October 5,2001

Hyperpolar Actions on Symmetric Spaces

E. Heintze, R.S. Palais, C.L. Terng* and G. Thorbergsson**

1. Introduction

An isometric action of a compact Lie group G on a Riemannian manifold M is called *polar* if there exists a closed, connected submanifold $\sum$ of M that meets all G-orbits and meets orthogonally. Such a $\sum$ is called a *section*. A section is automatically totally geodesic in M, and if it is also flat in the induced metric then the action is called *hyperpolar*. In this paper we study hyperpolar actions on compact symmetric spaces, prove some structure and classification theorems for them, and study their relation to polar actions on Hilbert space and to involutions of affine Kac-Moody algebras.

The origin of the theory of hyperpolar actions can be traced back at least to the 1950's, with Bott [B] and Bott and Samelson's [BS] introduction of the closely related concept of a "variationally complete" action. We will give a brief review below of what we know of the history of this subject, but first we recall some basic notation and terminology from the theory of symmetric spaces that will be needed throughout the paper.

Let G be a compact, semi-simple, connected Lie group, and $\mathcal{G} = TG_e$ its Lie algebra. A subgroup K of G is called a *symmetric subgroup* of G (and the pair (G, K) is called a *symmetric pair*) if there is an involution σ of G such that $(K_\sigma)_0 \subset K \subset K_\sigma$, where K_σ is the fixed point set of σ and $(K_\sigma)_0$ is the connected component of K_σ. In this case G/K equipped with a G-invariant metric induced from any Ad_G invariant inner product for $\mathcal{G}$ is a *symmetric space*. The corresponding orthogonal decomposition $\mathcal{G} = \mathcal{K} \oplus \mathcal{P}$ is called the *Cartan decomposition* associated to σ. It is completely determined by σ (independent of the inner product on $\mathcal{G}$); namely $\mathcal{K}$ and $\mathcal{P}$ are respectively the $+1$ and -1 eigenspaces of $d\sigma_e : \mathcal{G} \to \mathcal{G}$. It follows that $\mathcal{K}$ and $\mathcal{P}$ satisfy the characteristic bracket relations $[\mathcal{K}, \mathcal{K}] \subseteq \mathcal{K}$, $[\mathcal{K}, \mathcal{P}] \subseteq \mathcal{P}$, and $[\mathcal{P}, \mathcal{P}] \subseteq \mathcal{K}$.

1.1 Definition. If (G, K) is a symmetric pair and $\mathcal{G} = \mathcal{K} \oplus \mathcal{P}$ the corresponding Cartan decomposition, then the Adjoint representation of K on $\mathcal{P}$ is called an *s-representation*. "refsrep

Of course, up to equivalence the s-representation can be identified with the isotropy representation of the symmetric space G/K at eK (from which it gets its name).

* research support in part by NSF grant DMS 9304285
** research support in part by NSF grant DMS 9100179.

1.2 Remark. It is known (cf. [He], [Lo]) that if $M = G/K$ is a Riemannain symmetric space corresponding to a symmetric pair (G, K), then G is the connected component $\mathrm{Isom}_0(M)$ of the group of all isometries of M. “refIA

1.3 Remark. A compact Lie group G with a bi-invariant metric is a symmetric space, and $(G \times G, \Delta(G))$ is a symmetric pair. For $G \times G$ acts transitively on G by $(g_1, g_2) \cdot x = g_1 x g_2^{-1}$ and the isotropy group at e is $\Delta(G) = \{(g, g) \mid g \in G\}$; and $\sigma(x, y) = (y, x)$ is an involution of $G \times G$ with fixed point set $\Delta(G)$. These compact symmetric spaces are referred to as being of Type II or “group type”. “refIB

1.4 Remark. There is a well-known criterion for a Riemannian manifold M to be symmetric, namely the existence for each p in M of an isometry ϕ that fixes p and satisfies $d\phi_p(v) = -v$ for all v in TM_p. However it does *not* then nesessarily follow that if G is a transitive group of isometries of M and $K = G_p$ then (G, K) must be a symmetric pair. By 1.2 this is only so if $G = \mathrm{Isom}_0(M)$. For example, two possible ways of representing the symmetric space S^{2n-1} are as $SO(2n)/SO(2n-1)$ and as $SU(n)/SU(n-1)$. And while $(SO(2n), SO(2n-1))$ is a symmetric pair, $(SU(n), SU(n-1))$ is not. This distinction between symmetric space and symmetric pair will be important in what follows. “refIC

1.5 Remark. We have been slightly sloppy in the above remarks and, to avoid long and awkward statements of theorems, we will continue to allow ourselves the liberty of the same slight imprecision that we now describe. In general we can *not* really regard G as a group of diffeomorphisms of $M = G/K$, because the action of G on G/K may not be effective. The kernel, N, of this action is the intersection of all the conjugates gKg^{-1} of K, i.e., the largest subgroup of K that is normal in G, and this can be non-trivial. But for most purposes we can simply replace G by G/N and K by K/N. For example, if (G, K) is a symmetric pair then so is $(G/N, K/N)$, and what 1.2 really means is that $G/N = \mathrm{Isom}_0(M)$. (A typical example of such behavior is $M = S^n = Spin(n+1)/Spin(n) = SO(n+1)/SO(n)$.) In particular, if $Z(G)$ denotes the center of G, then $\Delta(Z(G))$ is the kernel of the action of $G \times G$ on G, so a connected group of isometries of G should really be regarded as a subgroup of $G \times G/\Delta(Z(G))$. Nevertheless, we will frequently speak as if any group of isometries of G were a subgroup of $G \times G$. “refID

We now turn to our historical remarks on the origins of the theory of polar and hyperpolar actions, which as we said begins with the papers [B] and [BS]. First we recall the definition of variational completeness. Let M be a complete Riemannian G-manifold. A geodesic γ in M is called *G-transversal* if $\gamma(t)$ is perpendicular to the orbit $G \cdot \gamma(t)$ for all t, and a Jacobi field J along a G-transversal geodesic γ is *G-transversal* if it is the deformation of G-transversal geodesics. The G-action on M is called *variationally complete* if every G-transversal Jacobi field J along γ satisfying the condition that there exists $0 < t_0$ such that $J(t)$ is tangent to the G-orbit at $\gamma(t)$ for $t = 0$ and $t = t_0$, is of the form $J(t) = \xi(\gamma(t))$ for some Killing vector field $\xi \in \mathcal{G}$. Bott and Samelson also gave an informal definition: "Intuitively we like to think of variational completeness as an absence of conjugate points in the orbit space M/G".

Now let X be a closed submanifold of $M \times M$, $P(M, X)$ the Hilbert manifold defined by

$$P(M, X) = \{\gamma \in H^1([0,1], M) \,|\, (\gamma(0), \gamma(1)) \in X\},$$

and $E : P(M, X) \to R$ the energy functional

$$E(\gamma) = \int_0^1 \parallel \gamma'(t) \parallel^2 dt.$$

The first main theorem in [BS] (Theorem I, page 976) can be stated as follows:

1.6 Theorem ([BS]). *Suppose the action of G on M is variationally complete, $x_0 \in M$ and N is a G-orbit. Then the energy functional $E : P(M, \{x_0\} \times N) \to R$ is a perfect Morse function and a homology basis can be constructed explicitly.* “refKa

The second main theorem in [BS] (Theorem II, page 986) states that:

1.7 Theorem ([BS]). *If (G, K) is a symmetric pair with Cartan decomposition $\mathcal{G} = \mathcal{K} \oplus \mathcal{P}$, then three natural actions associated to (G, K) are variationally complete; namely the action of K on the symmetric space G/K, the action of $K \times K$ on G, and finally the s-representation of K on $\mathcal{P}$ (or equivalently the isotropy representation of G/K).* “refBS

Shortly after, in [H1], R. Hermann generalized Theorem 1.7 as follows:

1.8 Theorem ([Hr1]). *If K_1 and K_2 are two symmetric subgroups of the same compact Lie group G, then the action of K_1 on G/K_2 is variationally complete and so is the action of $K_1 \times K_2$ on G.* “refHerm

A couple of years later still, L. Conlon in [Co1] associated to each automorphism σ of G, the subgroup $G(\sigma) := \{(g, \sigma(g)) \mid g \in G\}$ of $G \times G$. As a subgroup of $G \times G = \mathrm{Isom}_0(G)$, $G(\sigma)$ acts isometrically on G. As a consequence of Hermann's theorem, Conlon obtained:

1.9 Corollary ([Co1]). *Let G be a compact, connected Lie group with a bi-invariant metric and let σ be any automorphism of G. Then $G(\sigma)$ is a symmetric subgroup of $G \times G$, and hence the action of $G(\sigma)$ on G is variationally complete.* “refCo

PROOF. The map $(x, y) \mapsto (\sigma^{-1}(y), \sigma(x))$ is clearly an involution of $G \times G$, and its fixed point set is $G(\sigma)$. In particular $G(\mathrm{id}) = \Delta(G)$, and since $G \times G/\Delta(G) = G$, we can apply Hermann's theorem to the symmetric subgroups $K_1 = G(\sigma)$ and $K_2 = \Delta(G)$ of $G \times G$. ∎

Moreover, Conlon noted that all the known examples of variationally complete actions admit flat submanifolds that meet every orbit orthogonally. He called these flat submanifolds *K-transversal domains*, and the existence of such is exactly our definition of an action being hyperpolar. He proved:

1.10 Theorem ([Co3]). *If an isometric action of G on M is hyperpolar then it is variationally complete.* “refKb

Theorem 1.10 is generalized in [T1] where it is shown that if the action of G on M is polar and sections have no conjugate points then the action is variationally complete. This agrees with the intuitive definition of variational completeness mentioned above.

Conlon also made some progress on the classification of hyperpolar actions on Euclidean space ([Co4]), which was finally completed by J. Dadok ([Da]):

1.11 Theorem ([Da]). *Suppose H is a compact, connected Lie group, and $\rho : H \to SO(n)$ is a representation such that the action of H on R^n is hyperpolar. Then there exist a symmetric space G/K and a linear isometry $A : R^n \to \mathcal{P}$ such that A maps H-orbits isometrically onto K-orbits of the s-representation on $\mathcal{P}$ (where $\mathcal{G} = \mathcal{K} + \mathcal{P}$ is the Cartan decomposition).* “refDad

1.12 Remark. Let N be a K-orbit in $\mathcal{P}$ for the s-representation of K on $\mathcal{P}$, and $x_0 \in \mathcal{P}$. Then N is a generalized (partial) flag manifold. Identifying N as the submanifold of geodesic in $\mathcal{P}$ joining x_0 to a point in N, then the map defined by $r(\gamma) = \gamma(1)$ is a deformation retract of the path space $P(\mathcal{P}, \{x_0\} \times N)$ to N and the energy functional E on $P(\mathcal{P}, \{x_0\} \times N)$ restricted to N is the square of the Euclidean distance function $f_{x_0}(x) = \parallel x - x_0 \parallel^2$. It is known that $f_{x_0} : N \to R$ is a perfect Morse function, and the images of the Morse cycles explicitly constructed by Bott and Samelson under the map r are in fact the Schubert cycles. “refJh

It is known that (cf.[GR], [PS], [T1]) if N is an orbit of the action of $G(\sigma)$ on G for some automorphism σ on G, then the path space $P(G, \{e\} \times N)$ is an Adjoint orbit of the affine Kac-Moody group associated to σ, it can be viewed as an infinite dimenisonal complex (partial) flag manifold, the energy function E on the path space corresponds to the distance function in the affine Kac-Moody algebra, and the Morse cycles constructed for E are analogues of the Schubert cycles. In this paper, we prove that if K_1, K_2 are symmetric subgroups of G and N is an orbit of the $K_1 \times K_2$-action on G then the path space $P(G, \{e\} \times N)$ is an infinite dimensional analogue of the finite dimenisonal s-representations, i.e., there exist an affine algebra $\mathcal{U}$ and an involution σ on $\mathcal{U}$ such that $\mathcal{U}$ is the direct sum of the $+1$-eigenspace $\hat{\mathcal{K}}$ and -1-eigenspace $\hat{\mathcal{P}}$ of σ, the Adjoint action of $\hat{K}$ leaves $\hat{\mathcal{P}}$ invariant, and $P(G, \{e\} \times N)$ equipped with the Sobolev H^1-metric is isometric to an orbit of $\hat{K}$ on $\hat{\mathcal{P}}$ via the Adjoint action. Moreover, $P(G, \{e\} \times N)$ has the analogous geometric and topological properties as stated in Remark 1.12.

Next we recall the definition of infinite dimensional hyperpolar action ([T 2]). An action of a Hilbert Lie group $\mathcal{G}$ on a Hilbert space V is called *proper* if the map $\mathcal{G} \times V \to V \times V$ defined by $(g, v) \mapsto (g \cdot v, v)$ is proper, i.e., the preimage of a compact set is compact. The $\mathcal{G}$-action on V is called *Fredholm* if given any $v \in V$ the orbit map $\mathcal{G} \to V$ defined by $g \mapsto g \cdot v$ is a Fredholm map, i.e., the differential at every point g is a linear Fredholm operator. A proper, Fredholm, isometric $\mathcal{G}$-action on V is called *hyperpolar* if there exists a closed affine subspace $\sum$ of V that meets every $\mathcal{G}$-orbit orthogonally. It is known that if G is a compact, semi-simple Lie group and H is a closed subgroup such that the action of H on G is hyperpolar then the action of $P(G, H)$ on the Hilbert space $H^0([0.1], \mathcal{G})$ by gauge transformations is hyperpolar. Moreover, identifying $P(G, \{e\} \times G)$ with $H^0([0, 1], \mathcal{G})$ via $g \mapsto g^{-1}g'$, the path space $P(G, \{e\} \times N)$ is identified with a $P(G, H)$-orbit, where N is an H-orbit N in G.

