HYPERPOLAR ACTIONS AND k-FLAT HOMOGENEOUS SPACES

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Abstract. A closed, connected, k-dimensional submanifold of a compact Riemannian manifold M is called a k-flat of M if it is flat in the induced metric and totally geodesic. We call M "k-flat homogeneous" if every geodesic lies in some k-flat of M, and if the group of isometries of M acts transitively on pairs (σ, p) consisting of a k-flat σ and a point $p \in \sigma$. An isometric action on M is called hyperpolar if there exists a connected, closed, flat submanifold of M that meets all orbits orthogonally. We prove that the following three properties for a compact Riemannian manifold M are equivalent: (a) M is a Riemannian homogeneous manifold and admits a cohomogeneity k hyperpolar action with a fixed-point, (b) M is k-flat homogeneous, (c) M is a rank k symmetric space. Since 1-flat homogeneous is trivially equivalent to two-point homogeneous, the equivalence of (b) and (c) generalizes the well-known fact that two-point homogeneous spaces are the same as rank 1 symmetric spaces.

1. Introduction

An isometric action of a compact Lie group on a Riemannian manifold is called *polar* if there exists a connected, closed submanifold Σ (called a *section*) that meets all orbits orthogonally. A section is automatically totally geodesic, and if it is also flat in the induced metric then we will call the action *hyperpolar*. (Note that a flat section is the same thing as a "K-transversal domain" in the sense of Conlon [C]).

One of the goals of this paper is to give a structure and classification theory for hyperpolar actions with a fixed-point on compact, homogeneous Riemannian manifolds.

Recall that a connected, compact Riemannian manifold M is called *two-point homogeneous* if, given x_i, y_i in M such that the distance $d(x_1, x_2)$ is equal to the distance $d(y_1, y_2)$, there is an isometry φ of M such that $\varphi x_i = y_i$. Another goal of this paper is to give a generalization of the well-known fact that a two-point homogeneous space is a symmetric space of rank 1. (For a discussion of the history of this theorem, cf. [W2]; a simple and elegant proof will be found in [Sz]). To give a precise statement of our generalization, we need some further definitions.

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1.1 Definition. A k-dimensional closed and connected submanifold of a Riemannian manifold M is called a *k-flat* of M if it is totally geodesic and is flat in the induced metric. M is called *k-flat homogeneous* if every geodesic is contained in some k-flat, and if the group of isometries of M acts transitively on the set of pairs (x, τ), where τ is a k-flat and $x \in \tau$ (i.e., given two such pairs, (x_1, τ_1) and (x_2, τ_2), there exists an isometry φ of M such that $\varphi x_1 = x_2$ and $\varphi \tau_1 = \tau_2$).

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It is obvious that 1-flat homogeneous is equivalent to two-point homogeneous, and it also follows easily from the standard theory of symmetric spaces that a rank k-symmetric space is k-flat homogeneous. We show that the converse is also true. In fact, our main result is:

1.2 Theorem. *If M is a compact, connected Riemannian manifold, then the following three properties are equivalent:*

(a) M is a homogeneous Riemannian G-manifold, and there exists a closed subgroup H of G such that the action of H on M is hyperpolar of cohomogeneity k and has a fixed-point,

(b) M is k-flat homogeneous,

(c) M is a rank k symmetric space.

Next we give some idea of the proof of this theorem. It is not difficult to see that (a) and (b) are equivalent and, as we have said, it has long been known that (c) implies (a) (cf. Theorem 6.2, Chapter V of [He]). Thus, it suffices to prove that (a) implies (c). For this, we can assume that H is the isotropy subgroup of the fixed point, and we prove that if the action of H on a homogeneous manifold $M = G/H$ is hyperpolar with respect to some G-invariant metric on M, then it is also hyperpolar with respect to any normal G-invariant metric (for the definition of "normal", see (4) of section 2.1). Thus we may assume that the pair (G, H) satisfies the following conditions:

(i) G is a compact, connected Lie group equipped with a bi-invariant metric induced from an Ad_G-invariant inner product (,) on its Lie algebra $\mathfrak{g}$,

(ii) H is a closed subgroup of G, and the $\mathrm{Ad}_G(H)$-action on $\mathfrak{p} = \mathfrak{h}^\perp$ is polar with an abelian subalgebra in $\mathfrak{p}$ as a section.

Next we prove a decomposition theorem for the pairs (G, H) that satisfy conditions (i) and (ii). Namely, if the representation of H on $\mathfrak{p}$ is decomposed into irreducible H-spaces, then some finite cover of G/H can be decomposed accordingly as a direct product of isotropy irreducible homogeneous spaces. Using the classification theorem for isotropy irreducible homogeneous spaces, see [M], [W1], [Kr], [WZ], we prove that if (G, H) is a pair satisfying properties (i) and (ii), and if G/H is isotropy irreducible, then there is a finite cover $\pi : \tilde{G}/\tilde{H} \to G/H$ such that $\tilde{G}/\tilde{H}$ is an irreducible symmetric space. This shows that G/H is a locally symmetric space. Finally we prove that $\pi^{-1}(eH)$ lies in the center of the symmetric space $\tilde{G}/\tilde{H}$, which implies that G/H is a globally symmetric space.

If $M = G/H$ is 1-flat homogeneous—or equivalently if M is two-point homogeneous—then the isotropy representation of H is transitive on the unit sphere of TM_{eH}, so M is isotropy irreducible and hence, by Schur's Lemma, the invariant metric for M is unique up to scaling, and in particular it is automatically normal. Thus neither the result about changing metrics nor the decomposition theorem stated above is needed in this case, and only the simpler classification of transitive actions on spheres is required to prove that M is a locally symmetric space of rank 1. In fact, with these simplifications, our proof in this case reduces essentially to the classical proof.

2. Preliminary Results

In this section we will set up our notations and review some definitions and results from the theory of transformation groups and symmetric spaces.

Let G be a Lie group, $\mathfrak{g} = TG_e$ its Lie algebra, and X a smooth G-manifold. Each element ξ of $\mathfrak{g}$ will also be viewed as a vector field on X, namely the vector field generating the one-parameter group $\exp(t\xi)$ of diffeomorphisms of X. If $x \in X$ then G_x denotes the isotropy group at x and Gx the orbit of x. Clearly the tangent space to Gx at x is $T(Gx)_x = \{\xi(x) \mid \xi \in \mathfrak{g}\}$, and the Lie algebra of G_x is $\{\xi \in \mathfrak{g} \mid \xi(x) = 0\}$.

We call X a Riemannian G-manifold if the action of G on X is isometric. In this case, for each ξ in $\mathfrak{g}$ the corresponding vector field on X is a Killing field. The normal space to the orbit Gx at x will be denoted by $\nu(Gx)_x$, or simply by ν_x. A connected, closed submanifold Σ of X is called a *section* for (the action of G on) X, if Σ "meets every orbit", i.e., $G\Sigma = X$, and if Σ "meets orbits orthogonally", i.e., for each x in Σ, $T\Sigma_x \subseteq \nu_x$. If X admits a section then the action of G on X is called *polar*. If X admits a section that is flat in the induced metric, then the action of G on X is called *hyperpolar*.

2.1 Homogeneous compact Riemannian G-manifolds.

Let G be a compact Lie group, $(\,,\,)_o$ an Ad_G-invariant inner product on $\mathfrak{g}$, x a point in a homogeneous G-manifold M, $H = G_x$, and $\mathfrak{p}$ the orthogonal complement of $\mathfrak{h}$ in $\mathfrak{g}$ with respect to $(\,,\,)_o$. Then the map $gH \mapsto gx$ is a G-equivariant diffeomorphism of G/H with M that we will usually regard as an identification. We next recall some standard definitions and well-known facts: (cf. Chapter 7 of Besse's "Einstein Manifolds" [Be])

(1) We can identify $\mathfrak{p}$ with TM_x via $\xi \mapsto \xi(x)$, and then the isotropy representation of M at x is H acting on $\mathfrak{p}$ via the $\mathrm{Ad}_G(H)$-action.

(2) M is called *isotropy irreducible* if the isotropy representation at x is irreducible, i.e., if $\mathfrak{p}$ is irreducible under $\mathrm{Ad}_G(H)$. Similarly, M is called *strongly isotropy irreducible* if $\mathfrak{p}$ is irreducible under $\mathrm{Ad}_G(H_0)$.

(3) There exists a bijective correspondence between $\mathrm{Ad}_G(H)$-invariant inner products on $\mathfrak{p}$ and G-invariant metrics on M.

(4) The G-invariant metric on M corresponding to the restriction of an Ad_G-invariant inner product $(\,,\,)_o$ to $\mathfrak{p}$ is called the *normal metric* associated to $(\,,\,)_o$, and the corresponding Riemmanian G-manifold M is called a *normal* homogeneous Riemannian G-manifold.

