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1. Introduction. Let G be a Lie group and M a differentiable (*i.e.* C^∞) manifold. An *action* of G on M is a homomorphism $\varphi : g \to \varphi_g$ of G into the group of diffeomorphisms (*i.e.* non-singular, differentiable homeomorphisms) of M with itself such that the map $\Phi : (g, x) \to \varphi_g(x)$ of $G \times M$ into M is continuous (it then follows from a theorem of BOCHNER & MONTGOMERY [1] that Φ is automatically differentiable). A triple (G, M, φ) of such objects as those above we shall call a differentiable G-transformation group. If O is an open submanifold of M invariant under each φ_g then $g \to \varphi_g|O$ is an action of G on O and we will denote by (G, O, φ) the corresponding differentiable G-transformation group. If (G, M, φ) and (G, N, ψ) are two differentiable G-transformation groups then an *imbedding of* (G, M, φ) *in* (G, N, ψ) is an imbedding f of M into N such that $f \circ \varphi_g = \psi_g \circ f$ for all $g \in G$.

A particular class of differentiable G-transformation groups consists of the finite dimensional orthogonal representations of G, *i.e.* triples (G, M, φ) where V is a finite dimensional real Hilbert space and each φ_g is an orthogonal transformation of V. We shall prove the following theorem, and in fact something slightly more general.

Theorem. *A differentiable G-transformation group* (G, M, φ) *can be imbedded in a finite dimensional orthogonal representation of G provided G and M are compact.*

*After this work was completed, the author was informed of a more general theorem of G. D. MOSTOW, in which differentiability and compactness of M are not assumed. While MOSTOW's theorem is deeper and more difficult to prove, the simplicity gained by assuming differentiability seems to justify the publication of this paper. (Added in proof: MOSTOW's paper, which appeared under the title "Equivariant imbeddings in Euclidean space" in the May, 1957 issue of the *Annals of Mathematics*, contains a special proof for the differentiable case which, though very different from ours, is quite short.)

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In [2], page 215, E. CARTAN has proved this for the case that M is a compact, irreducible, symmetric Riemannian manifold and G acts transitively and isometrically on M.

2. The fundamental imbedding theorem. Let M be a Riemannian manifold, M_p the tangent space to M at $p \varepsilon M$, and $T(M) = \bigcup_{p\varepsilon M} M_p$ the tangent bundle of M with its usual manifold structure. Let D be the set of $v \varepsilon T(M)$ such that if $v \varepsilon M_p$ then the geodesic starting from p in the direction v can be extended to have length $||v||$ and let $\exp(v)$ be the point on this geodesic cutting off a length $||v||$. Then, as is well known, D is an open submanifold of $T(M)$ containing the zero vector field and exp is a differentiable map of D into M. If g is an isometry of M then clearly $\delta g(D) = D$ and, for each $v \varepsilon D, g(\exp v) = \exp \delta g(v)$, where δg denotes the differential of g.

Let Σ be a submanifold of M, Σ_p the tangent space to Σ at $p \varepsilon \Sigma$, and $\Sigma_p^\perp$ the orthogonal complement of Σ_p in M_p. Then $N(\Sigma) = \bigcup_{p\varepsilon\Sigma} \Sigma_p^\perp$, the *normal bundle of* Σ, is a submanifold of $T(M)$ of the same dimension as M. If $p \varepsilon \Sigma$ and 0_p is the zero vector at p then the restriction of exp to $N(\Sigma) \cap D$ is easily seen to have a non-zero differential at 0_p, and so by the implicit function theorem exp maps a neighborhood of 0_p in $N(\Sigma)$ diffeomorphically into M. If Σ is compact even more is true, well known, and rather easily demonstrated, namely Lemma 1 below.

Definition. If Σ is a submanifold of the Riemannian manifold M then we let

$$N(\Sigma, \epsilon) = \{v \varepsilon N(\Sigma) : ||v|| < \epsilon\}$$

and

$$S(\Sigma, \epsilon) = \{p \varepsilon M : \rho(p, \Sigma) < \epsilon\}$$

where ρ is the Riemannian metric function.