The above considerations naturally lead us to study the following closely related problems:

(1) the classification of hyperpolar actions on compact symmetric spaces,
(2) the geometry and topology of orbits of hyperpolar actions on compact symmetric spaces,
(3) the classification of hyperpolar actions on Hilbert space,
(4) the classification of isometric involutions of affine Kac-Moody algebras.

One of the main results of this paper is to reduce (1) to a problem in Lie algebra. First we prove that the action of a closed subgroup H of G on the symmetric space G/K is hyperpolar if and only if the action of $H \times K$ on G is hyperpolar. This reduces (1) to the classification of hyperpolar actions on compact Lie groups. It is known ([PT1]) that if the action of H on G is hyperpolar then the slice representation of H_e on the normal space $\nu(H \cdot e)_e$ is hyperpolar with an abelian subalgebra as a section. Although the converse is not in general true, we prove that it is true under the assumption that G is a compact, semi-simple Lie group equipped with the bi-invariant metric induced from the Killing form on $\mathcal{G}$. In fact, we prove that the action of a closed subgroup H of $G \times G$ on G is hyperpolar if the Lie algebra $\mathcal{H}$ of H satisfies the following condition (A):

(A) There exists $g_0 \in G$ such that the orthogonal complement of $\{g_0^{-1} x g_0 - y \,|\, (x, y) \in \mathcal{H}\}$ in $\mathcal{G}$ is an abelian subalgebra.

In particular, this proves that a cohomogeneity one action on G is automatically hyperpolar. Although this reduces (1) to the problem of finding all subalgebras $\mathcal{H}$ of $\mathcal{G} + \mathcal{G}$ satisfying condition (A), it remains a difficult open problem. To suggest the level of complexity of this problem, we note that finding all subgroups H_1 of G such that the action of $H_1 \times H_1$ on G has cohomogeneity one is equivalent to the problem of finding all the two-point homogeneous spaces.

There has been lot of progress made for problem (2) (for details see [TT]). We first note that the submanifold geometry of one principal orbit determines the whole orbit foliation of the hyperpolar action, and a principal orbit N of a hyperpolar action on symmetric space X satisfies the following conditions:

(a) the normal bundle $\nu(N)$ is globally flat and $\exp(\nu(N)_x)$ is contained in some flat of X for all $x \in N$,
(b) if v is a parallel normal field on N then $\Gamma(v(x)) = \Gamma(v(y))$ for all $x, y \in N$, where $\Gamma(v(x))$ is the focal data at x, defined as the set of all pairs (t, k) such that $\exp(tv(x))$ is a multiplicity k focal point of N with respect to x.

A submanifold N of a symmetric space X is called *equifocal* if N satisfies the conditions (a) and (b) above. It is proved in [TT] that there are equifocal submanifolds of symmetric spaces that are not orbits of any isometric action, and Theorem 1.6 holds for general equifocal submanifold in compact symmetric space. In fact, we proved that if N is an equifocal submanifold of a compact symmetric space X and $x_0 \in X$ is a non-focal point of N, then there exist a point $y \in N$ and an affine Coxeter group $\hat{W}$ acting on the normal space $\nu(N)_y$ such that $x_0 = \exp(v)$ for some $v \in \nu(N)_y$ and the energy functional $E : P(X, \{x_0\} \times N) \to R$ is a perfect Morse function whose critical set is equal to

$$\{\gamma_w(t) = \exp(t(w \cdot 0) + (1-t)v) \mid w \in \hat{W}\}.$$

Moreover, an analogue of the Schubert cycle is explicitly constructed for each critical point of E.

Now we turn to problem (3). We have mentioned above that if H is a closed subgroup of $G \times G$ and the action of H on G is hyperpolar then the action of the Hilbert Lie group $P(G, H)$ on the Hilbert space $H^0([0,1], \mathcal{G})$ by gauge transformations

$$g * u = gug^{-1} - g'g^{-1}$$

is hyperpolar. In particular, using examples in Theorem 1.7, 1.8 and 1.9 we conclude that the actions of $P(G, K \times K)$, $P(G, K_1 \times K_2)$ and $P(G, G(\sigma))$ on $H^0([0,1], \mathcal{G})$ are hyperpolar. However, problem (3) remains unsolved.

To explain problem (4), we need to give a rough description of affine Kac-Moody algebras (cf. [Ka], and a more detailed review in section 4). Let $\mathcal{G}$ be a compact, semi-simple Lie algebra, σ an order k automorphism of $\mathcal{G}$, and $L(\mathcal{G}, \sigma)$ the space of loops $u : [0, 2\pi] \to \mathcal{G}$ satisfying the condition $\sigma(u(t)) = u(t + \frac{2\pi}{k})$ for all t. Let $\hat{L}(\mathcal{G}, \sigma) = L(\mathcal{G}, \sigma) + Rc + Rd$ denote the two dimensional extension of $L(\mathcal{G}, \sigma)$ such that c is the center with 2-cocyle defined by

$$\omega(u, v) = \int_0^{2\pi} (u'(t), v(t))dt, \tag{1.1}$$

and $[d, u] = u'$. Then $\hat{L}(\mathcal{G}, \sigma)$ is an affine (Kac-Moody) algebra of type ℓ, where “refxa
ℓ is the order of $[\sigma] \in \mathrm{Aut}(\mathcal{G})/\mathrm{Int}(\mathcal{G})$. Moreover, all affine algebras can be obtained this way. Let $\langle\ ,\ \rangle$ denote the non-degenerate, ad-invariant bi-linear form on $\hat{L}(\mathcal{G}, \sigma)$ defined by

$$\langle u_1 + r_1c + s_1d, u_2 + r_2c + s_2d\rangle = r_1s_2 + r_2s_1 + \int_0^{2\pi} (u(t), v(t))dt, \tag{1.2}$$

where $(\ ,\ )$ is the negative of the Killing form on $\mathcal{G}$. Since the bi-linear form $\langle\ ,\ \rangle$ “refxb
has index one, the horosphere

$$V_0 = \{u + d - \tfrac{\langle u,u\rangle + 1}{2}\, c \mid u \in L(\mathcal{G})\}$$

is isometric to $L(\mathcal{G},\sigma)$ (equipped with the L^2-inner product) via the map

$$u + d - \tfrac{\langle u,u\rangle+1}{2}\, c \quad\mapsto\quad u.$$

Involutions on complex affine algebras have been classified (cf. [Le], [BR]). In this paper we are interested in finding all isometric involutions of the real affine algebra $\hat{L}(\mathcal{G},\sigma)$ that leave $Rc+Rd$ invariant. Since the classification results for complex affine algebras are rather complicated, it is not easy to translate them into the real case, and we give a direct and independent classification of such involutions. We first note that if ϕ is such an involution on $\hat{L}(\mathcal{G},\sigma)$ then the restriction of ϕ to $Rc+Rd$ is either equal to id or $-$id. Such ϕ will be called *involutions of the first and second kind* respectively. Moreover, the following statements are true:

- (i) $\hat{L}(\mathcal{G},\sigma)$ can be written as direct sum of $+1$ and -1 eigenspaces $\hat{\mathcal{K}}$ and $\hat{\mathcal{P}}$ of ϕ, which will be called the *Cartan decomposition* of $\hat{L}(\mathcal{G},\sigma)$ defined by ϕ,
- (ii) If ϕ is of the first kind then $\hat{\mathcal{K}}$ is a subalgebra of $\hat{L}(\mathcal{G},\sigma)$, and is isomorphic to some $\hat{L}(\mathcal{G}_0,\tau)$ for some subalgebra $\mathcal{G}_0$ of $\mathcal{G}$ and an automorphism τ on $\mathcal{G}_0$. In particular, this says that the $\mathrm{Ad}(\hat{K})$-orbit in $\hat{\mathcal{P}}$ is equivalent to the Adjoint action of some affine algebra.
- (iii) If ϕ is of the second kind then there exist an automorphism σ and involutions ρ_1,ρ_2 on $\mathcal{G}$ such that $\sigma\rho_1=\rho_2$ and the Adjoint action of $\hat{K}$ on $\hat{\mathcal{P}}$ is equivalent to the $\hat{K}_1$-action on $\hat{\mathcal{P}}_1$, where $\hat{L}(\mathcal{G},\sigma)=\hat{\mathcal{K}}_1+\hat{\mathcal{P}}_1$ is the Cartan decomposition of the involution ρ_1^* of the second kind defined by $\rho_1^*(u)(t)=\rho_1(u(-t))$.
- (iv) The Adjoint action leaves the Hilbert space $\hat{L}(\mathcal{G},\sigma)\cap V_0$ invariant. If ϕ is an involution of $\hat{L}(\mathcal{G},\sigma)$ of the second kind and $\hat{L}(\mathcal{G},\sigma)=\hat{\mathcal{K}}+\hat{\mathcal{P}}$ is the corresponding Cartan decomposition, then the Adjoint action of $\hat{\mathcal{K}}$ leaves the Hilbert space $\hat{\mathcal{P}}\cap V_0$ invariant.

These actions on Hilbert space obtained from involutions of the first and second kind will be called *affine s-representations*. Next we want to describe all affine s-representations, and state results analogous to those in Remark 1.12 for orbits of affine s-representations. It is known that (cf. [T2]) an affine s-representation arising from the Adjoint action of $\hat{L}(\mathcal{G},\sigma)$ is equivalent to the $P(G,G(\sigma))$-action on $H^0([0,1],\mathcal{G})$. Moreover, every Adjoint orbit in $V_0\cap\hat{L}(\mathcal{G},\sigma)$ can be viewed as an infinite dimensional analogue of a complex (partial) flag manifold, is isometric to $P(G,\{e\}\times N)$ for some $G(\sigma)$-orbit N of G, and the restriction of the height function

$$u + d - \tfrac{\langle u,u\rangle+1}{2}\, c \quad\mapsto\quad \langle u + d - \tfrac{\langle u,u\rangle+1}{2}\, c,\ \ -d+\tfrac{1}{2}\, c\rangle$$

to the Adjoint orbit corresponds to the perfect energy functional E on $P(G,\{e\}\times N)$. In this paper, we prove that an affine s-representation arising from an involution of $\hat{L}(\mathcal{G},\sigma)$ of the second kind is equivalent to the $P(G,K_1\times K_2)$-action on $H^0([0,1],\mathcal{G})$ for some symmetric subgroups K_1,K_2 of G. Moreover, the orbit of $\hat{K}$ on $\hat{\mathcal{P}} \cap V_0$ can be viewed as an infinite dimensional analogue of a real (partial) flag manifold, is isometric to $P(G, \{e\} \times N)$ for some $K_1 \times K_2$-orbit N of G, and the restriction of the above height function to the Adjoint orbit corresponds to the perfect energy functional E on $P(G, \{e\} \times N)$.

We also prove that given any symmetric subgroups K_1, K_2 of G, the action of $P(G, K_1 \times K_2)$ on $H^0([0,1], \mathcal{G})$ is equivalent to some affine s-representations of the second type. Since there exists a cohomogeneity one action on G that are not examples given by Theorem 1.8, the above discussion implies that Dadok's theoerm 1.11 is not true for Hilbert space, i.e., there exist hyperpolar actions on Hilbert space that are not affine s-representations.

This paper is organized as follows: In section 2 we prove that, given a closed subgroup H of $G \times G$, the condition (A) above is necessary and sufficient for the H-action on G to be hyperpolar. In section 3 we list the known examples of hyperpolar actions on G and give characterizations of some of these examples. In section 4 we construct all isometric involutions of affine algebras that leave $Rc + Rd$ invariant, and discuss their relations to hyperpolar actions on compact Lie groups and on Hilbert space. Finally, in section 5 we state some open problems.

The authors would like to thank the Max-Planck Institute in Bonn and Academia Sinica in Taipei for their hospitality while this paper was being prepared.

2. Algebraic criteria and rational tori

The main purpose of this section is to give a simple algebraic criterion for an isometric action on a symmetric space to be hyperpolar.

We will use the convention that for $g \in G$ and $y \in \mathcal{G}$, $gy := (L_g)_*(y)$, $yg := (R_g)_*(y)$, and $e^y := \exp(y)$, where L_g and R_g are left and right translation by g respectively. Recall that the *cohomogenity* of an action of G on M is the dimension of the orbit space M/G, or equivalently the codimension of a principal orbit.

2.1 Theorem. *Let G be a compact, semi-simple Lie group with the bi-invariant metric induced from the negative of the Killing form on $\mathcal{G}$. Let H be a closed subgroup of $G \times G$, acting isometrically on G by $(g_1, g_2) \cdot g = g_1 g g_2^{-1}$. Then the following statements are equivalent:* “refPn

(i) the H-action on G is hyperpolar,

(ii) there exists $g_0 \in G$ such that

$$g_0^{-1} \nu(H \cdot g_0)_{g_0} = \{g_0^{-1} x g_0 - y \,|\, (x, y) \in \mathcal{H}\}^{\perp}$$

is abelian,

(iii) $\nu(H \cdot e)_e$ contains a k-dimensional abelian subalgebra, where k is the cohomogeneity of the H-action on G.