An action of G on M is called *effective* (respectively, *almost effective*) if the kernel, N, of the group homomorphism $\rho : G \to \mathrm{Diff}(M)$ defined by $\rho(g)(x) = g \cdot x$ is $\{e\}$ (respectively, of dimension zero). Since N is a subgroup of H that is normal in G, by replacing G by G/N and H by H/N, we may assume that our homogeneous space G/H is effective whenever necessary. It is easy to see that the action of

G on G/H is effective (almost effective) if and only if H does not contain any (respectively, any non-discrete) normal subgroup of G.

We will use $I(M)$ to denote the group of isometries of M, and $I_o(M)$ for its identity component. Likewise G_o will denote the identity component of the Lie group G.

2.2 Compact symmetric spaces.

A Riemannian manifold M is called a *globally symmetric* if for each point $x \in M$ there exists an isometry s_x such that $s_x(x) = x$ and $D(s_x)_x = -\mathrm{id}$. (Since, in general, an isometry is determined by its differential at any point, s_x is unique.) The globally symmetric condition implies that the curvature tensor is covariant constant, and Riemannian manifolds that satisfy this weaker condition are called *locally symmetric*. Henceforth we will refer to globally symmetric Riemannian manifolds simply as symmetric spaces.

Let M be a connected, compact symmetric space, and G the group of *transvections*, i.e., the group generated by the $s_x s_y$ for all $x, y \in M$. Then the following are well-known facts (cf. [He], [L]):

(1) G acts transitively on M. We fix a point $p \in M$ and let $K = G_p$ denote its isotropy subgroup, so $M = G/K$. A pair (G, K) arising in this way will be called a *symmetric pair*.

(2) The map $\sigma : G \to G$ defined by $\sigma(g) = s_p g s_p$ is an involution (i.e., an automorphism of order two), and $(G_\sigma)_o \subseteq K \subseteq G_\sigma$, where G_σ is the fixed-point set of σ.

(3) Let $Z(M)$ denote the fixed-point set of the K_o-action on M, i.e.,

$$Z(M) = \{x \in M \mid k \cdot x = x, \ \forall\ k \in K_o\}.$$

We will call $Z(M)$ the *center* of M, as in Chapters IV and VI of Loos [L]. Loos shows that $Z(M)$ has a natural abelian group structure and acts freely on M.

(4) If F is a discrete subgroup of $Z(M)$ then $M' = M/F$ is also a symmetric space.

(5) Let $\mathfrak{p}$ denote the -1 eigenspace of $D\sigma_e$ on $\mathfrak{g}$. Then the isotropy representation of M at p is the $\mathrm{Ad}_G(K)$ action on $\mathfrak{p}$. Any representation equivalent to the isotropy representation of a symmetric space is called an *s-representation*, and in particular the representation of H on $\mathfrak{p}$ is called the s-representation of the symmetric pair (G, K).

(6) M is k-flat homogeneous.

(7) The decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ satisfies the following conditions:

$$[\mathfrak{k}, \mathfrak{k}] \subseteq \mathfrak{k}, \quad [\mathfrak{k}, \mathfrak{p}] \subseteq \mathfrak{p}, \quad [\mathfrak{p}, \mathfrak{p}] \subseteq \mathfrak{k}. \tag{$*$}$$

(8) A decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ of a Lie algebra $\mathfrak{g}$ satisfying the condition $(*)$ is called a *Cartan decomposition*. Such a decomposition defines a Lie algebra involution $D\sigma_e$ on $\mathfrak{g}$ by requiring that $\mathfrak{k}$ and $\mathfrak{p}$ are respectively the $+1$ and -1 eigenspaces of $D\sigma_e$.

(9) Let $\mathfrak{g}$ be a semi-simple Lie algebra, $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ a Cartan decomposition, and $\tilde{G}$ the simply-connected Lie group associated to $\mathfrak{g}$. Let $\tilde{K} = \exp(\mathfrak{k})$. Then $\tilde{G}/\tilde{K}$ is a simply-connected, symmetric space.

2.3 Polar actions.

Suppose M is a Riemannian G-manifold, the G-action on M is polar, and Σ is a section. Then the following are known ([PT1]):

(i) Σ is a totally geodesic submanifold of M. (Since totally geodesic submanifolds of $\mathbf{R}^n$ are automatically flat, it follows that polar representations are hyperpolar!)

(ii) $g\Sigma$ is also a section for each g in G so, since $G\Sigma = M$, there is a section through each point of M. Moreover, every section is of the form $g\Sigma$ for some $g \in G$.

(iii) Define the *normalizer* and *centralizer* of Σ in G by

$$N(\Sigma, G) = \{g \in G \mid g(\Sigma) = \Sigma\}, \quad Z(\Sigma, G) = \{g \in G \mid g(s) = s \quad \forall s \in \Sigma\}.$$

(Clearly $N(\Sigma, G)$ is the largest subgroup of G that acts on Σ, and $Z(\Sigma, G)$ is the kernel of this action.) The quotient $W(\Sigma, G) = N(\Sigma, G)/Z(\Sigma, G)$ is called the *generalized Weyl group* of the section Σ. It is a finite group acting effectively on Σ.

(iv) Recall that ν_x is an invariant subspace of the isotropy representation of G_x on TM_x, and the corresponding subrepresentation of G_x is called the *slice representation* at x. Every slice representation of M is polar; in fact if Σ is a section containing x then $T\Sigma_x$ is a section for the slice representation at x.

(v) The set M^o of points of M where the slice representation is trivial is called the set of *regular* points of M. It is a union of orbits, and these are called the *principal orbits* of M. M^o is an open, dense, connected subset of M, and is fibered by the principal orbits. The principal orbits all have the same (maximal) dimension, and their codimension, called the *cohomogeneity* of M, is the same as the dimension of any section. It follows that at a regular point p, $\exp(\nu_p)$ is the unique section through p.

2.4 Proposition. *Let M be a Riemannian G-manifold. A submanifold Σ of M is a section for the action of G if and only if it is a section for the action of G_o. In particular, the action of G on M is polar if and only if the action of G_o on M is polar.*

Proof. Since G_o-orbits are components of G-orbits, Σ meets G-orbits orthogonally, if and only if it meets G_o-orbits orthogonally. Clearly $G_o\Sigma = M$ implies $G\Sigma = M$, so it remains only to prove that if Σ is a slice for the action of G then it meets every G_o-orbit. To see this, let $p \in \Sigma$ be on a regular G-orbit. Since G_op is a connected component of Gp, $\nu(Gp)_p = \nu(G_op)_p$, so since Σ is totally geodesic and $T\Sigma_p = \nu(Gp)_p$, $\Sigma = \exp(\nu(G_op)_p)$. But whenever a Lie group H acts isometrically on a connected, complete Riemannian manifold X, it is well-known that for any $x \in X$, $\exp(\nu(Hx)_x)$ meets every H-orbit (cf. [PT1]). $\square$

Polar representations were classified up to "orbital equivalence" (see below) by Dadok [D]. We need some of his results, which we now state.

2.5 Theorem. *(Theorem 4 of [Da]) Suppose H is a connected, compact Lie group, $\rho : H \to \mathbf{SO}(V)$ a polar representation, and $V = V_1 \oplus V_2$ is a direct sum decomposition of V into H-invariant subspaces. Let $\mathfrak{a}$ be a section for V, $\mathfrak{a}_i = \mathfrak{a} \cap V_i$, and let $H_1 = Z_o(\mathfrak{a}_2, H)$ and $H_2 = Z_o(\mathfrak{a}_1, H)$ denote the identity components of the centralizers in H of $\mathfrak{a}_2$ and $\mathfrak{a}_1$ respectively. Then*

(i) $\mathfrak{a} = \mathfrak{a}_1 \oplus \mathfrak{a}_2$,
(ii) *the H_i-action on V_i is polar with $\mathfrak{a}_i$ as a section,*
(iii) *if $a = a_1 + a_2 \in \mathfrak{a}_1 \oplus \mathfrak{a}_2$, then $H \cdot a = (H_1 \cdot a_1) \times (H_2 \cdot a_2)$.*

2.6 Remark. Note that H and $H_1 \times H_2$ are *not* equal in general. Nevertheless, from the point of view of the geometry of orbits, 2.5 can be viewed as a decomposition theorem.

2.7 Definition. Let G_1 and G_2 be two Lie groups and let X_i be a Riemannian manifold on which G_i acts isometrically. We shall call these two actions *orbitally equivalent*, or ω*-equivalent* if there is an isometry of X_1 with X_2 mapping G_1 orbits to G_2 orbits.