Lemma 1. *If Σ is a compact submanifold of the Riemannian manifold M then for some $\epsilon \geqq 0$ $\exp$ maps $N(\Sigma, \epsilon)$ diffeomorphically onto $S(\Sigma, \epsilon)$.*

Now let G be a compact Lie group and let φ be an action of G on M such that each φ_g is an isometry of M. Let Σ be an orbit of M under G, *i.e.* Σ is of the form $G(p) = \{\varphi_g(p) : g \varepsilon G\}$ for some $p \varepsilon M$. Then, as is well known, Σ is a compact differentiable manifold, and in fact if $G_p = \{g \varepsilon G : \varphi_g(p) = p\}$ is the isotropy group at p then $gG_p \to \varphi_g(p)$ is a differentiable imbedding of G/G_p into M and onto Σ. The mapping $g \to (\delta\varphi_g)_p$ is an orthogonal representation of G_p in M_p. Clearly Σ_p is an invariant subspace of M_p under this representation, hence so also is $\Sigma_p^\perp$.

Definition. We denote by $U^{(\varphi,p)}$ the representation of G_p in $\Sigma_p^\perp$ defined by $U_g^{(\varphi,p)}v = \delta\varphi_g(v)$.

We now come to the fundamental imbedding theorem.

Theorem I. *Let M and N be Riemannian manifolds, G a compact group, and let φ and ψ be actions of G on M and N respectively such that each φ_g and each ψ_g is an isometry. Suppose $p \in M$ and $q \in N$ are such that $G_p = G_q$ and $U^{(\varphi,p)}$ is equivalent to a subrepresentation of $U^{(\varphi,q)}$. Then, letting Σ be the orbit of p in M under G and $\tilde{\Sigma}$ the orbit of q in N under G, there is an $\epsilon > 0$ such that $(G, S(\Sigma, \epsilon), \varphi)$ can be imbedded in $(G, S(\tilde{\Sigma}, \epsilon), \psi)$.*

Proof. Note that since each φ_g is an isometry and Σ is invariant under each φ_g it follows that $S(\Sigma, \epsilon)$ is invariant under each φ_g for any $\epsilon > 0$, so $(G, S(\Sigma, \epsilon), \varphi)$ makes sense.

We define a map F of $N(\Sigma)$ into $N(\tilde{\Sigma})$ which we will show to have the following two properties: (1) $F \circ \delta\varphi_g = \delta\psi_g \circ F$ for all $g \in G$ and (2) F is a differentiable imbedding of $N(\Sigma)$ in $N(\tilde{\Sigma})$. Let T set up an equivalence of $U^{(\varphi,p)}$ with a subrepresentation of $U^{(\psi,p)}$, *i.e.* T is an isometric linear map of $\Sigma_p^\perp$ into $\Sigma_q^\perp$ such that

$$T = U_g^{(\psi,q)} T U_{g^{-1}}^{(\varphi,p)}$$

for all $g \in G$. We define F on $\Sigma_{\varphi_g(p)}^\perp$ to be the one-to-one linear map into $\tilde{\Sigma}_{\psi_g(p)}^\perp$ given by $\delta\psi_g T \delta\varphi_{g^{-1}}$. If $\varphi_g(p) = \varphi_{\bar{g}}(p)$ then $h = g^{-1}\bar{g} \in G_p = G_q$ so

$$\delta\psi_{\bar{g}} T \delta\varphi_{\bar{g}^{-1}} = \delta\psi_g U_h^{(\psi,p)} T U_{h^{-1}}^{(\varphi,p)} \delta\varphi_{g^{-1}} = \delta\psi_g T \delta\varphi_g$$

so that F is well defined. Since F is a one-to-one map from $\Sigma_{\varphi_g(p)}^\perp$ to $\tilde{\Sigma}_{\psi_g(q)}^\perp$ and since $\psi_g(q) = \psi_g(q)$ if and only if $\varphi_g(p) = \varphi_g(p)$ (because $G_p = G_q$), it follows that F is one-to-one on all of $N(\Sigma)$. Since $\delta\varphi_h$ maps $\Sigma_{\varphi_g(p)}^\perp$ onto $\Sigma_{\varphi_{hg}(p)}^\perp$ it follows that

$$F \circ \delta\varphi_h = (\delta\psi_{hg} T \delta\varphi_{(hg)^{-1}}) \delta\varphi_h = \delta\psi_h \circ F$$