To prove this theorem, we first need a lemma:

2.2 Lemma. *Let G and H be as in Theorem 2.1. Suppose $\nu(H \cdot e)_e = \mathcal{A}$ is abelian. Then* “refPa

(i) $T(H \cdot e^a) \perp e^a \mathcal{A}$ for all $a \in \mathcal{A}$,

(ii) if $\exp(\mathcal{A})$ is closed, then the H-action on G is hyperpolar.

PROOF. It is obvious that

$$T(H \cdot e)_e = \{x - y \,|\, (x, y) \in \mathcal{H}\},$$
$$e^{-a} T(H \cdot e^a) = \{e^{-a} x e^a - y \,|\, (x, y) \in \mathcal{H}\}.$$

Since $\mathcal{A}$ is abelian, $\langle \ , \ \rangle$ is Ad-invariant, and $\nu(H \cdot e)_e = \mathcal{A}$, we get

$$\langle e^{-a} x e^a - y, b\rangle = \langle e^{-a} x e^a, b\rangle - \langle y, b\rangle = \langle x, b\rangle - \langle y, b\rangle = \langle x - y, b\rangle = 0$$

for all $(x, y) \in \mathcal{H}$ and $b \in \mathcal{A}$. This proves (i). Since H is compact, every orbit meets $\exp(\nu(H \cdot e)_e)$, and (ii) follows. ∎

It follows from the fact that $\exp(\nu(H \cdot g)_g)$ meets every H-orbit and Lemma 2.2 that statements (ii) and (iii) in Theorem 2.1 are equivalent. Moreover, if $g_0 \in G$ and $H \cdot g_0$ is a principal orbit, then e is a regular point of the action of

$$\{(g_0^{-1} g_1 g_0, g_2) \,|\, (g_1, g_2) \in H\}$$

on G. So we may assume that e is a regular point, and then Theorem 2.1 follows from:

2.3 Theorem. *Let G, H be as in Theorem 2.1. If $\nu(H \cdot e)_e = \mathcal{A}$ is abelian, then the H-action on G is hyperpolar.* “refPi

Thus, to prove Theorem 2.3, it suffices to show that $\exp(\mathcal{A})$ is a torus. To do this, we introduce the notion of a rational lattice. A *full lattice* Λ of an n-dimensional real vector space V is a discrete subgroup of V isomorphic to Z^n. Let T be a torus (i.e., a compact, connected, abelian Lie group), $\mathcal{T}$ its Lie algebra, and $\exp : \mathcal{T} \to T$ the exponential map. Then $\Lambda(T) = \ker(\exp)$ is a full lattice of $\mathcal{T}$, called the *lattice of* T. A basis for $\Lambda(T)$ over Z is called a Z-basis for Λ. The following Proposition is easy and well-known:

2.4 Proposition. *Let T be a torus, $\Lambda(T)$ the lattice of T, and $\mathcal{A}$ a linear subspace of $\mathcal{T}$. Then $\exp(\mathcal{A})$ is a subtorus of T if and only if $\mathcal{A} \cap \Lambda(T)$ is a full lattice of $\mathcal{A}$.* “refPc

2.5 Definition. A full lattice Λ of a (real) Euclidean vector space V will be called a *rational lattice* if the ratio of non-zero inner products of any two elements of Λ is rational. (It clearly suffices to check the rationality of ratios of non-zero inner products of elements from some Z basis for Λ). “refDi

2.6 Proposition. *Let Λ be a full lattice of an Euclidean vector space V, and Λ' a full, sublattice of Λ. Then Λ is rational if and only if Λ' is rational.* “refPd

PROOF. Of course, any sublattice of a rational lattice is rational. If Λ' has the same rank as Λ, then the integer matrix expressing the elements of a Z-basis for Λ' in terms of a Z-basis of Λ can be inverted, and the inverse is a matrix of rational numbers. So if Λ' is rational then Λ is also rational. ∎

We call a torus T a *Riemannian torus* if it has a specified translation invariant Riemannian metric. Such a metric is of course uniquely determined by the induced Euclidean structure $\langle\ ,\ \rangle$ on $\mathcal{T}$, and we will usually not distinguish between $\langle\ ,\ \rangle$ and the Riemannian structure. A Riemannian torus is called *rational* if its lattice $\Lambda(T)$ is rational in $\mathcal{T}$.

2.7 Proposition. *Any subtorus A of a rational torus T is rational.* “refPe

PROOF. For, as noted above, the lattice of A is a sublattice of the lattice of T. ∎

2.8 Theorem. *If $\mathcal{A}$ is a linear subspace of the Lie algebra $\mathcal{T}$ of a rational torus T, then $\exp(\mathcal{A})$ is a subtorus of T if and only if $\exp(\mathcal{A}^\perp)$ is.* “refPf

PROOF. Since rescaling an inner product does not change orthogonal complements, we can assume that the non-zero inner product of any two elements in $\mathcal{T}$ is rational. Assume $A = \exp(\mathcal{A})$ is a subtorus of T. Then $\Lambda(A) = \Lambda(T) \cap \mathcal{A}$ is a full lattice of $\mathcal{A}$. Choose a basis $a_1, \ldots, a_k$ for $\mathcal{A}$ with $a_i \in \Lambda(A)$, and extend it to a basis $a_1, \ldots, a_n$ for $\mathcal{T}$ with $a_{k+i} \in \Lambda(T)$. Let $v_1, \ldots, v_n$ denote the orthogonal basis for $\mathcal{T}$ obtained by applying the Gram-Schmidt process to $a_1, \ldots, a_n$, omitting the renormalization steps. That is, $v_1 = a_1$, and recursively,

$$v_j = a_j - \sum_{i=1}^{j-1} \left\langle a_j, \frac{v_i}{\langle v_i, v_i \rangle} \right\rangle v_i.$$

Then clearly $v_1, \ldots, v_k$ is an orthogonal basis for $\mathcal{A}$, and $v_{k+1}, \ldots, v_n$ is an orthogonal basis for $\mathcal{A}^\perp$. Moreover, since all $\langle a_i, a_j \rangle$ are rational, it follows by an obvious induction on j that v_j is a *rational* linear combination $\sum_{i \le j} r_{ij} a_i$ of a_i with $i \le j$. If m is the product of the denominators of the non-zero r_{ij}, then $mv_1, \ldots, mv_n$ is an *integral* linear combinations of the a_i, hence in $\Lambda(T)$. But of course $mv_{k+1}, \ldots, mv_n$ is still a basis for $\mathcal{A}^\perp$, so $\mathcal{A}^\perp \cap \Lambda(T)$ has rank $n-k$, i.e., it is a full lattice in $\mathcal{A}^\perp$. ∎

Next we want to show that any torus subgroup of G is rational in the induced metric. Since a subtorus of a rational torus is rational, and since every torus of G is included in a maximal torus, it will suffice to prove that a maximal torus T of G is rational. By Proposition 2.6, it will suffice to show that there exists a full sublattice Λ_0 of $\Lambda(T)$, that is rational. We will show below that Λ_0 can be chosen to be the co-root lattice.

Let $\mathcal{T}$ be a maximal abelian subalgebra of $\mathcal{G}$, $\mathcal{G}_C = \mathcal{G} \otimes_C C$, $R \subset \mathcal{T}^*$ the root space of $\mathcal{G}_C$, and let the root space decomposition of $\mathcal{G}_C$ be:

$$\mathcal{G}_C = \mathcal{T}_C \oplus \bigoplus_{\alpha \in R} \mathcal{G}_\alpha,$$

where $[a, z_\alpha] = i\alpha(a) z_\alpha$ for all $\alpha \in R$ and $z_\alpha \in \mathcal{G}_\alpha$. Dualizing α with respect to the Killing form gives elements $\overrightarrow{\alpha}$ of $\mathcal{T}$ satisfying $\alpha(X) = \langle X, \overrightarrow{\alpha} \rangle$, and the co-roots $\alpha^* \in \mathcal{T}$ are defined by $\alpha^* = 2\overrightarrow{\alpha}/\langle \overrightarrow{\alpha}, \overrightarrow{\alpha} \rangle$. The co-root lattice Λ_0 is then the lattice generated by the co-roots $\alpha^*, \alpha \in R$. It is part of the elementary theory of simple Lie algebras (cf. [He]) that Λ_0 is a full sublattice of $\Lambda(T)$, and also that if α and β are roots, then:

1) $\alpha(\alpha^*) = 2$
2) $\beta(\alpha^*) \in Z$
3) $\beta - \beta(\alpha^*)\alpha$ is also a root.

By 2) we have $2\langle \overrightarrow{\beta}, \overrightarrow{\alpha} \rangle / \langle \overrightarrow{\alpha}, \overrightarrow{\alpha} \rangle = \langle \overrightarrow{\beta}, \alpha^* \rangle = \beta(\alpha^*)$ is an integer, and $\| \overrightarrow{\alpha} \|^2 / \| \overrightarrow{\beta} \|^2 = \alpha(\beta^*)/\beta(\alpha^*)$ is rational for any $\alpha, \beta \in R$. So given any α, β, γ in R we have

$$\langle \alpha^*, \beta^* \rangle / \langle \gamma^*, \gamma^* \rangle = \frac{\langle \overrightarrow{\alpha}, \overrightarrow{\beta} \rangle}{\| \overrightarrow{\alpha} \|^2} \frac{\| \overrightarrow{\gamma} \|^2}{\| \overrightarrow{\beta} \|^2}$$

is rational. This proves that Λ_0 is rational, and we have proved:

2.9 Theorem. *If G is a compact, semi-simple Lie group equipped with the bi-invariant metric given by the Killing form on $\mathcal{G}$, then any torus subgroup of G is rational.* “refPg

2.10 Proof of Theorem 2.3. “refPq

It suffices to prove that $A = \exp(\mathcal{A})$ is closed, i.e., a torus. In any case $B = \overline{A}$ is a torus, so its Lie algebra $\mathcal{B}$ is abelian. Let $\mathcal{A}_1$ denote the orthogonal complement of $\mathcal{A}$ in $\mathcal{B}$, and $A_1 = \exp(\mathcal{A}_1)$. First we claim that B is transversal to the orbit $H \cdot e$. To see this, we note that by Lemma 2.2, $T(H \cdot a)_a \perp a\mathcal{A}$, i.e.,

$$(a^{-1}xa - y, \mathcal{A}) = 0, \quad \forall\, a \in A,\, (x, y) \in \mathcal{H}.$$

Since B is the closure of A, it follows by continuity that $T(H \cdot b)_b \perp b\mathcal{A}$ for all $b \in \mathcal{B}$. If $b \in (H \cdot e) \cap B$, then $TB_b = b\mathcal{B} \supset b\mathcal{A} = \nu(H \cdot b)_b$, which proves that B is transversal to $H \cdot e$. By transversality, $B \cap (H \cdot e)$ is a compact submanifold of G. Next we show that $B \cap (H \cdot e) = A_1$. In fact, since $T(B \cap (H \cdot e))_b \subset b\mathcal{B} \cap T(H \cdot e)_b$ and $T(B \cap (H \cdot e))_b \perp b\mathcal{A}$, we have that $T(B \cap (H \cdot e))_b = b\mathcal{A}_1$. Hence $B \cap (H \cdot e) = A_1$. Since B and A_1 are tori, Theorem 2.8 implies that A is a torus. ∎

In the remainder of this section, we will derive an algebraic criterion for an isometric action on a symmetric space to be hyperpolar. First, we reduce the classification of hyperpolar actions on G/K to that of G:

2.11 Proposition. *Let (G,K) be a symmetric pair, $M = G/K$ the corresponding symmetric space, and H a closed connected subgroup of G. Then the H-action on M is hyperpolar if and only if the action of $H \times K$ is hyperpolar on G.* “refBc

PROOF. Let $\pi : G \to G/K$ be the natural Riemannian submersion, $\bar{g} = \pi(g) = gK$, and $\mathcal{G} = \mathcal{K} + \mathcal{P}$ the Cartan decomposition. By conjugation, we may assume that $\bar{e}$ is a regular point with respect to the H-action. Identifying $T(G/K)_{\bar{e}}$ with $\mathcal{P}$, we have $\nu(H \cdot \bar{e})_{\bar{e}} = \nu((H \times K) \cdot e)_e$. If the H-action on G/K is hyperpolar, then $\nu(H \cdot \bar{e})_{\bar{e}}$ is abelian, which implies that $\nu((H \times K) \cdot e)_e$ is abelian, so by Theorem 2.3, the $H \times K$-action on G is hyperpolar. Conversely, if the action of $H \times K$ on G is hyperpolar and e is a regular point, then $A = \exp(\nu((H \times K) \cdot e)_e)$ is a torus. So $\pi(A)$ is a flat, closed submanifold of G/K that is a section of the H-action. ∎

As a consequence of Theorem 2.1 and the above Proposition, we have:

2.12 Corollary. *Let G be as in Theorem 2.1, (G,K) a symmetric pair, $M = G/K$ the corresponding symmetric space equipped with the quotient metric, and $\mathcal{G} = \mathcal{K} + \mathcal{P}$ the Cartan decomposition. Let $\pi : G \to G/K$ be the natural Riemannian submersion, and $\bar{g} = \pi(g)$. Let H a closed connected subgroup of G, and k the cohomogeneity of the H-action on G/K. Then the following statements are equivalent:* “refPl

(i) the H-action on G/K is hyperpolar,
(ii) $\nu(H \cdot \bar{e})_{\bar{e}}$ contains a k-dimensional abelian subalgebra,
(iii) there exists $g \in G$ such that $g_*^{-1}(\nu(H \cdot \bar{g})_{\bar{g}})$ is a k-dimensional abelian subalgebra.

