

# THE INITIAL VALUE PROBLEM FOR WEAKLY NONLINEAR PDE

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September 2014

**ABSTRACT.** We will discuss an extension of the pseudospectral method developed by Wineberg, McGrath, Gabl, and Scott for the numerical integration of the KdV initial value problem. Our generalization of their algorithm can be used to solve initial value problems for a wide class of evolution equations that are “weakly nonlinear” in a sense that we will make precise. This class includes in particular the other classical soliton equations (SGE and NLS). As well as being very simple to implement, this method exhibits remarkable speed and stability, making it ideal for use with visualization tools where it makes it possible to experiment in real-time with soliton interactions and to see how a general solution decomposes into solitons. We will analyze the structure of the algorithm, discuss some of the reasons behind its robust numerical behavior, and finally describe a fixed point theorem we have found that proves that the pseudospectral stepping algorithm converges.

## 1. INTRODUCTION

So-called pseudospectral methods are used for the numerical integration of evolution equations, using discrete Fourier transforms instead of finite differencing to evaluate spatial derivatives. An excellent early article is [FW]. A surprising fact is that these methods often work very well for nonlinear equations. The time-stepping for pseudospectral methods is accomplished by a classical differencing scheme that can in principle be either explicit or implicit, but for the usual stability reasons, an implicit method such as Crank-Nicolson (the trapezoidal rule) is usually preferred. However, when the equation is nonlinear, the solution of the implicit equations that arise can present a problem. One approach is to employ split-stepping; use Crank-Nicolson plus Gaussian elimination for the linear terms, but fall back to an explicit method for the nonlinear terms. An alternative approach, pioneered in [WMGS] and that we will refer to as the WMGS method, is to treat the linear and nonlinear terms together, write the implicit equation in fixed-point form, and then solve it by an iteration scheme.

WGMS originally developed their method to solve the initial value problems for the KdV and KP equations with periodic boundary conditions, and we became aware of their technique via an early version of [LS], in which Yi Li and D. H. Sattinger report on a modified WGMS algorithm. In this paper, we will discuss

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1991 *Mathematics Subject Classification.* Primary 35G25 and 65M70.

*Key words and phrases.* initial value problem, pseudospectral method, weakly nonlinear PDE.

During the preparation of the initial version of this paper, the author was a member of The Institute for Advanced Study, in Princeton, New Jersey

a generalization of the WGMS algorithm to treat the initial value problem for a fairly broad class of evolutionary PDE that are “weakly nonlinear”, in the sense that their nonlinear terms are a lower order perturbation of the linear part (see below for a precise definition) and we will prove a convergence theorem for the iteration method that is at the heart of the WGMS algorithm.

## 2. WEAKLY NONLINEAR PDE OF EVOLUTION

Let  $U$  denote a finite dimensional complex inner product space and  $V$  a vector space of  $U$ -valued functions on  $\mathbf{R}$ . Usually we will work in a fixed orthonormal basis  $(e_1, \dots, e_n)$  for  $U$  and use it to identify  $U$  with  $\mathbf{C}^n$ , so that elements  $u$  of  $V$  can be considered as  $n$ -tuples  $(u^1, \dots, u^n)$  of complex-valued functions. (If  $n = 1$  we shall say we are in the *scalar case* and we then identify  $u$  with  $u^1$ .)

We will specify  $V$  more precisely later, but the elements of  $V$  will admit derivatives up to a certain order, and they will in most cases be required to be  $2\pi$ -periodic, in which case we shall also consider them as functions on the unit circle in the complex plane. If  $u(t)$  is a curve in  $V$  we will also write  $u(x, t)$  for  $u(t)(x)$ . As usual we think of  $t$  as denoting time and  $x$  as space. We denote by  $D$  the differentiation operator  $\frac{\partial}{\partial x}$  and we also write  $u_x^i = Du^i$ ,  $u_{xx}^i = D^2u^i$ , etc., and of course  $u_t = \frac{\partial u}{\partial t}$ .

We will be considering “evolution equations” of the form  $u_t = F(u)$ , where  $F : V \rightarrow V$  should be thought of as a vector field on  $V$ , and its form will be a smooth function (usually polynomial) of the  $u^i$  and their derivatives,  $Du^j, D^2u^k, \dots$ . Usually  $F(u)$  will be the sum of a “dominant” linear differential operator, and a nonlinear part that we can consider as a “small perturbation” of this linear part. By a linear differential operator on  $V$  we will always mean an operator of the form  $u \mapsto \mathcal{L}(u) = (\mathcal{L}^1(u), \dots, \mathcal{L}^n(u))$  where  $\mathcal{L}^i(u) = \sum_{j=1}^n \mathcal{L}_j^i(D)u^j$ . Here each  $\mathcal{L}_j^i(X)$  is a polynomial with constant coefficients in an indeterminate  $X$ . In the scalar case  $\mathcal{L}(u) = \mathcal{L}(D)u$  and we will often use  $\mathcal{L}(D)u$  to denote  $\mathcal{L}(u)$  in the general case too.