which proves (1). It remains to show that F is differentiable and non-singular and therefore (since it is one-to-one) an imbedding. By the homogeneity property (1) it suffices to show that F is non-singular and differentiable on a set of the form $N(U) = \bigcup_{u \in U} \Sigma_u^\perp$ where U is a neighborhood of p in Σ. Since G acts transitively on Σ the map $g \to \varphi_g(p)$ of G onto Σ is a fiber map, equivalent in fact to the natural map of G onto G/G_p ; hence we can find a non-singular, differentiable, local cross-section t defined on a neighborhood U of p in Σ. Now $K : (u, v) \to \delta\varphi_{t(u)}(v)$ is clearly a diffeomorphism of $U \times \Sigma_p^\perp$ with $N(U)$ and $\tilde{F} : (u, v) \to \delta\psi_{t(u)}(Tv)$ is readily seen to be a differentiable, non-singular map of $U \times \Sigma_p^\perp$ into $N(\Sigma)$. Since clearly $F = \tilde{F} \circ K^{-1}$ on $N(U)$ we have the desired result.

Now using Lemma 1 choose ϵ so small that exp maps $N(\Sigma, \epsilon)$ diffeomorphically on $S(\Sigma, \epsilon)$ and $N(\tilde{\Sigma}, \epsilon)$ diffeomorphically on $S(\tilde{\Sigma}, \epsilon)$ and let $f = \exp \circ F \circ \exp^{-1}$ where $\exp^{-1}$ is the inverse of the restriction of exp to $N(\Sigma, \epsilon)$. Then clearly f is an imbedding of $S(\Sigma, \epsilon)$ in $S(\tilde{\Sigma}, \epsilon)$. Moreover using property (1) of F and the fact that $\exp \circ \delta\varphi_g = \varphi_g \circ \exp$ and $\exp \circ \delta\psi_g = \psi_g \circ \exp$ we get easily that $f \circ \varphi_g = \psi_g \circ f$ so that f is an imbedding of $(G, S(\Sigma, \epsilon), \varphi)$ in $(G, S(\tilde{\Sigma}, \epsilon), \psi)$ as was to be proved.

3. Imbedding in orthogonal representations.

Lemma a. *Let G be a compact Lie group and H a closed subgroup of G. There is a finite dimensional orthogonal representation θ of G in a space W and $w \in W$ such that*

$$H = \{g \in G : \theta_g(w) = w\}.$$

Proof. Let R be the real regular representation of G and let $\bar{f}$ be a continuous real function of G/H which takes on the value 1 only at H. Define f on G by $f(g) = \bar{f}(gH)$. Then it is trivial to verify that $H = \{g \in G : R_g(f) = f\}$. Let $L^2(G) = \bigoplus V_i$ be the decomposition of $L^2(G)$ into finite dimensional invariant subspaces irreducible under R and let f_i be the projection of f on V_i and

$$H_i = \{g \in G : R_g(f_i) = f_i\}.$$

Clearly H_i is a closed subgroup of G including H and $\bigcap H_i = H$. Now the closed subgroups of a compact Lie group satisfy the descending chain condition (at each step in a descending chain either the dimension or number of components must drop) so we can find $i_1, \cdots, i_n$ such that $H = \bigcap_j H_{i_j}$. Then let $W = \bigoplus_j V_{i_j}$, $w = \Sigma_j f_{i_j}$, and let θ be the restriction of R to W.

Lemma b. *Let G be a compact group, H a closed subgroup of G, and σ a finite dimensional unitary representation of H. Then there is a finite dimensional unitary representation π of G whose restriction to H contains σ as a subrepresentation.*

Proof. We can clearly assume that σ is irreducible, in which case the lemma is an immediate consequence of the Frobenius reciprocity theorem for induced representations of compact groups. See the italicized remark at the bottom of page 83 of [3].

Lemma c. *Let G be a compact group, H a closed subgroup of G, and U a finite dimensional orthogonal representation of H. Then there is a finite dimensional orthogonal representation δ of G whose restriction to H contains U as a subrepresentation.*

Proof. Let $\bar{U}$ be the complexification of U. When the space of $\bar{U}$ is regarded as a real vector space, $\bar{U}$ becomes an orthogonal representation containing U. By Lemma b we can find a finite dimensional unitary representation π of G whose restriction to H contains $\bar{U}$. Let δ be the orthogonal representation of G that π becomes when the space of π is regarded as a real vector space.