2.13 Corollary. *Let G/K be as in Corollary 2.12, and H a closed subgroup of G. If the H-action on G/K is of cohomogeneity one, then it is hyperpolar.* “refPm

2.14 Remark. It follows from the proof of Theorem 2.1 that this theorem still holds if G is equipped with a bi-invariant metric such that maximal tori of G are rational. “refPr

3. Examples and Characterizations

In this section, we give some known examples of hyperpolar actions on compact Lie groups and symmetric spaces, and give characterizations and conjugacy relations for some of these examples. We also discuss the maximality of H in G when the action of H on the symmetric space G/K is hyperpolar.

3.1 Examples. Example (2) is given in ([Co1]), which is a special case of the example (4) of Hermann ([Hr1]). “refBd

(1) The Adjoint Action. Let $H = \{(g,g) \,|\, g \in G\}$. Then the H-action on G is the Adjoint action of G, which is hyperpolar with maximal tori as sections.

(2) The σ-action. Let σ be an automorphism of G, and

$$H = G(\sigma) = \{(g, \sigma(g)) \,|\, g \in G\}.$$

The action of $G(\sigma)$ on G is called the σ-action, and is hyperpolar.

(3) The $K \times K$-action. If G/K is a symmetric space, then the action of $K \times K$ on G is hyperpolar.

(4) Hermann's examples. Let ϕ and ρ be involutions on G, and K_ρ and K_ϕ the fixed point set of ρ and ϕ respectively. Then the action of $H = K_\rho \times K_\phi$ on G is hyperpolar. To see this, we note that the isotropy subgroup is $H_e = \triangle(K_\phi \cap K_\rho)$ the diagonal group of $K_\phi \times K_\rho$. Furthermore, $T(H \cdot e)_e = \{X - Y \,|\, X \in \mathcal{K}_\rho \text{ and } Y \in \mathcal{K}_\phi\} = \mathcal{K}_\rho + \mathcal{K}_\phi$ and $\nu_e = \mathcal{P}_\rho \cap \mathcal{P}_\phi$, where $\mathcal{P}_\rho$ and $\mathcal{P}_\phi$ are the orthogonal complements of $\mathcal{K}_\rho$ and $\mathcal{P}_\phi$ respectively. Notice that on the Lie group $G^{\phi\rho} = \{g \in G \,|\, \phi\rho g = g\}$ we have the involution $\rho = \phi$ with fixed point group $K_\rho \cap K_\phi$ and Cartan decomposition $\mathcal{G}^{\phi\rho} = (\mathcal{K}_\rho \cap \mathcal{K}_\phi) \oplus (\mathcal{P}_\rho \cap \mathcal{P}_\phi)$. The cohomogeneity of the slice representation H_e on ν_e is therefore equal to the dimension of a maximal abelian subalgebra of $\mathcal{P}_\rho \cap \mathcal{P}_\phi$. Then it follows from Theorem 2.1 that the H-action on G is hyperpolar.

(4*) Let $\tau : G \times G \to G \times G$ be an involution, and H the fixed point set of τ. Then the H-action on G is hyperpolar. For the action of $H \times \triangle(G)$ is hyperpolar on $G \times G$ by (4) and hence the action of H on $G = G \times G/\triangle(G)$ is hyperpolar by Proposition 2.12.

(5) Cohomogeneity one actions. As noted in Corollary 2.13, any cohomogeneity one action on a symmetric space is hyperpolar. For example, if $\rho : K \to SO(n)$ is the isotropy representation of a rank 2 symmetric space, then the K-action on $S^{n-1} = SO(n)/SO(n-1)$ has cohomogeneity one. Hence the action of $\rho(K) \times SO(n-1)$ on $SO(n)$ is hyperpolar by Proposition 2.11. Notice that this example does not in general arise as a special case of Hermann's example (4). (For example, the isotropy representation of the Grassmannian $G_{2,m}(R)$ of two planes in R^m for $m > 2$: $\rho : SO(2) \times SO(m) \to SO(2m)$.)

3.2 Remark. The group H in Example 3.1 (3) is of the form $K \times K$, where "refBf
K is a symmetric subgroup of G. This example can be characterized as follows ([HPPT]): If L is a closed subgroup of G such that the action of $H = L \times L$ on G is hyperpolar, then G/L is a symmetric space of rank r, where r is the cohomogeneity of the H-action. But (G, L) is not necessarily a symmetric pair. For example, $(Spin(7), G_2)$ or $(G_2, SU(3))$.

Next we give a characterization of Example 3.1 (2):

3.3 Theorem. *Let G be a compact, simply connected Lie group, and H a* "refBe
compact, connected, simple subgroup of $G \times G$. If the H-action on G is non-transitive and hyperpolar, then G is simple and there is an automorphism σ of G such that $H = \{(g, \sigma(g)) \,|\, g \in G\}$.

First we need a Lemma.

3.4 Lemma. *Let K be a proper subgroup of a compact, connected, simple Lie group G of rank r. Then $\dim(G/K) > \sqrt{2}\, r$.* “refsb

PROOF. Let G be endowed with a bi-invariant metric and G/K with the quotient metric. Then G acts by isometries on G/K. The set N of elements of G that act trivially is a normal subgroup of G. So N is finite. It follows that $\dim G \leq \dim \mathrm{Iso}(G/K)$. Set $m = \dim(G/K)$. The dimension of the isometry group of a compact m-dimensional manifold is at most $m(m+1)/2$. It follows from the classification of compact simple Lie groups of rank r that their dimension is at least $r^2 + 2r$. Hence

$$r^2 + 2r \leq \frac{m(m+1)}{2},$$

which implies $m > \sqrt{2}\, r$. ∎

3.5 Proof of Theorem 3.3. “refBg

Let $p_i : H \to G$ be the projection onto the i-th factor for $i = 1, 2$. Since H is simple and $\ker p_i$ is a normal subgroup of H, $\ker p_i$ is either H or a finite subgroup of H. We have the following cases:
(i) $\ker p_2 = H$. Then $H = H_1 \times \{e\}$ for some subgroup H_1 of G. This means that H acts on G by left translations. All orbits are principal orbits, in particular the one through e. The section is a torus T and the Lie algebra G splits orthogonally into $\mathcal{G} = \mathcal{H}_1 + \mathcal{T}$, i.e., $\mathcal{T} = \mathcal{H}_1^{\perp}$. Since $\langle\ ,\ \rangle$ is ad-invariant, $\langle [\mathcal{H}_1, \mathcal{T}], \mathcal{T}\rangle = 0$ and $\langle [\mathcal{H}_1, \mathcal{T}], \mathcal{H}_1\rangle = 0$. So we have $[\mathcal{H}_1, \mathcal{T}] = 0$ and thus $\mathcal{T} = 0$ by the semisimplicity of G. Hence H acts transitively on G, a contradiction.

(ii) $\ker p_1 = H$. Then it follows similarly that H is transitive.

(iii) Both $\ker p_1$ and $\ker p_2$ are finite. Let $H' = p_1(H)$. Let $\mathcal{G} = \mathcal{G}_1 + \cdots + \mathcal{G}_k$ be the decomposition of $\mathcal{G}$ into simple ideals, and let G_i be the subgroup of G corresponding to $\mathcal{G}_i$. Let $H'_i \subset G_i$ be the projection of H' into G_i. Using that the codimension of the orbits of H cannot exceed the rank $r(G)$ of G, we get $\dim H + r(G) \geq \dim G$. Thus

$$(*) \qquad \sum_{i=1}^{k} \dim H'_i + \sum_{i=1}^{k} r(G_i) \geq \dim H + \sum_{i=1}^{k} r(G_i) \geq \sum_{i=1}^{k} \dim G_i.$$

If $H'_i \subset G_i$ is a proper subgroup of G_i, then $\dim H'_i + \sqrt{2} r(G_i) < \dim G_i$ by Proposition 3.4. By $(*)$ we thus have $H'_i = G_i$ for some i, and without loss of generality we can assume $i = 1$. Then $H' \to G_1$ is a covering map, and so in fact an isomorphism, since G and therefore all the G_i are simply-connected. It also follows that $p_i : H \to p_i(H)$ are isomorphisms and hence

$H = \{(g, \sigma(g)) \mid g \in H'\}$, where $\sigma : H' \to G$ is the injective homomorphism $\sigma = p_2 p_1^{-1}$. It is therefore left to prove that $H' = G$. Using (*),

$$\dim(H) + r(G) \geq \dim(G), \qquad \dim(G_i) \geq r(G_i)^2 + 2r(G_i)$$

and the fact that $H \simeq H' \simeq G_1$, we get

$$\sum_{i=1}^{k} r(G_i) \geq \sum_{i=2}^{k} \dim G_i \geq \sum_{i=2}^{k} (r(G_i)^2 + 2r(G_i)).$$

Hence

$$r(G_1) \geq \sum_{i=2}^{k} (r(G_i)^2 + r(G_i)).$$

In particular $r(H') = r(G_1) > r(G_i)$ for all $i \geq 2$. Therefore the projection π_i of H' to G_i, $i \geq 2$, cannot have finite kernel and thus π_i must be trivial. This implies $H' = G_1$.

Similarly we get $p_2(H) = G_i$ for some i. Since $p_2(H) \simeq H \simeq G_1$ and $r(G_i) < r(G_1)$ for $i \geq 2$ we must have $p_2(H) = G_1$ as well. Thus H acts only on G_1 and leaves the G_i, $i \geq 2$, fixed. Since the action of H is hyperpolar, G_i must be flat for $i \geq 2$. Hence $k = 1$ and $H' = G_1 = G$. ∎

Next we recall the notion of conjugacy for isometric actions.

3.6 Definition. Let X_1 be a Riemannian G_1-manifold, and X_2 a Riemannian G_2-manifold. Then X_1 and X_2 are called *conjugate* if there is an isomorphism $\phi : G_1 \to G_2$ and an isometry $f : X_1 \to X_2$ such that $\phi(g) \cdot f(x) = f(g \cdot x)$ for all $g \in G_1$ and all $x \in X_1$. “refOa

We now discuss the conjugacy of Examples 3.1 (4). Let G be a simple, compact Lie group. Recall that there is a standard list of (finitely many) involutions on G such that any involution on G can be conjugated to one of the standard list by an inner automorphism (cf. [Lo], p.101). For example, the standard list of involutions for the classical groups is:

$$\begin{aligned} A_n &= SU(n+1) : \quad \tau,\ \mathrm{Ad}(I_{p,q}),\ \tau\,\mathrm{Ad}(I_{p,q}) \\ B_n &= SO(2n+1) : \quad \mathrm{Ad}(I_{p,q}) \\ C_n &= Sp(n) : \quad \tau,\ \mathrm{Ad}(I_{p,q}) \\ D_n &= SO(2n) : \quad \mathrm{Ad}(I_{p,q}),\ \mathrm{Ad}(J_n), \end{aligned}$$

where $I_{p,q} = \mathrm{diag}(1, \ldots, 1, -1, \ldots, -1)$ is the diagonal matrix with p entries equal to 1 and q entries equal to -1, J_n is the $2n \times 2n$ matrix corresponding to the standard symplectic form on R^{2n}, and τ denotes complex conjugation.

In particular, if τ_1, τ_2 are involutions on G, then there exist σ_1, σ_2 from the standard list of involutions on G and $b_i \in G$ such that $\tau_i = \mathrm{Ad}(b_i)\sigma_i\,\mathrm{Ad}(b_i)^{-1}$. Note that $K_\sigma = \rho(K_\tau)$ if $\sigma = \rho\tau\rho^{-1}$ for some $\rho \in \mathrm{Aut}(G)$, where K_σ and K_τ are the fixed point sets of σ and τ respectively. So to classify the examples in 3.1(iv) up to conjugacy leads us naturally to the following result:

3.7 Proposition. *Suppose H_1 and H_2 are two subgroups of $G \times G$ and there exists a fixed $b \in G$ such that* “refOb

$$H_2 = \{(bh_1b^{-1}, h_2) \,|\, (h_1, h_2) \in H_1\}.$$

Then the H_1-actions and H_2-actions on G are conjugate via the isometry L_b given by the left multiplication $g \mapsto bg$. Similarly, if

$$H_2 = \{(h_1, bh_2b^{-1}) \,|\, (h_1, h_2) \in H_1\},$$

then the H_1 and H_2 actions on G are conjugate via the isometry $R_{b^{-1}}$, the right multiplication by b^{-1}.