Note that ω-equivalence is in general a relation between actions of possibly *different* groups. For example, the natural actions of $\mathbf{SO}(2n)$ on $\mathbf{R}^{2n}$ and of $\mathbf{SU}(n)$ on $\mathbf{C}^n$ are ω-equivalent.

2.8. Remark. Clearly any action that is ω-equivalent to a polar (hyperpolar) action is itself polar (hyperpolar). It was proved by Bott and Samelson [BS] that s-representations are polar, and Dadok's main result is that up to ω-equivalence there are no others.

2.9 Theorem. *(Dadok [D]) A polar representation of a compact, connected Lie group is ω-equivalent to an s-representation.*

The following results are easy consequences of 2.5.

2.10 Proposition. *Suppose $\rho : H \to \mathbf{SO}(V)$ is a polar representation, and $V = V_o \oplus V_1 \oplus \cdots \oplus V_r$ is a decomposition of V as a direct sum of H-invariant subspaces such that V_o is a trivial H-space and the V_i are non-trivial irreducible H-spaces for $1 \le i \le r$. Then V_i and V_j are inequivalent H-spaces for $1 \le i < j \le r$.*

Proof. We denote the infinitesimal action of an element h of $\mathfrak{h}$ on an element v of V by $h \cdot v$. Suppose V_1 is equivalent to V_2, and let $\varphi : V_1 \to V_2$ be an H-equivariant linear isomorphism. Then we have $\varphi(h \cdot x) = h \cdot \varphi(x)$ for all $h \in \mathfrak{h}$ and $x \in V_1$. Let $a_1 \in V_1$ be a regular element for the H-action on V_1. Then $a_2 = \varphi(a_1)$ is a regular element for the H-action on V_2. Let $\mathfrak{a}$ be a section of V containing $a_1 + a_2$, $\mathfrak{a}_i = \mathfrak{a} \cap V_i$, and let $H_1 = Z(\mathfrak{a}_2, H)$ be as in Dadok's Theorem 2.5. Then, since $h_1 \cdot a_2 = 0$ for all $h_1 \in \mathfrak{h}_1$ and $a_2 \in \mathfrak{a}_2$, $\varphi(h_1 \cdot a_1) = h_1 \cdot \varphi(a_1) = 0$. Since φ is an isomorphism, $h_1 \cdot a_1 = 0$, so V_1 is a trivial representation, a contradiction. □

Next we prove that whether or not a representation is polar is independent of the choice of H-invariant scalar product on the representation space.

2.11 Theorem. *Let $\rho : H \to GL(V)$ be a representation, and $(\,,\,)_k$, $k = 1, 2$, H-invariant inner products on V. If the H-action on $(V, (\,,\,)_1)$ is polar and $\mathfrak{a}$ is a section, then*
(1) the H-action on $(V, (\,,\,)_2)$ is also polar with $\mathfrak{a}$ as a section,
(2) the orthogonal complements of $\mathfrak{a}$ in V are the same with respect to both inner products; namely, if a is a point where $\mathfrak{a}$ meets a principal orbit, then both are equal to $T(Ha)_a$.

Proof. By Lemma 2.10, we may write V as a direct sum $V = V_o \oplus V_1 \oplus \cdots \oplus V_r$ of H-invariant subspaces such that V_o is a trivial H-space and $V_1, \ldots, V_r$ are non-trivial, inequivalent irreducible H-spaces. It follows that $V_o, \ldots, V_r$ are mutually orthogonal with respect to *any* H-invariant inner product on V. Since V_i is irreducible for $1 \leq i \leq r$, there exists $c_i > 0$ such that $(\,,\,)_2 = c_i(\,,\,)_1$ on V_i. Let $\mathfrak{a}_i = \mathfrak{a} \cap V_i$, $a = a_o + \ldots + a_r \in \mathfrak{a}$ a regular point, and P_i the tangent plane of the orbit Ha_i at a_i. Since the H-action on $(V, (\,,\,)_1)$ is polar, by 2.5 H_i on V_i is polar with $\mathfrak{a}_i$ as section. Hence $\mathfrak{a}_i \perp P_i$ with respect to $(\,,\,)_1$. Note that $\mathfrak{a}_o = V_o$, while if $i > 0$ then, since the two inner products on V_i are proportional, $\mathfrak{a}_i$ is also orthogonal to P_i with respect to $(\,,\,)_2$, so (1) and (2) follow. $\square$

Now we review some elementary properties of Killing vector fields and totally geodesic, flat submanifolds.

2.12 Proposition. *([Be] Proposition 7.87) Let M be a homogeneous G-manifold, $x \in M$, and $H = G_x$. Let $\mathfrak{p}$ denote the orthogonal complement of $\mathfrak{h}$ in $\mathfrak{g}$ with respect to an $\mathrm{Ad}(G)$-invariant inner product, $(\,,\,)_o$, on $\mathfrak{g}$, and let (M, ds_o^2) be the normal homogeneous Riemannian G-manifold associated to $(\,,\,)_o$. If we identify TM_x with $\mathfrak{p}$ as in 2.1, then the Riemann tensor R of (M, ds_o^2) at x satisfies:*

$$(R(\xi, \eta)(\xi), \eta)_o = ([\xi, \eta]_{\mathfrak{h}}, [\xi, \eta]_{\mathfrak{h}})_o + \frac{1}{4}([\xi, \eta]_{\mathfrak{p}}, [\xi, \eta]_{\mathfrak{p}})_o,$$

where $[\xi, \eta]_{\mathfrak{h}}$ and $[\xi, \eta]_{\mathfrak{p}}$ are respectively the $\mathfrak{h}$ and $\mathfrak{p}$ components of $[\xi, \eta]$.

2.13 Corollary. *With the same assumptions as above, if Σ is a totally geodesic submanifold of M containing x, then Σ is flat if and only if $T\Sigma_x$ is an abelian subalgebra of $\mathfrak{p}$.*

As a consequence of 2.13 and 2.3, we have

2.14 Proposition. *If G/H is a compact, normal homogeneous G-manifold such that the action of H on G/H is hyperpolar, then the action of H on $\mathfrak{p} = \mathfrak{h}^{\perp}$ is polar with abelian subalgebras of $\mathfrak{p}$ as sections.*

We end this section by giving a sufficient condition for a totally geodesic, flat submanifold of a Riemannian G-manifold M to be a section. (This is actually a special case of a theorem of R. Hermann [H2].)

2.15 Theorem. *If Σ is a compact, connected, flat, and totally geodesic submanifold of a Riemannian H-manifold M and Σ is orthogonal to some H-orbit at one point, then Σ meets H-orbits orthogonally. If in addition the dimension of Σ is the cohomogeneity of the H-action on M, then Σ is a section for M and the H-action on M is hyperpolar.*

We recall that a necessary and sufficient condition for a vector field ξ on M to be a Killing field on M is that $\langle \nabla_u \xi, v \rangle = -\langle u, \nabla_v \xi \rangle$ for all $u, v \in TM_x$. We will need two easy facts concerning Killing fields.

2.16 Lemma. *A Killing vector field ξ on a compact, connected, flat Riemannian manifold τ has constant length. In particular, if ξ vanishes at one point then it is identically zero.*

Proof. The universal cover of τ is the Euclidean space $\mathbf{R}^n$. The lifting $\tilde{\xi}$ of ξ to $\mathbf{R}^n$ is a Killing vector field on $\mathbf{R}^n$, so there is a skew-adjoint operator A on $\mathbf{R}^n$ and $b \in \mathbf{R}^n$ such that $\tilde{\xi}(x) = Ax + b$. Since τ is compact, ξ is bounded, and hence so is $\tilde{\xi}$. This implies that $A = 0$, so $\tilde{\xi}$ is a constant vector field on $\mathbf{R}^n$, and so the length of ξ is constant. $\square$

2.17 Lemma. *Let τ be a totally geodesic submanifold of M, ξ a Killing vector field on M, and ξ^τ the vector field on τ defined by $\xi^\tau(x) =$ the projection of $\xi(x)$ onto $T\tau_x$. Then ξ^τ is a Killing vector field on τ.*

Proof. Let $\bar{\nabla}$ denote the Levi-Civita connection of M, ∇ the induced connection on $T\tau$, $\xi^\perp$ the projection of ξ to the normal bundle $\nu(\tau)$, and $u, v \in T\tau_x$. Since τ is totally geodesic, $\bar{\nabla}_u \xi^\perp \in \nu(\tau)_x$ and $\bar{\nabla}_u \xi^\tau = \nabla_u \xi^\tau$. It follows that $\langle \bar{\nabla}_u \xi, v\rangle = \langle \nabla_u \xi^\tau, v\rangle = -\langle u, \bar{\nabla}_v \xi\rangle = -\langle u, \nabla_v \xi^\tau\rangle$, so ξ^τ is a Killing vector field of τ. $\square$

2.18 Proof of Theorem 2.15. Since each $\xi \in \mathfrak{h}$ is a Killing field on M, it follows from 2.16 and 2.17 that if ξ is orthogonal to Σ at one point of Σ then ξ is orthogonal to Σ at every point of Σ. Recalling that $T(Hs)_s = \{\xi(s) \mid \xi \in \mathfrak{h}\}$ it now follows that if $T\Sigma_s \subseteq \nu(Hs)_s$ holds for one point, s, of Σ it also holds at every other point of Σ.