Lemma d. *Let G be a compact Lie group, H a closed subgroup of G, and U a finite dimensional orthogonal representation of H. Then there is a finite dimensional orthogonal representation ψ of G in a space V and a vector $v \in V$ such that*

$$H = G_v = \{g \in G : \psi_g(v) = v\}$$

and $U^{\psi, v}$ contains a subrepresentation equivalent to U.

Proof. Let θ be the representation of Lemma a and δ the representation of Lemma c, and let $\psi = \theta \oplus \delta$ so that $V = W \oplus$ Space of δ. Then if we take $v = (w, 0)$ the conclusions are readily verified.

Theorem II. *Let G be a compact Lie group, (G, M, φ) a differentiable G-transformation group, and Σ any orbit in M. Then there is an invariant neighborhood O of Σ in M and a finite dimensional orthogonal representation ψ of G in a space V for which there exists an imbedding f of (G, O, φ) in (G, V, ψ).*

Proof. Since G is compact we can find a Riemannian structure for M relative to which each φ_g is an isometry. Let $p \in \Sigma$ and choose ψ by Lemma d where $H = G_p$ and $U = U^{(\epsilon, p)}$. By Theorem I for some $\epsilon > 0$ we can find an imbedding f of $(G, S(\Sigma, \epsilon), \varphi)$ in $(G, S(\tilde{\Sigma}, \epsilon), \psi)$ where $\tilde{\Sigma}$ is the orbit of v. Now $O = S(\Sigma, \epsilon)$ is an invariant neighborhood of Σ in M and f is *a fortiori* an imbedding of (G, O, φ) in (G, V, ψ).

Theorem III. *Let G be a compact Lie group and let (G, M, φ) be a differentiable G-transformation group. If O is any relatively compact, open, invariant submanifold of M then there is a differentiable mapping f of M into the space V of a finite dimensional orthogonal representation ψ of G which is equivariant* (i.e. *satisfies $\varphi_g \circ f = \psi_g \circ f$ for all $g \in G$) and is such that $f \mid O$ is an imbedding of (G, O, φ) in (G, V, ψ).*

Proof. Let $O_1, \cdots, O_n$ be a covering of $\bar{O}$ by a finite number of invariant open submanifolds of M and $f_i : O_i \to V_i$ an imbedding of (G, O_i, φ) in a finite dimensional orthogonal representation (G, V_i, ψ_i) of G. The existence of such follows from Theorem II and the compactness of $\bar{O}$. Let $W_1, \cdots, W_n$ be an open covering of $\bar{O}$ with $\bar{W}_i \subset O_i$. We can assume that each W_i is invariant, otherwise replace W_i by $\{\varphi_g(w) : g \in G, w \in W_i\}$. Let g_i be a differentiable real valued function on M which is identically unity on W_i and identically zero on $M - O_i$. We can assume that each g_i is invariant under the action φ of G on M (otherwise replace $g_i(x)$ by $\int_G g_i(\varphi_g(x))\, d\mu(g)$ where μ is the normalized Haar measure). Let $V_0 = R^n$ and let ψ_0 be the identity representation of G on V_0. Define $f_0 : M \to V_0$ by $\bar{f}_0(x) = (g_1(x), \cdots, g_n(x))$, and define $\bar{f}_i : M \to V_i$ by $\bar{f}_i(x) = g_i(x) f_i(x)$ for $x \in O_i$ and $\bar{f}_i(x) =$ the zero vector of V_i for $x \in M - O_i$. Let $V = V_0 \oplus \cdots \oplus V_n$, $\psi = \psi_0 \oplus \cdots \oplus \psi_n$ and define $f : M \to V$ by $f(x) = (\bar{f}_0(x), \cdots, \bar{f}_n(x))$. Clearly f is differentiable and equivariant, and since f_i is an imbedding of O_i it follows that $\bar{f}_i$ is non-singular on W_i and hence that f is non-singular on the union of the W_i and so on O. If $x, y \in O$ and $f(x) = f(y)$ then for some i $x \in W_i$ so $g_i(x) = 1$, hence, since $\bar{f}_0(x) = \bar{f}_0(y)$, $g_i(y) = g_i(x) = 1$ so $y \in O_i$. Thus x and y both belong to O_i, where f_i is one-to-one. Moreover $f_i(x) = g_i(x) f_i(x) = \bar{f}_i(x) = \bar{f}_i(y) = g_i(y) f_i(y) = f_i(y)$ and it follows that $x = y$, so f is one-to-one on O.

We note that the theorem of the introduction is a special case of Theorem III.

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