PROOF. In the first case note that $bh_1b^{-1}bxh_2^{-1} = L_b(h_1xh_2^{-1})$ implies that the actions are conjugate. The other case follows similarly. ∎

3.8 Corollary. *Let G be a simple, simply connected, compact Lie group, and τ_1, τ_2 involutions on G. Then there exist σ_1, σ_2 from the standard list of involutions on G such that the actions of $K_{\tau_1} \times K_{\tau_2}$ and $K_{\sigma_1} \times K_{\sigma_2}$ on G are conjugate.* “refOc

Next we discuss the conjugacy of Examples 3.1 (2). Note that the σ-action is hyperpolar even when σ is not of finite order. We will prove that every σ-action on G is conjugate to a τ-action for some automorphism τ of order 1, 2 or 3.

In order to proceed, we need to review the diagram automorphism group of a simply connected, compact, semisimple Lie group G (cf. [Lo] p. 44 and [He] Chap 9, 10). It is known that each automorphism of the Dynkin digram of G gives rises to an automorphism of G, which will be called a *diagram automorphism*. Let $E(G)$ denote the group of diagram automorphisms. Then $E(G)$ is isomorphic to $\mathrm{Aut}(R)/W$, where R is the root system of $\mathcal{G}$ and W the corresponding Weyl group. Let $\mathrm{Aut}(G)$ and $\mathrm{Int}(G)$ denote the group of automorphisms and inner automorphisms of G respectively. Then $\mathrm{Int}(G)$ is the identity component of $\mathrm{Aut}(G)$ and the quotient group is isomorphic to the group $E(G)$. Furthermore, $\mathrm{Aut}(G)$ is the semidirect product of $\mathrm{Int}(G)$ and $E(G)$. It is known that $E(G)$ is as follows (see [Lo], p. 99):

$$E(G) = 1, \text{ if } G = A_1, B_r, C_r, E_7, E_8, F_4, G_2$$
$$E(G) = Z_2, \text{ if } G = A_r\,(r > 1), D_r (r \neq 4), E_6\,.$$
$$E(G) = S_3, \text{ if } G = D_4$$

Note that every element in Z_2 or S_3 has order $1, 2$ or 3. For $\sigma \in \mathrm{Aut}(G)$, we let $[\sigma]$ denote the left coset $\mathrm{Int}(G)\sigma$ in $\mathrm{Aut}(G)/\mathrm{Int}(G)$. Using that $\mathrm{Aut}(G)$ is the semidirect product of $\mathrm{Int}(G)$ and $E(G)$, one sees easily that there exists an automorphism $\mu \in [\sigma]$ of the same order as $[\sigma]$ in $E(G)$ (i.e, of order 1, 2 or 3). So there is an element $b \in G$ such that $\sigma = \mathrm{Ad}(b)\mu$. Hence

$$G(\sigma) = \{(g, \sigma(g)) \,|\, g \in G\} = \{(g, b\mu(g)b^{-1}) \,|\, g \in G\}.$$

Then it follows from Proposition 3.7 that the σ-action and the μ-action on G are conjugate. This proves the following theorem:

3.9 Theorem. *Let G be a simply connected, compact Lie group, $\sigma \in \mathrm{Aut}(G)$. Then there exists a diagram automorphism μ of G that represents the same element as σ in $\mathrm{Aut}(G)/\mathrm{Int}(G)$ and has order 1, 2, or 3. Moreover, the σ-action on G is conjugate to the μ-action.* “refOd

As a consequence, Examples 3.1 (2) on a simply connected, semi-simple Lie group G are classified up to conjugacy by the diagram automorphisms of G.

3.10 Remark. Let G be a non-connected, compact Lie group, G_0 the identity component, and G_1 some other component. Let $h_1 \in G_1$, $\sigma : G_0 \to G_0;\ g \to h_1 g h_1^{-1}$, and $f : G_1 \to G_0;\ h \to h h_1^{-1}$. Then $f(ghg^{-1}) = g f(h) \sigma(g)^{-1}$. This proves that the action of G_0 on G_1 by conjugation is hyperpolar, and is conjugate to the σ-action on G_0. This also gives the generalization of the maximal torus theorem to nonconnected groups given in [Se]. “refBi

Motivated by Dadok's classification of polar representations [Da], we define the notion of ω-equivalence—a weaker equivalence relation than conjugacy.

3.11 Definition. Let X_1 be a Riemannian G_1-manifold, and X_2 a Riemannian G_2-manifold. Then X_1 and X_2 are called *ω-equivalent* (or orbital equivalent), if there exists an isometry $f : X_1 \to X_2$ such that $f(G_1 \cdot x) = G_2 \cdot f(x)$. “refBj

We will show below that a group whose action is hyperpolar is often maximal in the sense that the action of a larger group is either ω-equivalent or transitive.

3.12 Theorem. *Let $H \subset L \subset SO(n)$ be connected, closed subgroups. If the action of H on R^n is irreducible and polar, then the L-action is either transitive on S^{n-1} or is ω-equivalent to the H-action.* “refmaxi

PROOF. We will use some basic results from the theory of isoparametric submanifolds that can be found in [PT 2]. The proof will be by induction on $k = \dim(R^n/H)$. The theorem is obviously true if $k = 1$. Now assume the theorem is proved for $k-1$, and assume $\dim(R^n/H) = k$. Choose $p \in S^{n-1}$ such that the slice representation of H_p has rank $k-1$ and its associated Dynkin diagram is connected. This can be done by chosing p to be a non-zero point on the intersection of the hyperplanes $\ell_2, \ldots, \ell_k$, where $\ell_1, \ldots, \ell_k$ are hyperplanes corresponding to simple roots and ℓ_1 corresponds to the left-end vertex of the Dynkin diagram. Then it follows from the slice theorem that the slice representation of H_p is irreducible on $\nu_0 = \nu(Hp)_p \cap (\mathbf{R}p)^\perp$. Since L_p leaves $\nu_1 = \nu(Lp)_p \cap (\mathbf{R}p)^\perp$ invariant, $\nu_1 \subset \nu_0$ and $H_p \subset L_p$, ν_1 is an invariant subspace of ν_0 under H_p. But H_p on ν_0 is irreducible. So one of the following cases is true:

(1) $\nu_1 = 0$. This implies that $\nu(Lp)_p = \mathbf{R}p$ is one-dimensional. Then the L-action is transitive on S^{n-1}.

(2) $\nu_1 = \nu_0$. Then $\nu = \nu(Hp)_p = \nu(Lp)_p$ and Hp and Lp have the same dimension. But $Hp \subset Lp$, so $Hp = Lp$. By the slice theorem we have that H_p is irreducible and polar with cohomogeneity $k-1$ on ν_0. Also $H_p \subset L_p \subset SO(\nu_0)$. So, by the induction hypothesis, the L_p-action on ν_0 is polar and either ω-equivalent to the H_p-action or transitive on the unit sphere in ν_0. In the

first case it follows that the dimension of the orbits of H and L are equal and hence the orbits coincide since $H \subset L$. In the second case the orbits of L are hypersurfaces in spheres and it follows that L is polar. The orbits of both the H- and the L-action give rise to isoparametric foliations with the orbit $Hp = Lp$ in common. It is proved in [HOT] that a full leave in an isoparametric foliation determines the foliation. This proves that the orbits of H and L coincide, i.e., H and L are ω-equivalent. ∎

3.13 Theorem. *Let $X = G/K$ be a compact, connected and irreducible symmetric space, and $H \subset L \subset G$ closed, connected subgroups. Suppose the H-action on X is hyperpolar and there exists a $p \in X$ such that $Hp = Lp$. Then the L-action on X is ω-equivalent to the H-action.* “refmaxiii

PROOF. If A is a section of H passing through p, then A meets one orbit of L, $Lp = Hp$, orthogonally. Corollary 2.12 implies that the action of L on X is hyperpolar with A as a section. The principal orbits of H and L are therefore equal since their dimensions are equal and $H \subset L$. The principal orbits of a hyperpolar action determine the other orbits. It follows that the H- and L-actions are ω-equivalent. ∎

It is known that (cf. [Co3], [PT1]) if the action of H on the compact symmetric space G/K is hyperpolar and A is a section through eK then $W(A) = N(A)/Z(A)$ is a finite group, where $N(A)$ and $Z(A)$ are the normalizer and centralizer of A in H respectively. Let $\mathcal{A}$ denote the Lie algebra of A, and $\Lambda = \exp^{-1}(e) \cap \mathcal{A}$ the unit lattice in $\mathcal{A}$. Then $W(A)$ acts on Λ and the corresponding semi-direct product $\hat{W}(A) = W(A) \ltimes \Lambda$ is an affine Weyl group. The group $W(A)$ and $\hat{W}(A)$ will be called the *generalized Weyl group* and the *affine Coxeter group associated to the* H-action.

3.14 Corollary. *Let $X = G/K$ be a compact, connected and irreducible symmetric space, and $H \subset L \subset G$ closed, connected subgroups. Suppose the H-action on X is hyperpolar and the affine Coxeter group associated to the H-action is irreducible. Then the L-action on X is either transitive or is ω-equivalent to the H-action.* “refmaxiv

PROOF. It is known that the slice representation of a polar action is polar ([PT1]). Since the associated affine Coxeter group is irreducible, there exists $p \in X$ such that the slice representation of H_p on $\nu = \nu(Hp)_p$ is irreducible. Since $H \subset L$, $\nu_1 = \nu(Lp)_p \subset \nu$. Because ν is an irreducible H_p-space, we have the following two cases:

(1) $\nu_1 = 0$, this implies that the L-action is transitive on X.

(2) $\nu_1 = \nu$, then $Hp = Lp$. So by Theorem 3.13, the H and L actions are ω-equivalent. ∎

4. Involutions of Kac-Moody algebras and polar actions on Hilbert space

In this section, we classify isometric, involutive automorphisms of affine algebras, and prove that up to ω-equivalence the affine s-representations associated to these involutions are either the $P(G, G(\sigma))$-actions or $P(G, K_1 \times K_2)$-actions on $H^0([0,1], \mathcal{G})$, where σ is some automorphism and K_1, K_2 are some symmetric subgroups of G.

First we review some basic facts about affine algebras (cf.[Ka]). Let $\mathcal{G}$ be the Lie algebra of a compact, semi-simple Lie group, $L(\mathcal{G}) = H^0(S^1, \mathcal{G})$, $t \in [0, 2\pi]$ the angle variable in S^1, and let u' denote du/dt (for those u for which du/dt is defined, i.e., $u \in H^1(S^1, \mathcal{G})$). Let $[u, v](t) = [u(t), v(t)]$. Then $L(\mathcal{G})$ is a Lie algebra. There are three types of affine algebras, which are two-dimensional extensions of $L(\mathcal{G})$ (cf. [Ka], Ch 7 and 8). We begin with the algebras of type one, which are extensions of $L(\mathcal{G})$:

$$\hat{L}(\mathcal{G}) = L(\mathcal{G}) + Rc + Rd$$

with the bracket operation defined on a dense subspace of $\hat{L}(\mathcal{G})$ by

$$\begin{aligned} &[u, v]^\wedge = [u, v] + \omega(u, v)c, \\ &[d, u]^\wedge = u' \\ &[c, u]^\wedge = [c, d]^\wedge = 0, \end{aligned}$$

where ω is the 2-cocyle defined on $L(\mathcal{G})$ by formula (1.1). Let $\langle\, , \,\rangle$ be the bi-linear form (1.2) defined on $\hat{L}(\mathcal{G})$. Then $\langle\, , \,\rangle$ is non-degenerate and ad-invariant, i.e., for all $\xi, \eta, \zeta \in \hat{L}(\mathcal{G})$ we have

$$\langle [\xi, \eta]^\wedge, \zeta \rangle = \langle \xi, [\eta, \zeta]^\wedge \rangle.$$

It is known that (cf. [PS]) there exists a group $\hat{L}(G)$ and a torus bundle $\pi : \hat{L}(G) \to L(G)$ such that $d\pi_e$ is the natural projection of $\hat{L}(\mathcal{G})$ to $L(\mathcal{G})$. Moreover, the Adjoint action of $\hat{L}(G)$ on its Lie algebra is given as follows (cf. [Ka], [PS]):

(1) For $\hat{g} \in \pi^{-1}(g)$ and $u \in L(\mathcal{G})$ then

$$\mathrm{Ad}(\hat{g})(d + u - \frac{\parallel u \parallel^2 + 1}{2} c) = d + g * u - \frac{\parallel g * u \parallel^2 + 1}{2} c,$$

where $g * u = gug^{-1} - g'g^{-1}$ is the gauge transformation and $\parallel u \parallel$ is the L^2-norm of u.

(2) Let $\{\exp(sd) \mid s \in [0, 2\pi]\}$ denote the one parameter circle subgroup in $\hat{L}(G)$ generated by the element d. Then

$$\mathrm{Ad}(\exp(sd))(u)(t) = u(t + s).$$

Although the bracket $[\ ,\ ]^\wedge$ is defined on a dense subset of $\hat{L}(\mathcal{G})$ and $\hat{L}(G)$ is not a Lie group, the Adjoint action is well-defined on $\{d + u - \frac{\parallel u \parallel^2 + 1}{2} c \mid u \in L^2(\mathcal{G})\}$.

Next we recall some explicit constructions of automorphisms on affine algebras in terms of automorphisms on $\mathcal{G}$.