Suppose that Σ contains at least one regular point p. Then $T\Sigma_p = \nu_p$ and since Σ is connected and totally geodesic, $\Sigma = \exp \nu_p$. On the other hand for any orbit Gq, $\exp(\nu_q)$ meets every orbit, see 2.5(iii) in [PT1]. It follows that Σ meets every orbit. Hence Σ is a section since it meets the orbits perpendicularly.

Let us now assume that Σ does not contain any regular point of the action of H on M. Among all isotropy groups occurring along Σ, let H denote one that is minimal. Let

$$\tilde{M} = M_{(H)} = \{p \in M \mid G_p \text{ is conjugate to } H\}.$$

The set $\tilde{M}$ is a locally closed submanifold of M; the so-called H-stratum. It is a basic fact that

$$\dim(\tilde{M}/G) < \dim(M/G) = \operatorname{cohom}(M);$$

see Theorem 3.8, p. 184, of [Br]. Clearly, $\pi : \tilde{M} \to \tilde{M}/G$ restricts to an immersion $\Sigma \to \tilde{M}/G$. Hence $\dim(\Sigma) < \operatorname{cohom}(M)$, which is a contradiction. It follows that Σ must contain a regular point, and this completes the proof of the theorem. $\square$

3. Classification of polar pairs

In this section, we will prove that if G/H is a normal homogeneous manifold such that the action of H on G/H is hyperpolar, then G/H is a symmetric space. To prove this, we define the following related notion of polar pairs and classify them.

3.1 Definition. A pair (G, H) is called a *polar pair* if it satisfies the following conditions:

(a) G is a compact, connected Lie group equipped with a bi-invariant metric induced from an $\mathrm{Ad}(G)$-invariant inner product $\langle\,,\,\rangle$ on the Lie algebra $\mathfrak{g}$,
(b) H is a closed subgroup of G,
(c) the action of G on G/H is almost effective,
(d) the $\mathrm{Ad}_G(H)$-action of H on $\mathfrak{p} = \mathfrak{h}^\perp$ is polar with abelian subalgebras as sections. (We refer to these as *abelian sections*).

3.2 Definition. A polar pair (G, H) is called irreducible if G/H is strongly isotropy irreducible, i.e., if the representation of the connected component H_0 of H on $\mathfrak{p}$ is irreducible.

3.3 Remark. By 2.4, H on $\mathfrak{p}$ is polar if and only if H_o on $\mathfrak{p}$ is polar, and hence (G, H) is a polar pair if and only if (G, H_o) is a polar pair. If G is semi-simple, $\tilde{G}$ the simply-connected Lie group corresponding to $\mathfrak{g}$, and $\tilde{H}$ the subgroup $\exp(\mathfrak{h})$ of $\tilde{G}$, then (G, H) is a polar pair if and only if $(\tilde{G}, \tilde{H})$ is a polar pair.

3.4 Example. By 2.14, if G/H is a compact, normal homogenous G-manifold such that the action of H on G/H is hyperpolar, then (G, H) is a polar pair.

3.5 Example. Since s-representations are polar, if M is a compact symmetric space, then the symmetric pair (G, K) associated to M is a polar pair.

3.6 Example. $\mathbf{Spin}(7)$ acts on $\mathbf{R}^8$ by the spin representation, and it is transitive on the unit sphere $\mathbf{S}^7$ with isotropy group $\mathbf{G}_2$. Moreover, the isotropy representation of $\mathbf{S}^7 = \mathbf{Spin}(7)/\mathbf{G}_2$ is the irreducible $\mathbf{G}_2$-representation on $\mathbf{R}^7$, which is transtive on $\mathbf{S}^6$. Hence the $\mathbf{G}_2$-action on $\mathbf{S}^7$ is hyperpolar (the normal geodesic to a principal orbit is a section), and $(\mathbf{Spin}(7), \mathbf{G}_2)$ is a polar pair, but *not* a symmetric pair. Similarly, the seven dimensional representation of $\mathbf{G}_2$ is transitive on the unit sphere $\mathbf{S}^6$ with isotropy group $\mathbf{SU}(3)$. The isotropy representation of $\mathbf{S}^6 = \mathbf{G}_2/\mathbf{SU}(3)$ is the standard representation of $\mathbf{SU}(3)$ on $\mathbf{C}^3 = \mathbf{R}^6$ which is of course polar. Hence $(\mathbf{G}_2, \mathbf{SU}(3))$ is another example of a polar pair that is not a symmetric pair.

3.7 Theorem. *Suppose (G, H) is a polar pair, and $\mathfrak{p} = \mathfrak{p}_1 \oplus \cdots \oplus \mathfrak{p}_r$ is a direct sum decomposition of $\mathfrak{p} = \mathfrak{h}^\perp$ into irreducible H_o-spaces. Let*

$$\mathfrak{g}_i = \mathfrak{p}_i + [\mathfrak{p}_i, \mathfrak{p}_i], \quad \mathfrak{h}_i = \mathfrak{h} \cap \mathfrak{g}_i.$$

Then

(1) $\mathfrak{g} = \mathfrak{g}_1 \oplus \cdots \oplus \mathfrak{g}_r$ and $\mathfrak{h} = \mathfrak{h}_1 \oplus \cdots \oplus \mathfrak{h}_r$ are direct sum decompositions of $\mathfrak{g}$ and $\mathfrak{h}$ respectively into ideals,
(2) if $I = \{i \mid \mathfrak{p}_i$ is a trivial H-space$\}$, then $\mathfrak{h}_i = 0$, $\mathfrak{g}_i = \mathfrak{p}_i$ for all $i \in I$ and $\mathfrak{p}_o = \oplus\{\mathfrak{p}_i \mid i \in I\}$ is the center $\mathfrak{z}$ of $\mathfrak{g}$,
(3) $Z = \exp(\mathfrak{z}) = T^m$ is a torus, and $H_j = \exp(\mathfrak{h}_j)$ and $G_j = \exp(\mathfrak{g}_j)$ are closed, connected subgroups of G for $j \notin I$,
(4) for $j \notin I$, (G_j, H_j) is an irreducible polar pair with G_j semi-simple,
(5) G_i and G_j commute if $i \neq j$.

To prove this theorem, we need the following Lemma:

3.8 Lemma. *Let (G, H) be a polar pair, $\mathfrak{p} = \mathfrak{h}^\perp$, and $X, Y \in \mathfrak{p}$. Then $[X, Y] \perp \mathfrak{h}$ if and only if $[X, Y] = 0$, so in particular the normal space ν_X of the H-orbit through X is the set $\mathfrak{z}(X) \cap \mathfrak{p} = \{Y \in \mathfrak{p} \mid [X, Y] = 0\}$, where $\mathfrak{z}(X)$ denotes the centralizer of X in $\mathfrak{g}$.*

Proof. Let $\mathfrak{a}$ be an abelian section containing X for the H-action on $\mathfrak{p}$, and let H_X be the isotropy group of X. Since the H-action on $\mathfrak{p}$ is polar, it follows from the

Slice Theorem for isoparametric submanifolds (cf. [PT2], [HOT]) that

$$\nu_X = \bigcup_{h \in H_X} \mathrm{Ad}(h)(\mathfrak{a}).$$

Because $\mathfrak{a}$ is abelian and $X \in \mathfrak{a}$, we have

$$\bigcup_{h \in H_X} \mathrm{Ad}(h)(\mathfrak{a}) \subseteq \mathfrak{z}(X) \cap \mathfrak{p}.$$

This implies that $\nu_X \subseteq \mathfrak{z}(X) \cap \mathfrak{p}$. Conversely, let $Y \in \mathfrak{z}(X) \cap \mathfrak{p}$. Since $\langle\, ,\, \rangle$ is $\mathrm{Ad}(G)$-invariant,

$$\langle Y, [\mathfrak{h}, X] \rangle = \langle [X, Y], \mathfrak{h} \rangle = 0.$$

This proves $Y \in \nu_X$, and the lemma follows. $\square$

3.9 Proof of Theorem 3.7.

First we note that

$$[\mathfrak{p}_i, \mathfrak{p}_j] = 0, \quad \text{if } i \neq j.$$

This follows directly from 3.8, since:

$$\langle [\mathfrak{p}_i, \mathfrak{p}_j], \mathfrak{h} \rangle = \langle \mathfrak{p}_i, [\mathfrak{p}_j, \mathfrak{h}] \rangle = \langle \mathfrak{p}_i, \mathfrak{p}_j \rangle = 0.$$

We will prove each statement of the theorem seperately below.

