm for $B(H)$, hence L^p is a subspace of $B(H)$, and by Lemmas (3) and (4) it is an ideal. Moreover if $A, B \in L^p$ then by Lemmas (1) and (4)

$$\|AB\|_p \leq \|A\|_p \|B\|_\infty \leq \|A\|_p \|B\|_p$$

so L^p is a normed algebra under the norm $\|\ \|_p$, and by Lemma (1) again its topology is finer than that induced from $B(H)$.

Suppose $\{A_n\}$ is a Cauchy sequence in L^p. Then by Lemma (1) $\{A_n\}$ is Cauchy in $B(H)$, hence $A \longrightarrow A$ in $B(H)$, so $(A_n\phi, \psi) \longrightarrow (A\phi, \psi)$ for any $\phi, \psi \in H$. Now by Lemma (2) given $\varepsilon > 0$ there exists $N(\varepsilon)$ such that for all orthonormal sequences $\{\phi_n\}$ and $\{\psi_n\}$

$$(\sum_n |((A_j - A_k)\phi_n, \psi_n)|^p)^{1/p} < \varepsilon$$

if $j, k > N(\varepsilon)$. Letting $k \longrightarrow \infty$ and using Lemma (2) once more, $\|A - A_j\|_p < \varepsilon$ if $j > N(\varepsilon)$, proving the completeness of L^p.

If P is an orthogonal projection onto an n dimensional subspace E of H then $|P|^p = P$. Choosing an orthonormal basis $\{\phi_m\}$ such that $\{\phi_1, \dots, \phi_n\}$ is a basis for E,

$$\|P\|_p^p = \operatorname{tr} P = \sum_m (P\phi_m, \phi_m) = n$$

so $P \in L^p$. If $A \in B(H)$ has its range in E then $A = PA \in L^p$ since L^p is an ideal, hence the ideal $\mathscr{F}$ of $B(H)$ consisting of all $A \in B(H)$ such that rank $A = \dim A(H) < \infty$ is included in L^p. Then

$$\bigcup_n B(H_n) \subseteq \mathscr{F} \subseteq L^p.$$

Given $\theta \in H$ define $F_\theta \in \mathscr{F}$ by $F_\theta(x) = (x, \theta)\theta$. Then F_θ is a positive operator with the single eigenvalue $\|\theta\|^2$, hence

$$\|F_\theta^{1/2}\|_p = \|\theta\|.$$

Moreover if θ is a unit vector and $C_n = F_\theta(1 - P_n)$ then

$$C_n^* C_n = (1 - P_n)F_\theta(1 - P_n) = F_{(1-P_n)\theta},$$

so

$$\|F_\theta(1 - P_n)\|_p = \|F_{(1-P_n)\theta}^{1/2}\|_p = \|(1 - P_n)\theta\|.$$

If $A \in L^p$ is positive there is an orthonormal basis $\{\theta_n\}$ such that $A\theta_n = \lambda_n\theta_n$ with $\sum_n \lambda_n^p = \|A\|_p^p < \infty$. Let $A_m = \sum_{k \le m} \lambda_k F_{\theta_k}$. Then

$$\|A - A_m\|_p^p = \sum_{k>m} \lambda_k^p.$$

Given $\varepsilon > 0$ we can therefore choose m so large that $\|A - A_m\|_p < \varepsilon/3$. Then

$$\|(A - A_m)P_n\|_p \le \|A - A_m\|_p \|P_n\|_\infty < \varepsilon/3$$

for all n. Since

$$\|A_m(1 - P_n)\|_p \le \sum_{k \le m} \lambda_k \|F_{\theta_k}(1 - P_n)\|_p = \sum_{k \le m} \lambda_k \|(1 - P_n)\theta_k\|$$

and since

$$\|(1 - P_n)\theta_k\| \longrightarrow 0$$

as $n \longrightarrow \infty$ for each k, if $m > N(\varepsilon)$ then

$$\|A_m(1 - P_n)\|_p < \varepsilon/3.$$

Then for $n > N(\varepsilon)$

$$\|A(1 - P_n)\|_p \le \|A - A_m\|_p + \|A_m(1 - P_n)\|_p + \|(A - A_m)P_n\| < \varepsilon$$

so

$$\|A(1 - P_n)\|_p \longrightarrow 0.$$

If $A \in L^p$ is not necessarily positive, then

$$\|A(1 - P_n)\|_p = \|U(A)|A|(1 - P_n)\|_p \le \|U(A)\|_\infty \||A|(1 - P_n)\|_p \longrightarrow 0.$$

Then by Lemma (4)

$$\|(1 - P_n)A\|_p = \|(A^*(1 - P_n))^*\| \longrightarrow 0$$

and

$$\|P_nA(1 - P_n)\|_p \leq \|P_n\|_\infty \|A(1 - P_n)\|_p \leq \|A(1 - P_n)\|_p \longrightarrow 0.$$

Hence

$$\|A - P_nAP_n\|_p \leq \|A(1 - P_n)\|_p + \|P_nA(1 - P_n)\|_p \longrightarrow 0,$$

i.e.

$$Q_n(A) \longrightarrow A \text{ in } L^p.$$

Q.E.D.

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