4.1 Proposition. *Let σ be an automorphism of $\mathcal{G}$ of order k, m respectively. Then $\hat{\sigma} : \hat{L}(\mathcal{G}) \to \hat{L}(\mathcal{G})$ defined by* “refJe

$$\hat{\sigma}(u)(t) = \sigma(u(-\tfrac{2\pi}{k} + t)), \ \ \hat{\sigma}(c) = c, \ \ \hat{\sigma}(d) = d$$

is an automorphism of order k, is an isometry with respect to the ad-invariant form $\langle \ , \ \rangle$, and the fixed point set of $\hat{\sigma}$ is

$$\hat{L}(\mathcal{G}, \sigma) = L(\mathcal{G}, \sigma) + Rc + Rd,$$

where

$$L(\mathcal{G}, \sigma) = \{u \in L(\mathcal{G}) \mid u(t) = \sigma(u(-\tfrac{2\pi}{k} + t))\}.$$

4.2 Proposition. *Let ρ be an involution on $\mathcal{G}$. Then $\rho^* : \hat{L}(\mathcal{G}) \to \hat{L}(\mathcal{G})$ defined by* “refJf

$$\rho^*(u)(t) = \rho(u(-t)), \quad \rho^*(c) = -c, \quad \rho^*(d) = -d,$$

is an involution, and is an isometry with respect to the ad-invariant form $\langle \ , \ \rangle$.

If the order of $[\sigma]$ in $\mathrm{Aut}(\mathcal{G})/\mathrm{Int}(\mathcal{G})$ is equal to ℓ, then $\hat{L}(\mathcal{G}, \sigma)$ is called an *affine algebra of type ℓ*. The following two propositions give constructions of automorphisms of $\hat{L}(\mathcal{G}, \sigma)$:

4.3 Proposition. *Let σ be an order k automorphism and ρ an involution on $\mathcal{G}$. Then $\rho^* : \hat{L}(\mathcal{G}) \to \hat{L}(\mathcal{G})$ leaves $\hat{L}(\mathcal{G}, \sigma)$ invariant if and only if $\sigma\rho$ is an involution, or equivalently if and only if $\sigma\rho = \rho\sigma^{-1}$.* “refKc

PROOF. It suffices to prove that $\rho^*(L(\mathcal{G}, \sigma)) \subset L(\mathcal{G}, \sigma)$. To see this, we assume that $u \in L(\mathcal{G})$ satisfies the condition that $u(t) = \sigma(u(-2\pi/k + t))$, and we want to show that

$$\rho^*(u)(t) = \sigma(\rho^*(u)(-2\pi/k + t)).$$

By the definition of $\rho^*(u)$, we see that the left hand side (LHS) and the right hand side (RHS) of the above equation are equal to

$$LHS = \rho(u(-t)),$$
$$RHS = \sigma(\rho(u(2\pi/k - t))) = \sigma(\rho(\sigma(u(-t)))).$$

Hence $RHS = LHS$ if and only if $\sigma\rho\sigma = \rho$. ∎

4.4 Proposition. *Let σ and ρ be finite order automorphisms on $\mathcal{G}$ of order k and m respectively, with $k = 2$ or 3. Then the map $\hat{\rho} : \hat{L}(\mathcal{G}) \to \hat{L}(\mathcal{G})$ leaves $\hat{L}(\mathcal{G}, \sigma)$ invariant if and only if $\rho\sigma = \sigma\rho$. Moreover, if $\rho\sigma = \sigma\rho$ and $\hat{\mathcal{K}}$ denotes the set of fixed points of $\hat{\rho}$ on $\hat{L}(\mathcal{G}, \sigma)$, then* “refKh

(1) if $m = kn$ for some n, then $\hat{\mathcal{K}} = \hat{L}(\mathcal{G}_0, \theta)$, where $\mathcal{G}_0$ is the fixed point set of $\sigma^{-1}\rho^n$ and $\theta = \rho \mid \mathcal{G}_0$, or

(2) if k is not a factor of m, then there exist integers a, b such that $ak + bm = 1$ and $\hat{\mathcal{K}} = \hat{L}(\mathcal{G}, \rho^a\sigma^b)$.

PROOF. The first part of the Proposition is obvious. To prove the rest, let $u \in L(\mathcal{G}) \cap \hat{\mathcal{K}}$. Then we have

$$\sigma(u(t)) = u(\tfrac{2\pi}{k} + t), \qquad \rho(u(t)) = u(\tfrac{2\pi}{m} + t).$$

(1) If $m = kn$, then

$$\rho^n(u(t)) = u(2n\pi/m + t) = u(2n\pi/kn + t) = u(2\pi/k + t) = \sigma(u(t)).$$

This implies that $\hat{K} = \hat{L}(K_\tau, \theta)$, where $\tau = \sigma^{-1}\rho^n$, and $\theta = \rho \,|\, K_\tau$.

(2) If k does not divide m, then since k is a prime, k, m are relatively prime. Hence there exist integers a, b such that $ak + bm = 1$, or equivalently $a/m + b/k = 1/km$. Then we have

$$u(2\pi/km + t) = u(2\pi(a/m + b/k) + t) = \rho^a\sigma^b(u(t)).$$

This proves that $\hat{K} = \hat{L}(\mathcal{G}, \rho^a\sigma^b)$. ∎

The following theorem gives a characterization of the involutions on the affine algebras constructed in Proposition 4.3 and 4.4:

4.5 Theorem. *Let $\mathcal{G}$ be the Lie algebra of a compact Lie group such that $\mathcal{G}_C = \mathcal{G} \otimes C$ is simple, σ an order k diagram automorphism of $\mathcal{G}$, and ϕ an involution of the affine algebra $\hat{L}(\mathcal{G}, \sigma)$. Suppose ϕ is an isometry with respect to the ad-invariant form $\langle , \rangle$ and ϕ leaves $Rc + Rd$ invariant. Then ϕ must be one of the following two types:* “refJe

(1) there exist $s \in [0, 2\pi]$ and an involution ρ on $\mathcal{G}$ such that $\sigma\rho = \rho\sigma$ and ϕ is the restriction of $\mathrm{Ad}(\exp(sd)) \circ \hat{\rho}$ to $\hat{L}(\mathcal{G}, \sigma)$,

(2) there exist $s \in [0, 2\pi]$ and an involution ρ of $\mathcal{G}$ such that $\sigma\rho\sigma = \rho$ and ϕ is the restriction of $\mathrm{Ad}(\exp(sd)) \circ \rho^*$ to $\hat{L}(\mathcal{G}, \sigma)$.

PROOF. It follows easily from the fact that Rc is the center and ϕ is an isometry that the restriction of ϕ to $Rc + Rd$ is either equal to id or $-$ id. Now we assume that $\phi(c) = -c, \phi(d) = -d$, and we will prove below that statement (2) is true. A similar proof will also show that (1) is true if $\phi(c) = c$ and $\phi(d) = d$.

Let $\mathcal{G}$ the complex Lie algebra $\mathcal{G} \otimes C$, and $\tau : \mathcal{G} \to \mathcal{G}$ the conjugation map. Then σ can be extended uniquely to an order k automorphism of $\mathcal{G}$, which will still be denoted by σ. Let $\epsilon = e^{\frac{2\pi}{k}}$, and $\mathcal{G}_j$ the ϵ^j-eigenspace of σ. Then

$$\mathcal{G} = \mathcal{G}_0 + \mathcal{G}_1 + \ldots + \mathcal{G}_{k-1}.$$

For $j \in Z$, we define $\mathcal{G}_j = \mathcal{G}_i$ if $0 \le i \le k-1$ and $i \equiv j \pmod{k}$.

Given $u(t) = \sum u_n e^{int} \in L(\mathcal{G})$, then $u \in L(\mathcal{G}, \sigma)$ if and only if

$$\tau(u_n) = u_{-n}, \qquad u_n \in \mathcal{G}_n \qquad \forall\, n.$$

Since

$$\phi([d,[d,u]^\wedge]^\wedge) = [d,[d,\phi(u)]^\wedge]^\wedge,$$

ϕ leaves the eigenspaces of d^2 invariant. So there exist conjugate linear maps $F_n : \mathcal{G}_n \to \mathcal{G}_{-n}$ such that

$$\phi(xe^{int} + \tau(x)e^{-int}) = F_n(x)e^{-int} + \tau(F_n(x_n))e^{int}.$$

where $x \in \mathcal{G}_n$. Since ϕ is an automorphism of $\hat{L}(\mathcal{G},\sigma)$, we have

$$[F_n(X), F_m(Y)] = F_{n+m}([X,Y]), \qquad X \in \mathcal{G}_m, Y \in \mathcal{G}_n, \qquad (*_{m,n})$$

$$F_{-n} \circ F_n = \mathrm{id}\,.$$

We divide the proof of (2) into the following steps:

(i) There exist a non-trivial simple ideal $\mathcal{H}$ in $\mathcal{G}_0$ and $a \in C$ such that $F_{nk}(x) = a^n F_0(x)$ for all $x \in \mathcal{H}$.

To see this, we set $A_n = F_0^{-1} F_{nk}$. Then A_n is complex linear map on $\mathcal{G}_0$. Applying F_0^{-1} to equation $(*_{0,nk})$, we get

$$A_n([x,y]) = [x, A_n(y)]. \qquad (**)$$

Since σ is a diagram automorphism, there is a non-trivial simple ideal $\mathcal{H}$ of $\mathcal{G}_0$ (cf. [He]). Then equation (**) implies that $A_n(\mathcal{H}) \subset \mathcal{H}$. By Schur's lemma, there exists $a_n \in C$ such that $A_n \,|\, \mathcal{H} = a_n\,\mathrm{id}$. But equation $(*_{m,n})$ implies that $a_n a_m = a_{n+m}$. So we have $A_n \,|\, \mathcal{H} = a^n\,\mathrm{id}$ with $a = a_1$. This implies that $F_n(x) = a^n F_0(x)$ for all $x \in \mathcal{H}$.

(ii) $F_{j+nk} = a^n F_j$.

For $0 \le j < k$, we let

$$\mathcal{G}_{j,a} = \{X \in \mathcal{G}_j \,|\, F_{j+nk}(X) = a^n F_j(X)\ \forall n \in Z\}$$

$$\mathcal{G}_a = \sum_{j=0}^{k-1} \mathcal{G}_{j,a}.$$

It is easy to check that $\mathcal{G}_a$ is an ideal of $\mathcal{G}$. But we have proved that $\mathcal{G}_{0,a} \supset \mathcal{H}$ in (i). So $\mathcal{G}_a$ is a non-zero ideal of the simple algebra $\mathcal{G}$, which implies that $\mathcal{G}_a = \mathcal{G}$. Hence we have proved that

$$F_{j+nk}(X) = a^n F_j(X), \qquad \forall\ X \in \mathcal{G}_C, 0 \le j < k.$$

(iii) Let $\beta = a^{-\frac{1}{k}}$, and $\rho : \mathcal{G} \to \mathcal{G}$ be the map defined by

$$\rho(X) = \beta^j F_j(X), \quad \text{if } X \in \mathcal{G}_j.$$

Then ρ is an automorphism of $\mathcal{G}$. To see this, we let $X \in \mathcal{G}_i$ and $Y \in \mathcal{G}_j$:
(a) if $i+j < k$, then

$$[\rho(X), \rho(Y)] = [\beta^i F_i(X), \beta^j F_j(Y)] = \beta^{i+j}[F_i(X), F_j(Y)]$$
$$= \beta^{i+j} F_{i+j}([X,Y]) = \rho([X,Y]),$$

(b) if $i+j > k-1$, then

$$[\rho(X), \rho(Y)] = [\beta^i F_i(X), \beta^j F_j(Y)] = \beta^{i+j} F_{i+j}([X,Y])$$
$$= \beta^{i+j} a F_{i+j-k}([X,Y]) = \rho([X,Y]).$$

(iv) ρ is an involution and $\sigma\rho\sigma = \rho$.

It follows from the construction of ρ that $\sigma\rho\sigma = \rho$. So it remains to prove that ρ is an involution. To see this, we compute directly for $X \in \mathcal{G}_i$:

$$\rho(\rho(X)) = \rho(\beta^i F_i(X)) = \beta^i \rho(F_i(X))$$
$$= \beta^i \beta^{k-i} F_{k-i}(F_i(X)) = \beta^k a F_{-i}(F_i(X)) = X.$$

(v) Let $X \in \mathcal{G}_n$, and $u(t) = Xe^{int} + \tau(X)e^{-int}$. Then

$$\phi(u) = \beta^{-n}\rho(X)e^{-int} + \beta^n \rho(\tau(X))e^{int}. \qquad (***)$$

In particular, this implies that $\beta^{-n} = \bar{\beta}^n$, i.e., $|\beta| = 1$. So there exists $s \in [0, 2\pi]$ such that $\beta = e^{is}$.

(vi) $\phi = \mathrm{Ad}(\exp(sd))\rho^*$.