(1) We want to show that $\mathfrak{g}$ decomposes into a direct sum of ideals $\mathfrak{g}_i$. (Notice that this is not the case for every polar isotropy representation of a homogeneous space G/H. An easy counter-example is $G = \mathbf{SU}(n+1)$ and $H = \mathbf{SU}(n)$. Here $\mathfrak{p}$ has a trivial factor, although G being simple cannot split.)

We first prove that

$$\mathfrak{g} = \mathfrak{p} + [\mathfrak{p}, \mathfrak{p}].$$

(This sum is in general not direct.) Let us assume that this does not hold. Then there is a non-zero X orthogonal to $\mathfrak{p} + [\mathfrak{p}, \mathfrak{p}]$. Clearly $X \in \mathfrak{h}$ and $\langle X, [\mathfrak{p}, \mathfrak{p}] \rangle = 0$ so, since $[X, \mathfrak{p}] \subseteq \mathfrak{p}$ and $\langle [X, \mathfrak{p}], \mathfrak{p} \rangle = 0$, $[X, \mathfrak{p}] = 0$. But this implies that $\exp tX$ acts as the identity on G/H contradicting the assumption that H does not contain any non-discrete normal subgroup of G.

We next want to prove that $\mathfrak{g}_i$ is an ideal and that $[\mathfrak{g}_i, \mathfrak{g}_j] = 0$ for $i \neq j$. Notice that since $[\mathfrak{p}_i, \mathfrak{p}_j] = 0$ for $i \neq j$, we have $\mathfrak{g} = \mathfrak{g}_1 + \cdots + \mathfrak{g}_r$. To prove that $\mathfrak{g}_i$ is an ideal one verifies:

(i) $[\mathfrak{p}_j, [\mathfrak{p}_i, \mathfrak{p}_i]] = 0$ for $i \neq j$ by the Jacobi identity.

(ii) $[\mathfrak{p}_i, [\mathfrak{p}_i, \mathfrak{p}_i]] \subseteq \mathfrak{g}_i$, since by the Ad-invariance of the metric, for $i \neq j$ we have

$$\langle \mathfrak{p}_j, [\mathfrak{p}_i, [\mathfrak{p}_i, \mathfrak{p}_i]] \rangle = 0, \quad \langle [\mathfrak{p}_j, \mathfrak{p}_j], [\mathfrak{p}_i, [\mathfrak{p}_i, \mathfrak{p}_i]] \rangle = 0.$$

(iii) $[[\mathfrak{p}_j, \mathfrak{p}_j], [\mathfrak{p}_i, \mathfrak{p}_i]] = 0$ for $i \neq j$ by the Jacobi identity.

(iv) $[[\mathfrak{p}_i, \mathfrak{p}_i], [\mathfrak{p}_i, \mathfrak{p}_i]] \subseteq \mathfrak{g}_i$ since, by Ad-invariance, for $i \neq j$ we have

$$\langle \mathfrak{p}_j, [[\mathfrak{p}_i, \mathfrak{p}_i], [\mathfrak{p}_i, \mathfrak{p}_i]] \rangle = 0, \quad \langle [\mathfrak{p}_j, \mathfrak{p}_j], [[\mathfrak{p}_i, \mathfrak{p}_i], [\mathfrak{p}_i, \mathfrak{p}_i]] \rangle = 0.$$

It now follows that $\mathfrak{g}_i$ is an ideal for all $1 \le i \le r$, and that $[\mathfrak{g}_i, \mathfrak{g}_j] = 0$ if $i \neq j$.

Since for $i \neq j$ we have

$$\langle \mathfrak{p}_i, [\mathfrak{p}_j, \mathfrak{p}_j] \rangle = \langle [\mathfrak{p}_i, \mathfrak{p}_j], \mathfrak{p}_j \rangle = 0,$$

$\mathfrak{g}_i$ is orthogonal to $\mathfrak{g}_j$, so we have an orthogonal direct sum decomposition

$$\mathfrak{g} = \mathfrak{g}_1 \oplus \cdots \oplus \mathfrak{g}_r.$$

Now set $\mathfrak{h}_i = \mathfrak{g}_i \cap \mathfrak{h}$. We would like to show that $\mathfrak{h} = \mathfrak{h}_1 \oplus \cdots \oplus \mathfrak{h}_r$. This will follow when we have proved that in a decomposition of $X \in \mathfrak{h}$ into $X = X_1 + \cdots + X_r$, $X_i \in \mathfrak{g}_i$, the component $X_i \in \mathfrak{h}$.

First notice that $\langle X, \mathfrak{p} \rangle = 0$. So for all i we have

$$\langle X_1 + \cdots + X_r, \mathfrak{p}_i \rangle = 0.$$

Since the $\mathfrak{g}_i$ are orthogonal ideals we have that $\langle X_i, \mathfrak{p}_j \rangle = 0$ for $i \neq j$, and hence $\langle X_i, \mathfrak{p}_i \rangle = 0$ for every i. It follows that

$$\langle X_i, \mathfrak{p} \rangle = \langle X_i, \mathfrak{p}_1 \rangle + \cdots + \langle X_i, \mathfrak{p}_r \rangle = 0$$

which proves that $X_i \in \mathfrak{h}$. This finishes the proof of (1).

(2) If $\mathfrak{p}_i$ is an irreducible trivial H_o-space, then $\mathfrak{p}_i$ is of dimension 1. Hence $\mathfrak{g}_i = \mathfrak{p}_i + [\mathfrak{p}_i, \mathfrak{p}_i] = \mathfrak{p}_i$ and $\mathfrak{h}_i = 0$. To prove the second part of the statement, we note that the almost-effectiveness of the action of G on G/H implies that $\mathfrak{z} \cap \mathfrak{h} = 0$. Let $\mathfrak{p}_0 = \oplus\{\mathfrak{p}_i \mid i \in I\}$. Since $\mathfrak{p}_i = \mathfrak{g}_i$ is one-dimensional for all $i \in I$ and $[\mathfrak{g}_i, \mathfrak{g}_j] = 0$ for all $i \neq j$, $\mathfrak{p}_0 \subseteq \mathfrak{z}$. Conversely, let $z \in \mathfrak{z}$. Write $z = z_1 + \cdots + z_r$ with $z_k \in \mathfrak{g}_k$. Suppose $k \notin I$, i.e., $\mathfrak{p}_k$ is a non-trivial H_o-space. Then we have

$$0 = [z, \mathfrak{g}_k] = [z_1 + \cdots + z_r, \mathfrak{g}_k] = [z_k, \mathfrak{g}_k].$$

But $[z_k, \mathfrak{g}_j] = 0$ for all $j \neq k$. Hence $z_k \in \mathfrak{z}$, which implies that $[z_k, \mathfrak{h}] = 0$. Write $z_k = h_k + x_k \in \mathfrak{h}_k \oplus \mathfrak{p}_k$. Then $[x_k, \mathfrak{h}] = [z_k, \mathfrak{h}]_{\mathfrak{p}} = 0$, which implies that $x_k = 0$ (because $\mathfrak{p}_k$ is a non-trivial H_o-space). So we have $z_k = h_k \in \mathfrak{h} \cap \mathfrak{z}$. But $\mathfrak{z} \cap \mathfrak{h} = 0$, so $z_k = 0$ if $k \notin I$, and $z \in \oplus\{\mathfrak{g}_i \mid i \in I\} = \mathfrak{p}_0$. This proves (2).