This follows from (v) and formula (***). ∎

Next we recall a theorem proved in [T2]:

4.6 Theorem ([T2]). *Let G be a compact, semi-simple Lie group equipped with a bi-invariant metric, and H a closed subgroup of $G \times G$. Let $P(G,H)$ denote the space of H^1-paths $g : [0,1] \to G$ such that $(g(0), g(1)) \in H$, and let $P(G,H)$ act on $H^0([0,1], \mathcal{G})$ by gauge transformations:* “refJa

$$g * u = gug^{-1} - g'g^{-1}.$$

Let $\phi : H^0([0,1], \mathcal{G}) \to G$ be the holonomy map defined by $\phi(u) = g(1)$, where $g : [0,1] \to G$ is the solution to the initial value problem

$$g^{-1}g' = u, \qquad g(0) = e.$$

Then

(i) the orbit $P(G,H) * u = \phi^{-1}(H \cdot \phi(u))$,

(ii) if the action of H on G is hyperpolar and A is a section through e, then the action of $P(G,H)$ on $H^0([0,1], \mathcal{G})$ is polar with the space $\mathcal{A}$ of constant paths in the Lie algebra $\mathcal{A}$ of A as a section.

Applying this theorem to Examples 3.1, we obtain

4.7 Corollary. *The actions of the following groups on the Hilbert space $H^0([0,1],\mathcal{G})$ via gauge transformations are hyperpolar:* “refJd

(1) $P(G, G(\sigma))$, where σ is an automorphism on G and $G(\sigma) = \{(g, \sigma(g)) \,|\, g \in G\}$,
(2) $P(G, K \times K)$, where K is the fixed point set of some involution σ on G,
(3) $P(G, K_1 \times K_2)$, where K_i is the fixed point set of the involution σ_i for $i = 1, 2$,
(4) $P(G, H)$, if the cohomogeneity of the H-action on G is one.

As a consequence of Theorem 4.6 (i), we see that if the actions of H_1 and H_2 on G are ω-equivalent then the actions of $P(G, H_1)$ and $P(G, H_2)$ are ω-equivalent. The following Proposition shows that if the actions of H_i on G are conjugate, then so are the actions of $P(G, H_i)$ on $H^0([0,1],\mathcal{G})$.

4.8 Proposition. *Let H_1 and H_2 be two closed subgroups of $G \times G$. If there exists an element $(a, b) \in G \times G$ such that* “refOe

$$H_2 = \{(ah_1a^{-1}, bh_2b^{-1}) \,|\, (h_1, h_2) \in H_1\},$$

then the $P(G, H_1)$-action and $P(G, H_2)$-action on $V = H^0([0,1],\mathcal{G})$ are conjugate.

PROOF. It suffices to prove the proposition for $b = e$. Since G is connected and compact, there exists $c \in \mathcal{G}$ such that $a = \exp(c)$. Given $g \in P(G, H_1)$, i.e., $g : [0,1] \to G$ such that $(g(0), g(1)) \in H_1$, let

$$\tilde{g}(t) = e^{(1-t)c} g(t) e^{-(1-t)c}.$$

It is easy to see that $\tilde{g} \in P(G, H_2)$ because

$$\tilde{g}(0) = \exp(c) g(0) \exp(-c) = a g(0) a^{-1}, \quad \tilde{g}(1) = g(1).$$

For $u \in V$, a direct computation implies that

$$\tilde{g} * u = e^{(1-t)c}\{g(e^{-(1-t)c} u e^{(1-t)c} - c)g^{-1} - g' g^{-1}\} e^{-(1-t)c} + c.$$

Let $V = H^0([0,1],\mathcal{G})$, and $F : V \to V$ the map defined by

$$F(u) = \exp((1-t)c) u \exp(-(1-t)c) + c.$$

Then F is an isometry of V and

$$\tilde{g} * F(u) = F(g * u),$$

i.e., the $P(G, H_1)$-action and $P(G, H_2)$-action on V are conjugate via the isometry F. ∎

As an immediate consequence of the above Proposition, Corollary 3.8 and Theorem 3.9, we obtain

4.9 Corollary. *The following statements are true:* “refOf

(i) Given two involutions τ_1, τ_2 on G, the $P(G, K_{\tau_1} \times K_{\tau_2})$-action on V is conjugate to $P(G, K_{\sigma_1} \times K_{\sigma_2})$-action on V for some involutions σ_1, σ_2 from the standard list of involutions on$G$, where K_{τ_i} and K_{σ_i} denote the fixed point sets of τ_i and σ_i respectively.

(ii) Given $\tau \in \mathrm{Aut}(G)$, the $P(G, G(\tau))$-action on V is conjugate to the action of $P(G, G(\mu))$ for some diagram automorphism μ. Moreover, the order of μ is 1, 2 or 3, and $[\tau]$ and $[\mu]$ are the same element in $\mathrm{Aut}(G)/\mathrm{Int}(G)$.

4.10 Proposition ([PS]). *Let $V_0 = \{u + rc + d \,|\, u \in L(\mathcal{G}), r = \frac{\|u\|^2+1}{2}\}$. Then* “refJb

(1) the map $u + rc + d \mapsto u$ defines an isometry from V_0 to $H^0([0, 2\pi], \mathcal{G})$,

(2) the Adjoint action of $\hat{L}(G)$ on $\hat{L}(\mathcal{G})$ leaves V_0 invariant, and induces an action of $L(G)$ on V_0,

(3) the induced action of $L(G)$ on V_0 is conjugate to the $P(G, G(\mathrm{id}))$ action on $H^0([0, 2\pi], \mathcal{G})$, where $G(\mathrm{id}) = \{(g, g) \,|\, g \in G\}$.

Given an order k automorphism σ and involution ρ of G, we can define in the same way as for $L(\mathcal{G})$ an automorphism $\hat{\sigma}$ and an involution ρ^* on $L(G)$. Moreover, (cf. [T1]):

(1) both $\hat{\sigma}$ and ρ^* can be lifted to the affine group $\hat{L}(G)$,

(2) the Lie algebra of the fixed point set $\hat{L}(G, \sigma)$ of $\hat{\sigma}$ on $\hat{L}(G)$ is $\hat{L}(\mathcal{G}, \sigma)$ (here we still use σ to denote the automorphism $d\sigma_e$ on $\mathcal{G}$),

(3) the loop part of $\hat{L}(G, \sigma)$, i.e., $\pi\hat{L}(G, \sigma)$, is

$$L(G, \sigma) = \{g \in H^1(S^1, G) \,|\, g(t) = \sigma(g(-2\pi/k + t))\}.$$

4.11 Proposition ([T2]). *Let σ be an order k automorphism of G, $\hat{\sigma}$ the automorphism on $\hat{L}(G)$ associated to σ, and $\hat{L}(G, \sigma)$ the fixed point set of $\hat{\sigma}$. Let V_0 be as in Proposition 4.10, and $V(\sigma) = V_0 \cap \hat{L}(\mathcal{G}, \sigma)$. Then* “refJc

(1) the Adjoint action of $\hat{L}(G, \sigma)$ on its Lie algebra $\hat{L}(\mathcal{G}, \sigma)$ leaves $V(\sigma)$ invariant, and hence induces an action of $L(G, \sigma)$ on $V(\sigma)$,

(2) let $H^0([0, \frac{2\pi}{k}], \mathcal{G})$ denote the Hilbert space with the inner product defined by

$$\langle u, v \rangle = k \int_0^{\frac{2\pi}{k}} (u(t), v(t)) dt,$$

then the map

$$V(\sigma) \to H^0([0, \tfrac{2\pi}{k}], \mathcal{G}), \qquad u + rc + d \mapsto u \,|\, [0, \tfrac{2\pi}{k}]$$

is an isometry,

(3) the induced action of $L(G, \sigma)$ on $V(\sigma)$ is conjugate to the $P(G, G(\sigma))$-action on $H^0([0, \frac{2\pi}{k}], \mathcal{G})$.

4.12 Remark. We saw in Corollary 4.9 that the action of $P(G, G(\tau))$ on $H^0([0,1], \mathcal{G})$ is conjugate to that of $P(G, G(\mu))$ if μ is an automorphism of G such that $[\tau]$ and $[\mu]$ are equal in $\mathrm{Aut}(G)/\mathrm{Int}(G)$. This is related to Theorem 8.5 on p.133 in [Ka] where it is proved that if σ is an automorphism of finite order, then the the affine algebra $\hat{L}(\mathcal{G}_C, \sigma)$ is isomorphic to the affine algebra $\hat{L}(\mathcal{G}_C, \mu)$ where μ is the diagram automorphism of G such that $[\sigma]$ and $[\mu]$ are equal in $\mathrm{Aut}(\mathcal{G}_C)/\mathrm{Int}(\mathcal{G}_C)$. The order k of the diagram automorphism μ coincides with the order of $[\tau] = [\sigma]$ in $\mathrm{Aut}(\mathcal{G}_C)/\mathrm{Int}(\mathcal{G}_C)$ and the type of the corresponding Kac-Moody algebra $\hat{L}(\mathcal{G}_C, \sigma)$ is k. Note that τ in Corollary 4.9 (ii) does not have to be of finite order. “refCb

4.13 Proposition ([T2]). *Let ρ be an involution of $\mathcal{G}$, ρ^* the induced involution on $\hat{L}(\mathcal{G})$, and $\hat{L}(\mathcal{G}) = \hat{\mathcal{K}} + \hat{\mathcal{P}}$ the decomposition of $\hat{L}(\mathcal{G})$ into ± 1-eigenspaces of ρ^*, i.e.,* “refJk

$$\mathcal{K} = \{u \in L(\mathcal{G}) \mid \rho(u(-t)) = u(t)\},$$
$$\mathcal{P} = \{u \in L(\mathcal{G}) \mid \rho(u(-t)) = -u(t)\} \ + Rc + Rd.$$

Then

(1) the Adjoint action of $\hat{K}$ leaves $V_0 \cap \hat{\mathcal{P}}$ invariant, where V_0 is as in Proposition 4.10,

(2) the map $u + rc + d \mapsto u \,|\, [0, \pi]$ is an isometry from $V_0 \cap \hat{\mathcal{P}}$ to $H^0([0,\pi], \mathcal{G})$, where $H^0([0,\pi], \mathcal{G})$ is equipped with the inner product

$$\langle u, v\rangle = 2\int_0^\pi (u(t), v(t))\, dt,$$

(3) the Adjoint action of $\hat{K}$ on $V_0 \cap \hat{\mathcal{P}}$ is conjugate to the action of $P(G, K \times K)$ on $H^0([0,\pi], \mathcal{G})$.

Next we want to show that the third family of examples in Corollary 4.7 also arises from some involutions of affine algebras.

4.14 Proposition. *Let σ be an automorphism of order k and ρ an involution on G. Suppose the automorphism $\varphi = \sigma\rho$ is an involution. Then the restriction of ρ^* to $\hat{L}(G, \sigma)$ gives an involution on $\hat{L}(G, \sigma)$. Moreover,* “refJg

(i) the fixed point set $\tilde{K}$ of ρ^* on $\hat{L}(G, \sigma)$ is

$$\tilde{K} = \{g \in H^1(S^1, G) \mid g(-t) = \rho(g(t)),\ \sigma(g(-\tfrac{2\pi}{k} + t)) = g(t)\},$$

which is isomorphic to the Hilbert Lie group $P(G, K_\rho \times K_\varphi)$,

(ii) the Adjoint action of $\tilde{K}$ leaves $V(\sigma) = V \cap \hat{L}(\mathcal{G}, \sigma)$ invariant, where V is as in Proposition 4.10,

(iii) the $\mathrm{Ad}(\tilde{K})$ action on $V(\sigma)$ is conjugate to the action of $P(G, K_\rho \times K_\varphi)$ on $H^0([0, \frac{\pi}{k}], \mathcal{G})$.