Since G is compact, $\mathfrak{g}$ is the direct sum of the center $\mathfrak{z}$ and a semi-simple ideal, so (3) and (4) follow from (2). Finally, (5) follows from the fact that $[\mathfrak{g}_i, \mathfrak{g}_j] = 0$ for $i \neq j$. $\square$

3.10 Corollary. *Let (G, H) be a polar pair, and $\mathfrak{p} = \mathfrak{p}_o \oplus \cdots \oplus \mathfrak{p}_m$ a decomposition, where $\mathfrak{p}_o$ is a trivial H_0-space and $\mathfrak{p}_i$ is a non-trivial, irreducible H_0-space for $1 \le i \le m$. Then there exist a polar pair $(\tilde{G}, \tilde{H})$, and a surjective group homomorphism $\rho : \tilde{G} \to G$ such that*

(1) the kernel of ρ is a finite group, $\rho(\tilde{H}) = H_o \subseteq H$, and ρ is a local isometry,

(2) the map $\pi : \tilde{G}/\tilde{H} \to G/H$ defined by $\pi(\tilde{g}\tilde{H}) = \rho(\tilde{g})H$ is ρ-equivariant, a finite cover, and a local isometry with respect to the normal invariant metrics on $\tilde{G}/\tilde{H}$ and G/H induced from the bi-invariant metric of $\tilde{G}$ and G respectively,

(3) $\tilde{G}/\tilde{H}$ can be written as the direct product $T \times \tilde{G}_1/\tilde{H}_1 \times \cdots \times \tilde{G}_r/\tilde{H}_r$, where T is a flat torus of dimension equal to $\dim(\mathfrak{p}_o)$, and each $(\tilde{G}_i, \tilde{H}_i)$ is an irreducible polar pair such that $\tilde{G}_i$ is simply-connected, semi-simple, $\tilde{H}_i$ is connected, and the isotropy representation of $\tilde{G}_i/\tilde{H}_i$ is ω-equivalent to the H_0-action on $\mathfrak{p}_i$.

Proof. We may assume that $\mathfrak{p}_i$ is a non-trivial H_0-space if $i \leq m$, and that $\mathfrak{p}_i$ is trivial if $m < i \leq r$ in 3.7. Then $\mathfrak{z}$, the center of $\mathfrak{g}$, is equal to $\mathfrak{p}_o = \oplus\{\mathfrak{p}_i \mid m < i \leq r\}$. If we take $T = \exp(\mathfrak{z})$, let $\rho_i : \tilde{G}_i \to G_i$ be the simply-connected cover of G_i, and $\tilde{H}_i = \exp(\mathfrak{h}_i)$ in $\tilde{G}_i$ for $i \leq m$, then $\rho(z, g_1, \ldots, g_m) = \rho_1(g_1) \cdots \rho_m(g_m)$ is a well-defined map of $\tilde{G} = T \times \tilde{G}_1 \times \cdots \times \tilde{G}_m$ to G with the required properties. $\square$

3.11 Lemma. *If (G, H) is a polar pair then*

$$2 \dim H + \operatorname{rank} G - \dim G \geq 0,$$

with equality if and only if the H action on G/H has principal orbits of dimension equal to $\dim H$ *and codimension equal to* $\operatorname{rank} G$.

Proof. If $\mathfrak{a}$ is an abelian section for the action of H on $\mathfrak{p}$ then clearly $\operatorname{rank} G \geq \dim(\mathfrak{a})$. But since $\dim(\mathfrak{a})$ is the cohomogeneity of the H action on $\mathfrak{p}$, if p is a regular point then $\dim(\mathfrak{a}) = \dim(\mathfrak{p}) - \dim(Hp) \geq \dim(\mathfrak{p}) - \dim(H)$, so $\operatorname{rank}(G) \geq \dim(\mathfrak{p}) - \dim(H)$. Finally, since $\dim(\mathfrak{p}) = \dim(G) - \dim(H)$, it follows that $2 \dim H + \operatorname{rank} G - \dim G \geq 0$. $\square$

3.12 Classification Theorem. *If (G, H) is an irreducible polar pair with G semi-simple and simply-connected and H connected, then (G, H) is either a symmetric pair associated to some irreducible symmetric space of compact type or else it is isomorphic to* $(\mathbf{Spin}(7), \mathbf{G}_2)$ *or* $(\mathbf{G}_2, \mathbf{SU}(3))$.

Proof. Let (G, H) be an irreducible polar pair, so by definition 3.2, G/H is a strongly isotropy irreducible space. A purely conceptual proof of this theorem seems unlikely because of the existence of the two special cases, and we will proceed by using the classification theory for strongly isotropy irreducible spaces developed in [M], [W1], [Kr], and [WZ]. We now assume that (G, H) is *not* a symmetric pair and prove it must be either $(\mathbf{Spin}(7), \mathbf{G}_2)$ or $(\mathbf{G}_2, \mathbf{SU}(3))$. We note that Wolf has proved that G must be a compact simple group (cf. [W1], Theorem 1.1).

To complete the proof of the theorem we go through the lists given in [W1] and [WZ] of almost effective, strongly isotropy irreducible spaces G/H for which (G, H) is not a symmetric pair, and in each case compute $2 \dim(H) + \operatorname{rank}(G) - \dim(G)$. This computation is tedious but straigthforward from the well-known fact that the classical compact groups $\mathbf{SU}(n)$, $\mathbf{SO}(2n+1)$, $\mathbf{Sp}(n)$ and $\mathbf{SO}(2n)$ have respectively dimensions $n^2 - 1$, $2n^2 + 2n$, $2n^2 + n$, and $2n^2 - n$, and ranks $n - 1$, n, n, and n, while the exceptional groups $\mathbf{E}_6$, $\mathbf{E}_7$, $\mathbf{E}_8$, $\mathbf{F}_4$, and $\mathbf{G}_2$ have ranks given by their subscripts and dimensions 78, 133, 248, 52, and 14 respectively.

The resulting table is given on the following page. We have used the list from [WZ] for classical G and the list from [W1] for exceptional G. For each of the eight infinite families it is easy to check that the formulas for $2 \dim(H) + \operatorname{rank}(G) - \dim(G)$ all give negative values in their valid ranges, while for the 28 other cases only $(\mathbf{SO}(7), \mathbf{G}_2)$ and $(\mathbf{G}_2, \mathbf{SU}(3))$ show non-negative values of $2 \dim(H) + \operatorname{rank}(G) - \dim(G)$.

One final remark should be made about the table. For the proof of the theorem we should consider pairs (G, H) with G simply connected. The exceptional groups in the G column of our table are indeed the simply-connected versions, but for consistency with the list in [WZ], for groups of types A_n, B_n, C_n, and D_n, we

TABLE OF NON-SYMMETRIC, STRONGLY ISOTROPY IRREDUCIBLE PAIRS

G	H	Range	$2\dim(H)+\mathrm{rank}(G)-\dim(G)$
$\mathbf{SU}(pq)$	$\mathbf{SU}(p)\times\mathbf{SU}(q)/\Delta\mathbf{Z}_{(p,q)}$	$p\geq q\geq 2,\ p\neq 2$	$pq-(p^2-2)(q^2-2)$
$\mathbf{SU}(n(n-1)/2)$	$\mathbf{SU}(n)/\mathbf{Z}_{(2,n)}$	$n\geq 5$	$-(n-1)(n+1)(n+2)(n-4)/4$
$\mathbf{SU}(n(n+1)/2)$	$\mathbf{SU}(n)/\mathbf{Z}_{(2,n)}$	$n\geq 3$	$-(n-1)(n+1)(n-2)(n+4)/4$
$\mathbf{SU}(16)$	$\mathbf{Spin}(10)$		-150
$\mathbf{SU}(27)$	$\mathbf{E}_6$		-546
$\mathbf{Sp}(n)$	$\mathbf{SO}(n)\times\mathbf{Sp}(1)/\Delta\mathbf{Z}_{(2,n)}$	$n\geq 3$	$-(n-2)(n+3)$
$\mathbf{Sp}(2)$	$\mathbf{Sp}(1)$		-2
$\mathbf{Sp}(7)$	$\mathbf{Sp}(3)$		-56
$\mathbf{Sp}(10)$	$\mathbf{SU}(6)/\mathbf{Z}_3$		-130
$\mathbf{Sp}(16)$	$\mathbf{SO}'(12)$		-380
$\mathbf{Sp}(28)$	$\mathbf{E}_7$		-1302
$\mathbf{SO}(4n)$	$\mathbf{Sp}(n)\times\mathbf{Sp}(1)/\Delta\mathbf{Z}_2$	$n\geq 3$	$-4n^2+6n+6$
$\mathbf{SO}(2n^2-n-1)$	$\mathbf{Sp}(n)/\mathbf{Z}_2$	$n\geq 3$	$\lfloor(-4n^4+4n^3+15n^2-3)/2\rfloor$
$\mathbf{SO}((n(n+1)-2)/2)$	$\mathbf{SO}(n)/\mathbf{Z}_{(2,n)}$	$n\geq 3, n\neq 4$	$\lfloor(-n^4-2n^3+15n^2-12)/8\rfloor$
$\mathbf{SO}(n)$	$\mathbf{K}$ simple, centerless	$n=\dim\mathbf{K}\geq 8$	$\lfloor(-n^2+6n)/2\rfloor$
$\mathbf{SO}(16)$	$\mathbf{Spin}(9)$		-40
$\mathbf{SO}(42)$	$\mathbf{Sp}(4)/\mathbf{Z}_2$		-768
$\mathbf{SO}(26)$	$\mathbf{F}_4$		-208
$\mathbf{SO}(70)$	$\mathbf{SU}(8)/\mathbf{Z}_4$		-2254
$\mathbf{SO}(128)$	$\mathbf{SO}'(16)$		-7824
$\mathbf{SO}(7)$	$\mathbf{G}_2$		$10\quad (>0)!$
$\mathbf{G}_2$	$\mathbf{SO}(3)$		-6
$\mathbf{G}_2$	$\mathbf{SU}(3)$		$4\quad (>0)!$
$\mathbf{F}_4$	$\mathbf{SO}(3)\times\mathbf{G}_2$		-14
$\mathbf{F}_4$	$(\mathbf{SU}(3)\times\mathbf{SU}(3))/\mathbf{Z}_3$		-16
$\mathbf{E}_6$	$\mathbf{SU}(3)/\mathbf{Z}_3$		-56
$\mathbf{E}_6$	$\mathbf{G}_2$		-44
$\mathbf{E}_6$	$\mathbf{SU}(3)\times\mathbf{G}_2$		-28
$\mathbf{E}_6$	$(\mathbf{SU}(3)\times\mathbf{SU}(3)\times\mathbf{SU}(3))/\mathbf{Z}_3$		-24
$\mathbf{E}_7$	$\mathbf{SU}(3)/\mathbf{Z}_3$		-110
$\mathbf{E}_7$	$\mathbf{Sp}(3)\times\mathbf{G}_2$		-56
$\mathbf{E}_7$	$\mathbf{Spin}(3)\times\mathbf{F}_4$		-16
$\mathbf{E}_7$	$(\mathbf{SU}(3)\times\mathbf{SU}(6))/\mathbf{Z}_3$		-40
$\mathbf{E}_8$	$\mathbf{G}_2\times\mathbf{F}_4$		-108
$\mathbf{E}_8$	$\mathbf{SU}(9)/\mathbf{Z}_3$		-80
$\mathbf{E}_8$	$(\mathbf{SU}(3)\times\mathbf{E}_6)/\mathbf{Z}_3$		-68