PROOF. Let $g \in \tilde{K}$. Then $g(0) = \rho(g(0))$ implies that $g(0) \in K_\rho$; and

$$g(\pi/k) = \sigma(g(-\pi/k)) = \sigma(\rho(g(\pi/k))) = \varphi(g(\pi/k)),$$

where the last equality is true because $\varphi = \sigma\rho$. So $g(\pi/k) \in K_\varphi$. Next we claim that for $g \in \tilde{K}$, g is determined by $g\,|\,[0, \pi/k]$. To see this, for $t \in [0, \pi/k]$ we have

$$g(\tfrac{\pi}{k} + t) = \sigma(g(-\tfrac{\pi}{k} + t)) = \sigma(\rho(g(\tfrac{\pi}{k} - t)).$$

This proves (i). Statements (ii) and (iii) follow. ∎

4.15 Remark. Let φ, ρ be two involutions on G such that $\sigma = \varphi\rho$ has finite order. Then Proposition 4.14 implies that the $P(G, K_\rho \times K_\varphi)$-action on $H^0([0,1], \mathcal{G})$ is essentially the Adjoint action of $\tilde{K}$ on $\hat{\mathcal{P}}$, where $\tilde{K}$ is the fixed point set of the involution ρ^* on $\hat{L}(G, \sigma)$ and $\hat{\mathcal{P}}$ is the (-1)-eigenspace of ρ^* on $\hat{L}(\mathcal{G}, \sigma)$. In other words, the $P(G, K_\rho \times K_\varphi)$-action is the isotropy representation of the "symmetric space" $\hat{L}(G, \sigma)/\tilde{K}$. “refKe

4.16 Proposition. *Let G be a compact, simple Lie group, σ_1, σ_2 two involutions on G, and K_1, K_2 the corresponding fixed point sets. If the action of $K_1 \times K_2$ on G is not transtive, then there exists $x \in G$ such that the composition of the involutions $\mathrm{Ad}(x)\sigma_1 \mathrm{Ad}(x)^{-1}$ and σ_2 has finite order.* “refCc

PROOF. It is proved in [Co2] that there exists $x \in G$ such that the two involutions $\sigma_1^x = \mathrm{Ad}(x)\sigma_1 \mathrm{Ad}(x)^{-1}$ and σ_2 commute except when $(\mathcal{G}, \mathcal{K}_1, \mathcal{K}_2)$ is one of the following three triples:

$$\begin{cases} (su(2n), sp(n), su(2q-1) \times su(2(n-q)+1)) \\ (so(2n), su(n), so(2q+1) \times so(2(n-q)-1) \\ (so(8), so(3) \times so(5), \omega(so(3) \times so(5))), \end{cases}$$

where ω is the triality automorphism of $so(8)$. In particular, this says that the order of $\sigma_1^x \circ \sigma_2$ is equal to 2 if $(\mathcal{G}, \mathcal{K}_1, \mathcal{K}_2)$ is not one the above three triples. But for the above three triples, it is easy to check that the order of $\sigma_1 \circ \sigma_2$ is equal to $4, 4$ and 3 respectively. ∎

Recall that in the Introduction, we call the action of $L(G, \sigma)$ on $V(\sigma)$ given in Proposition 4.11 an affine s-representation of the first kind, and the action of $\hat{K}$ on $V(\sigma)$ in Proposition 4.14 an affine s-representation of the second kind. Since $\sigma_1^x = \mathrm{Ad}(x)\sigma_1 \mathrm{Ad}(x)^{-1} = \mathrm{Ad}(x\sigma_1(x)^{-1}) \circ \sigma_1$, $[\sigma_1^x] = [\sigma_1]$ in $\mathrm{Aut}(\mathcal{G})/\mathrm{Int}(\mathcal{G})$. As consequence of Corollary 4.9, Proposition 4.14 and 4.16, we have

4.17 Corollary. *Let G be a simply connected, compact, simple Lie group, and K_1, K_2 symmetric subgroups of G. If the action of $K_1 \times K_2$ is not transtive on G, then the action of $P(G, K_1 \times K_2)$ on $H^0([0,1], \mathcal{G})$ is w-equivalent to an affine s-representation of the second kind.* “refCd

5. Open problems

The classification of hyperpolar actions on symmetric spaces and on Hilbert space remains unsolved. In this section, we will give some open problems, which we think are needed for achieving such classifications. In order to explain our approach, we first give an outline of a proof different from the one given by Dadok for Theorem 1.11. It is not difficult to see that if $\rho : G \to SO(n)$ is a decomposible polar representation then there exist polar representations $\rho_1 : G_1 \to SO(V_1)$ and $\rho_2 : G_2 \to SO(V_2)$ such that the G-action on R^n is ω-equivalent to the product action of $G_1 \times G_2$ on $V_1 \times V_2$. So the classification of polar representations reduces to that of irreducible ones. By Theorem 3.12 the classification of irreducible polar representations can be carried out by checking whether the action of H on S^{n-1} is polar for a closed subgroup H of $SO(n)$, where H is one of the following cases:

(i) H is a maximal subgroup of $SO(n)$,

(ii) H is a maximal subgroup of a closed subgroup G on $SO(n)$, where G acts on S^{n-1} transtively.

This can be done because the H's in (i) and (ii) are known (cf. [Bor1,2], [MS], [Dy], [Kr]).

Since we have a maximality result similar to Theorem 3.12 for hyperpolar actions on symmetric spaces, it is a natural to study the problem of decomposablity of a hyperpolar action. To explain this notion, we make the following definition:

5.1 Definition. An isometric action of G on M is called *decomposable* if there exist a Riemannian G_1-manifold M_1 and G_2-manifold M_2 such that the action of G on M is ω-equivalent to the product action of $G_1 \times G_2$ on $M_1 \times M_2$; and say that the action on M decomposes as the product of two isometric actions. “refEa

5.2 Decomposition Conjecture. Let G be a compact, simply connected Lie group, and H a closed subgroup of $G \times G$. Suppose the action of H on G is hyperpolar, A is a section through e, and $W(A) = N(A)/Z(A)$ is the generalized Weyl group associated to the H-action, where $N(A)$ and $Z(A)$ are the normalizer and centralizer of A in H respectively. Then the following statements are equivalent: “refEb

(a) the H-action on G decomposes as product of two isometric actions,

(b) the H-action on G decomposes as the product of two hyperpolar actions,

(c) the action of $W(A)$ on A is decomposable.

The above conjecture is true for the following two families of hyperpolar actions:

(i) $H = G(\sigma)$ for some automorphism σ of G. By Theorem 3.9, we may assume that s is a diagram automorphism of G. Then this conjecture is true because of the decomposability of affine Kac-Moody algebras (cf. [Ka]).

(ii) $H = K \times K$, where K is a closed subgroup of G. This is proved in [HPTT].

5.3 Conjecture. If the action of H on G is hyperpolar and indecomposible, then G is simple. “refEc

If the above two conjectures are true, then the classification of hyperpolar actions on a compact, simply connected, simple Lie group G is reduced to the classification of indecomposable ones. A closed subgroup H of the isometry group of a symmetric space X is called *transitive* if the action of H on X is transitive. It follows from Corollary 3.14 that every indecomposable hyperpolar action on the symmetric space G/K is ω-equivalent to one of the following:

(i) H_1-action on G/K, where H_1 is a maximal subgroup of some transitive, maximal subgroup H of G and the H_1-action on G is hyperpolar,

(ii) H-action on G/K, where H is maximal in G and the H-action on G/K is not transtive but is hyperpolar.

Since the classification of maximal subgroups of G is known (cf. [Dy], [Kr]), the classification of indecomposable hyperpolar actions on the symmetric space G/K can be achieved following the above two steps.

Similarly, every indecomposable hyperpolar action on G is ω-equivalent to one of the following:

(i) H_1-action on G, where H_1 is a maximal subgroup of some transitive, maximal subgroup H of $G \times G$ and the H_1-action on G is hyperpolar,

(ii) H-action on G, where H is maximal in $G \times G$ and the H-action on G is not transtive but is hyperpolar.

So in order to classify hyperpolar actions on G, we pose the following two problems:

(a) The classification of all maximal subgroups of $G \times G$.

(b) The classification of all maximal subgroups of a transitive sugroup of $G \times G$.

Since the geometry of the orbit foliation of a variationally complete actions is very similar to that of a hyperpolar action on a symmetric space, the following is a natural question:

5.4 Question. Is a variationally complete action on a compact symmetric space hyperpolar? “refEe

We suspect there is also a decomposition theorem for polar actions on a Hilbert space V. In fact, we think a more general conjecture should be true. To explain this, we recall that a submanifold M of V is *isoparametric* if the normal bundle is globally flat, the restriction of η to any finite disk normal bundle is proper and Fredholm, and the shape operators along any parallel normal field are conjugate, where $\eta : \nu(M) \to V$ is the end point map defined by $\eta(v) = x + v$ if $v \in \nu(M)_x$. It is known that the principal orbits of polar action on Hilbert space are isoparametric ([T2]). We have also associated to each isoparametric submanifold M an affine Coxeter group W such that W acts on each normal plane $x_0 + \nu(M)_{x_0}$ isometrically and the action is equivariant under the normal parallel translation. An isoparametric submanifold M of V is called *decomposable* if there exist two closed subspaces V_1, V_2 of V and isoparametric submanifolds M_i in V_i for $i = 1, 2$ such that $V = V_1 + V_2$ and M is equal to the product of $M_1 \times M_2$. Otherwise, M is called *indecomposable*.

5.5 Conjecture. If M is an isoparametric submanifold of a Hilbert space V, “refEd then M is indecomposable if and only if the affine Coxeter group associated to M is irreducible.

In particular, this will give the complete decomposition theorem, i.e., every isoparametric submanifold in V can be written as direct sum of irreducible ones.

Note that if the action of H on G is hyperpolar then the action of the path group $P(G, H)$ on the Hilbert space $H^0([0,1], \mathcal{G})$ via gauge transformation is polar. Moreover, a principal orbit of the $P(G, H)$-action is decomposable as an isoparametric submanifold if and only if the $P(G, H)$-action is decomposable, which is also equivalent to the condition that the H-action on G is decomposable. So Conjecture 5.5 implies Conjecture 5.2.

References

[BR] Bausch, J. and Rousseau, G., *Involutions de prèmiere espèce des algèbres affines* . Revue de l'Institut Elie Cartan, no. 11, Nancy, 1989.

[Bor1] Borel, A., Some remarks about Lie groups transitive on spheres and tori , *Bull. Amer. Math. Soc.* **55** (1949), 580–587.

[Bor2] Borel, A., Le plan projectif des octaves et les sphères comme espaces homogènes, *C.R. Acad. Sc. Paris* **230** (1950), 1378–1380.

[B] Bott, R., An application of the Morse theory to compact Lie groups, *Bulletin de la Société Mathématique de France* **84** (1956), 251–282.

[BS] Bott, R and Samelson, H., Applications of the theory of Morse to symmetric spaces, *Amer. J. Math.* **80** (1958), 964–1029.

[Co 1] Conlon L., The topology of certain spaces of paths on a compact symmetric space, *Trans. Amer. Math. Soc.* **112** (1964), 228–248.

[Co 2] Conlon L., Remarks on commuting involutions , *Bull. Amer. Math. Soc.* **22** (1969), 255–257.

[Co 3] Conlon L., Variational completeness and K-transversal domains , *J. Differential Geometry* **5** (1971), 135–147.

[Co 4] Conlon L., A class of variationally complete representations , *J. Differential Geometry* **7** (1972), 149–160.

[Da] Dadok J., Polar coordinates induced by actions of compact Lie groups, *Trans. Amer. Math. Soc.* **288** (1985), 125–137.

[Dy] Dynkin, E.B., The maximal subgroups of the classical groups , *AMS Translation from Trudy Moskov. Mat.* **6** (1957), 245–378.

[GR] Garland, H. and Raghunathan, M.S., A Bruhat decomposition for the loop space of a compact group: A new approach to results of Bott, *Proc. Nat. Acad. Sci. USA* **72** (1975), 4716–4717.

[HOT] Heintze, E., Olmos, C. and Thorbergsson, G., Submanifolds with constant principal curvatures and normal holonomy groups, *International J. Math.* **2** (1991), 167–175.

[HPPT] Heintze, E., Palais, R.S., Terng, C.L. and Thorbergsson, G., Hyperpolar Actions and k-flat Homogeneous Spaces, to appear in Crelle

[He] Helgason, S., *Differential Geometry, Lie Groups, and Symmetric Spaces* . Academic Press, NY, 1978.

[Hr 1] Hermann, R., Variational completeness for compact symmetric spaces, *Proc. Amer. Math. Soc.* **11** (1960), 544–546.

[Hr 2] Hermann, R., Totally geodesic orbits of groups of isometries, *Nederl. Akad. Wetensch.* **65** (1962), 291-298.

[Ka] Kac, V.G., *Infinite dimensional Lie algebras, Third Edition.* Cambridge University Press, Cambridge, 1990.

[Kr] Krämer, Eine Klassifikation bestimmter Untergruppen kompakter zusammenhängender Liegruppen , *Comm. in Algebra* **3** (1975), 691–737.

[Le] Levstein, F., A classification of involutive automorphisms of an affine Kac-Moody Lie algebra, *J. Algebra* **114** (1988), 489–518.

[Lo] Loos, O., *Symmetric Spaces, I,II*. Benjamin Inc., New York, 1969.

[MS] Montgomery, D. and Samelson, H., Transformation groups on spheres , *Annals of Math.* **44** (1943), 454–470.

[PT1] Palais, R.S. and Terng, C.L., A general theory of canonical forms, *Trans. Amer. Math. Soc.* **300** (1987), 771–789.

[PT2] Palais R.S., and Terng C.L., *Critical Point Theory and Submanifold Geometry.* Lecture Notes in Math., vol. 1353, Springer-Verlag, Berlin and New York, 1988.

[PS] Pressley, A. and Segal, G., *Loop Groups* . Clarendon Press, Oxford, 1986.

[Se] Segal, G., The representation ring of a compact Lie group, *Publ. Math. IHES* **34** (1968), 13–128.

[T1] Terng, C.L., Variational completeness and infinite dimensional goemtry, *Proceedings of Leeds Conference, Geometry and Topology of submanifolds* **3** (1990), 279–293.

[T2] Terng, C.L., Polar Actions on Hilbert Space, to appear in Journal of Geometric Analysis

[TT] Terng, C.L. and Thorbergsson, G., Equifocal submanifolds in Symmetric spaces, preprint

Ernst Heintze
Institut für Mathematik
Universität Augsburg
Universitätsstraße 14
D-86159 Augsburg, Germany
email: heintze@uni-augsburg.de

Richard S. Palais
Department of Mathematics
Brandeis University
Waltham, MA 02254-9110
email: palais@math.brandeis.edu

Chuu-lian Terng
Department of Mathematics
Northeastern University
Boston, MA 02115
email: terng@neu.edu

Gudlaugur Thorbergsson
Mathematisches Institut
Universität Köln
Weyertal 86-90
D-50931 Köln, Germany
email: gthorbergsson@mi.uni-koeln.de