(In the above table, (p,q) is the GCD of p and q, $\lfloor a\rfloor$ is the greatest integer less than or equal to a, and $\mathbf{SO}'(4n)$ denotes the quotient $\mathbf{Spin}(4n)/\mathbf{Z}_2$ not isomorphic to $\mathbf{SO}(4n)$.)

have used the classical versions, $\mathbf{SU}$, $\mathbf{Sp}$, and $\mathbf{SO}$, and while $\mathbf{SU}(n)$ and $\mathbf{Sp}(n)$ are simply-connected, $\mathbf{SO}(n)$ has a two-sheeted simply-connected cover, $\pi : \mathbf{Spin}(n) \to \mathbf{SO}(n)$. It is easy to see that $\mathbf{SO}(n)/H$ is strongly isotropy irreducible if and only if $\mathbf{Spin}(n)/\bar{H}$ is strongly isotropy irreducible, where $\pi(\bar{H}) = H$. Thus the only non-symmetric, irreducible polar pairs (G, H) with G semi-simple, simply-connected and H a connected closed subgroup of G are $(\mathbf{Spin}(7), G_2)$ and $(G_2, \mathbf{SU}(3))$ with quotients diffeomorphic to S^7 and S^6 respectively, and both pairs are polar by 3.6. $\square$

Because $\mathbf{Spin}(7)/\mathbf{G}_2 = \mathbf{S}^7$ and $\mathbf{G}_2/\mathbf{SU}(3) = \mathbf{S}^6$ are symmetric spaces (although not symmetric pairs), it follows from 3.10 and 3.12 that:

3.13 Corollary. *If (G, H) is a polar pair with G simply-connected and semi-simple and H connected, then G/H is a simply-connected symmetric space of compact type.*

3.14 Theorem. *If (G, H) is a polar pair and the H-action on $\mathfrak{p}$ is of cohomogeneity k, then G/H is a compact symmetric space of rank k.*

Proof. Using the notation in 3.10, we see from 3.12 that $\tilde{M} = \tilde{G}/\tilde{H}$ is a symmetric space, so $M = G/H$ is a locally symmetric space. Using 2.2 (4), to prove that M is globally symmetric it suffices to prove that $\pi^{-1}(p) \subseteq Z(\tilde{M})$, where $p = eH \in M$.

If $(\tilde{G}_i, \tilde{H}_i)$ is a symmetric pair with $\tilde{H}_i$ connected, then by 2.2 (3) the center $Z(\tilde{M}_i)$ of $\tilde{M}_i = \tilde{G}_i/\tilde{H}_i$ is $\tilde{M}_i^{\tilde{H}_i}$, the fixed-point set of $\tilde{H}_i$ on $\tilde{M}_i$. Note that the fixed-point set of $\mathbf{G}_2$ on $\mathbf{S}^7$ (resp., of $\mathbf{SU}(3)$ on $\mathbf{S}^6$) is $\{p, -p\}$, which is also the center $Z(\mathbf{S}^7)$ of $\mathbf{S}^7$ (resp., the center $Z(\mathbf{S}^6)$ of $\mathbf{S}^6$). So even if $\mathbf{Spin}(7)/\mathbf{G}_2$ or $\mathbf{G}_2/\mathbf{SU}(3)$ is one of the factors in $\tilde{M}_i$ we still have $Z(\tilde{M}_i) = \tilde{M}_i^{\tilde{H}_i}$. It is known that $Z(\tilde{M}) = T \times Z(\tilde{M}_1) \times \cdots \times Z(\tilde{M}_m)$, so $Z(\tilde{M})$ is equal to $\tilde{M}^{\tilde{H}}$. Using $\pi(\tilde{g} \cdot x) = \rho(\tilde{g})\pi(x)$, we see that if $\pi(y) = p$ then $\tilde{H} \cdot y \subseteq \pi^{-1}(p)$. Since π is a finite cover and $\tilde{H}$ is connected, $\tilde{H} \cdot y = y$, i.e., $y \in \tilde{M}^{\tilde{H}} = Z(\tilde{M})$. $\square$

As consequence of the proof of 3.14, we also obtain

3.15 Corollary. *If (G, H) is an irreducible polar pair, then (G, H) must be either the symmetric pair for an irreducible symmetric space, or one of the following pairs: $(\mathbf{Spin}(7), \mathbf{G}_2)$, $(\mathbf{SO}(7), \mathbf{G}_2)$, $(\mathbf{G}_2, \mathbf{SU}(3))$, or $(\mathbf{S}^1, \mathbf{Z}_n)$.*

It follows from 3.4 and 3.14 that

3.16 Corollary. *If G/H is a compact, normal Riemannian homogeneous space such that the action of H on G/H is hyperpolar, then G/H is a symmetric space (although (G, H) is not necessarily a symmetric pair).*

In the next section we will prove the same conclusion without assuming that the Riemannian homogeneous space G/H is necessarily normal.

4. k-flat homogeneous spaces

The main result of this section is the following characterization of compact, k-flat homogeneous manifolds.

4.1 Theorem. *A compact k-flat homogeneous space is a symmetric space of rank k.*

4.2 Remark. Notice that it follows from Theorem 4.1 that a compact manifold can only be k-flat homogeneous for one k, in contrast to the n-dimensional Euclidean space, which is k-flat homogeneous for all $1 \leq k \leq n$.

The following Proposition follows directly from the definition of k-flat homogeneity.

4.3 Proposition. *Let M be a compact, Riemannian manifold. Then M is k-flat homogeneous if and only if the following three conditions are satisfied:*
(i) every geodesic is contained in some k-flat,
(ii) $G = I(M)$ acts transtively on the set of k-flats of M,
(iii) there exists a k-flat τ such that the normalizer $N(\tau, G)$ acts transtively on τ.

4.4 Proposition. *Let M be a compact Riemannian manifold such that $I(M)$ acts transtively on the set of geodesics of M. Then M is 1-flat homogeneous, or equivalently two-point homogeneous.*

Proof. If M is one-dimensional the result is trivial. If $\dim(M) > 1$, then the fact that $G = I(M)$ is transitive on geodesics implies that $\dim(G) > 0$. Since G is compact it then follows that there is a circle subgroup $\Gamma \subseteq G$. Let γ be an Γ-orbit in M of maximal length. It is well-known (cf. [H1]) that γ is a closed geodesic, hence γ is a 1-flat, and obviously $N(\gamma, G)$ includes Γ, and so acts transtively on γ. It then follows from 4.3 that M is 1-flat homogeneous. □

4.5 Remark. Although condition (iii) is a consequence of (i) and (ii) if $k = 1$, this is no longer so if $k > 1$. The Klein bottle S is a counter-example for $k = 2$, and more generally, if M is m-flat homogeneous, then $M \times S$ satisfies conditions (i) and (ii) for $k = m + 2$, but is not k-flat homogeneous.

4.6 Theorem. *If (M, ds^2) is a compact Riemannian manifold then the following two statements are equivalent:*
(i) M is k-flat homogeneous.
(ii) M is a homogeneous Riemannian G-manifold and the action of some subgroup H of G is hyperpolar with k-dimensional sections and has a fixed-point.

Proof. We first prove that (i) implies (ii). Let $G = I(M)$, $H = G_x$, and τ a k-flat through x. We claim that $H\tau = M$. For if $p \in M$ then there exist a geodesic γ joining x to p, and a k-flat σ containing γ, so by definition of k-flat homogeneity, there exists $g \in G$ such that $g(x) = x$ and $g(\tau) = \sigma$, proving $H\tau = M$. Since M is compact and τ is totally geodesic, flat and orthogonal to the orbit $Hx = \{x\}$ at x, it follows from 2.15 that τ is a flat section for the H-action on M and so this action is hyperpolar.

We next prove that (ii) implies (i). Let γ be a geodesic. We have to show that γ is contained in a k-flat. Let x be a fixed-point of H and let $g \in G$ be such that $g\gamma$ passes through x. Let τ be a k-flat that is a section of H. Then $H(T_x\tau) = T_xM$, so that there is a $h \in H$ for which $g\gamma$ is contained in $h\tau$. It follows that γ lies in the k-flat $g^{-1}h\tau$. Now let (x_1, τ_1) and (x_2, τ_2) be such that $x_i \in \tau_i$ and τ_i is a k-flat.

By the homogeneity of M there are g_1 and $g_2 \in G$ such that $g_i(x_i) = x$ where x is a fixed-point of H. As in the first part of the proof it follows that $g_i\tau_i$ is a flat section. By 2.3 (ii), H is transitive on the set of sections. Hence there is an $h \in H$ such that $hg_1\tau_1 = g_2\tau_2$, i.e., $g_2^{-1}hg_1(\tau_1) = \tau_2$ and $g_2^{-1}hg_1(x_1) = x_2$. It follows that M is k-flat homogeneous. $\square$

4.7 Proposition. *Let M be a homogeneous Riemannian G-manifold, and H a closed subgroup of G_p. If the action of H on M is hyperpolar, then the H_o-action and the $(G_p)_o$-action on M are ω-equivalent.*

Proof. Since H-orbits are submanifolds of G_p orbits, it will suffice to prove that the two actions have the same cohomogeneity. If τ is a flat section for the H-action, then τ is a flat, totally geodesic submanifold and $H\tau = M$. Because $H \subseteq G_p$, we have $G_p\tau = M$. Since τ is perpendicular to the orbit $G_p p = \{p\}$, by 2.15 τ is a section for the action of G_p. Thus both H and G_p acting on M have cohomogeneity $\dim(\tau)$. $\square$

4.8 Remark. If $M = G/H$ equipped with a normal metric is k-flat homogeneous, then by 4.6 and 3.16, M is a symmetric space. However, the metric on a k-flat homogeneous space in general need not be normal, and to prove 4.1, we need a non-linear analogue of 2.11. First a lemma.

4.9 Lemma. *If (τ, ds_1^2) is a flat, compact, homogeneous Riemannian N-manifold, and ds_2^2 is another N-invariant metric on τ, then ds_2^2 is also flat.*

Proof. Since τ is compact, flat and homogeneous, it follows that it is a torus and the universal cover $\tilde{\tau}$ of τ is $\mathbf{R}^n$. We may assume that N acts on τ effectively (we can always quotient out the kernel of the action). Then the Lie algebra $\mathfrak{n}$ of N is the abelian Lie algebra $\mathbf{R}^n$. So there exists local coordinate system $(u_1, \ldots, u_k)$ on τ such that the coordinate vector fields are Killing fields. It follows that any N-invariant metric on τ is of the form $\sum c_{ij} du_i \otimes du_j$ for some constant positive matrix (c_{ij}), and hence is flat. $\square$

4.10 Theorem. *Let (M, ds^2) be a k-flat homogeneous space, $G = I(M, \mathrm{ds}^2)$ and $H = G_x$. Let ds_o^2 be the normal homogeneous metric on M associated to some $\mathrm{Ad}(G)$-invariant inner product $\langle\, , \rangle_o$ on $\mathfrak{g}$. Then*

(1) (M, ds_o^2) is also k-flat homogeneous,
(2) ds^2 and ds_o^2 have the same set of k-flats,
(3) the H-action on (M, ds_o^2) is hyperpolar,
(4) (M, ds_o^2) is a symmetric space of rank k.

Proof. Let τ be a k-flat through x for (M, ds^2), and $N = \{g \in G \mid g(\tau) = \tau\}$. By 4.3, $N = N(\tau, G)$ acts transitively on τ.

We first claim that the action of H on (M, ds_o^2) is polar. To prove this, we let $\mathfrak{a} = T\tau_x$. Then 2.11 implies that $[\mathfrak{h}, \mathfrak{a}] \perp \mathfrak{a}$ with respect to both metrics. Note that $g \in G$ is an isometry with respect to both ds^2 and ds_o^2. For $g \in N$, we have $(g_*)_x(\mathfrak{a}) = T\tau_{gx}$. So $(g_*)_x([\mathfrak{h}, \mathfrak{a}]) \perp T\tau_{gx}$ with respect to both metrics. By 4.6, the H-action on (M, ds^2) is hyperpolar with τ as a section. So $T\tau_{gx}$ is the normal space to the orbit Hgx at gx with respect to ds^2 if it is a principal orbit. Hence we have

$(g_*)_x([\mathfrak{h}, \mathfrak{a}]) = T(Hgx)_{gx}$. This proves that an orbit Hgx is perpendicular to τ at gx with respect to ds_o^2, i.e., the H-action on (M, ds_o^2) is polar and τ is a section. Since N acts transtively on τ, 4.9 implies that τ is flat in the metric induced from ds_o^2.

Clearly (2) and (3) are consequence of the proof of (1), and (4) follows from (3) and 3.16. $\Box$

4.11 Proof of 4.1.

Let $G = I(M, \mathrm{ds}^2)$, $H = G_p$, $\langle\, ,\, \rangle_o$ an $\mathrm{Ad}(G)$-invariant inner product on $\mathfrak{g}$, and ds_o^2 the associated normal G-invariant metric on M. By 4.10, (M, ds_o^2) is k-flat homogeneous, and it follows from 4.8 that (M, ds_o^2) is a symmetric space. It remains to prove that (M, ds^2) is also a symmetric space. To do this, we use the same notation as in 3.10. Let $\tilde{h}$ and $\tilde{h}_o$ be the lifting of ds^2 and ds_o^2 to $\tilde{M}$ respectively. Then:

(1) both $\tilde{h}$ and $\tilde{h}_o$ are $\tilde{G}$-invariant,

(2) $\tilde{M} = T \times \tilde{M}_1 \times \cdots \times \tilde{M}_m$, where (T, g_0) is a flat torus, $(\tilde{M}_i, g_i)$ is a simply-connected, irreducible symmetric space of compact type, and the metric $\tilde{h}_o$ on $\tilde{M}$ is the product metric $\tilde{h}_o = g_o + g_1 + \cdots + g_m$.

Let $p_o = e\tilde{H}$. By 2.11, there exist positive constants c_i such that $\mathrm{ds}^2 \mid \mathfrak{p}_i = c_i\langle\, ,\, \rangle_o \mid \mathfrak{p}_i$. Let g_o^* be the homogeneous flat metric on T induced from $\tilde{h}$. Then the $\tilde{G}$-invariant metrics $\tilde{h}$ and $\tilde{h}^* = g_o^* + c_1 g_1 + \cdots + c_m g_m$ agree at p_o, which implies that $\tilde{h} = \tilde{h}^*$. But $(\tilde{M}, \tilde{h}^*)$ is a symmetric space, so $(\tilde{M}, \tilde{h})$ is a symmetric space. Moreover, $(\tilde{M}, \tilde{h}_o)$ and $(\tilde{M}, \tilde{h})$ have the same center $\tilde{M}^{\tilde{H}}$, which contains $\pi^{-1}(p)$ as a discrete subgroup. Hence by 2.2 (4), (M, ds^2) is also a symmetric space. $\Box$

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