The simplest kind of nonlinear operator that we shall consider is a zero order nonlinear operator, by which we will mean a map of the form  $u \mapsto G(u) = (G^1(u), \dots, G^n(u))$ , where  $G^i(u)(x) = G^i(u^1(x), \dots, u^n(x))$  and  $G^i(Y_1, \dots, Y_n)$  is either a constant coefficient polynomial on  $\mathbf{C}^n$  or more generally an entire function of these variables (i.e., given by a power series that converges for all values of  $(Y_1, \dots, Y_n)$ ). Of course, care must be taken to make sure that if  $u \in V$  then also  $G(u) \in V$ . When we come to the rigorous proofs, we will assume that  $V$  is one of the Sobolev Hilbert spaces  $H^m(\mathbf{R}, U)$  for  $m > \frac{1}{2}$ , and since it is well-known that  $H^m(\mathbf{R}, \mathbf{C})$  is a Banach algebras, it follows easily that  $G$  is a smooth map of  $H^m(\mathbf{R}, U)$  to itself. The most general kind of nonlinearity that we will consider will be one that can be factored into a composition of the form  $M(D)G(u)$  where  $M(D)$  is a linear differential operator as above and  $G(u)$  is a zero order nonlinearity.

If  $L(X) = \sum_{m=1}^{\ell} a_m X^m$  is a complex polynomial, then the differential operator  $L(D)$  is called *formally skew-adjoint* if  $\langle L(D)u_1, u_2 \rangle = -\langle u_1, L(D)u_2 \rangle$  whenever  $u_1$  and  $u_2$  are smooth maps of  $\mathbf{R}$  into  $U$  with compact support. Here  $\langle u, v \rangle$  denotes the  $L^2$  inner product, i.e.,  $\langle u, v \rangle := \int_{-\infty}^{\infty} \langle u(x), v(x) \rangle dx$ . Integration by parts shows that  $D$  is skew-adjoint. Moreover an odd power of a formally skew-adjoint operator (and  $i$  times an even power) is clearly again formally skew-adjoint, so it follows that  $L(D)$  is formally skew-adjoint if and only if the coefficients  $a_m$  are real for  $m$  odd

and imaginary for  $m$  even, i.e., if and only if  $L(ik)$  is imaginary for all real  $k$ , and it is this last condition that we shall use.

**Definition.** A system of partial differential equation of the form:

$$(WNWE) \quad u_t^i = L^i(D)u^i + M^i(D)G^i(u).$$

is called a *weakly nonlinear wave equation* if:

- 1) Each  $L^i(D)$  is a formally skew-adjoint operator and the polynomials  $L^i(X)$  all have the same degree,  $\ell$ ,
- 2) degree  $M^i(X) < \ell$ ,
- 3)  $G^i(0) = 0$ , so that  $u(x, t) \equiv 0$  is a solution of (WNWE).

In what follows we will denote the minimum difference,  $\ell - \text{degree } M^i(X)$ , by  $q$ . For the most part, we will be dealing with the case  $n = 1$ , in which case we put  $L = L^1$  and  $M = M_1^1$ , so  $\ell = \text{degree } L$  and  $q = \text{degree } L - \text{degree } M$ , and a weakly nonlinear wave equation has the form:

$$(WNWE) \quad u_t = L(D)u + M(D)G(u).$$

Two important examples are the Korteweg-deVries Equation:

$$(KdV) \quad u_t = -u_{xxx} - uu_x = -D^3u - \frac{1}{2}D(u^2),$$

and the Nonlinear Schrödinger Equation:

$$(NLS) \quad u_t = iu_{xx} + i|u|^2u = iD^2u + i|u|^2u.$$

In the former case we have  $L(X) = -X^3$ ,  $M(X) = -\frac{1}{2}X$ ,  $G(X) = X^2$ , and in the latter,  $L(X) = iX^2$ ,  $M(X) = i$ ,  $G(X) = |X|^2X$ . In the next section we will see that the Sine-Gordon Equation:

$$(SGE) \quad u_{tt} = u_{xx} + \sin u$$

also can be regarded as a weakly nonlinear wave equation.

### 3. THE SINE-GORDON EQUATION

A natural reduction of the linear wave equation to a system of first order PDE is  $\frac{\partial u}{\partial t} = \frac{\partial v}{\partial x}$ ,  $\frac{\partial v}{\partial t} = \frac{\partial u}{\partial x}$ . in fact,  $\frac{\partial^2 u}{\partial t^2} = \frac{\partial}{\partial t} \frac{\partial}{\partial x} v = \frac{\partial}{\partial x} \frac{\partial}{\partial t} v = \frac{\partial^2}{\partial x^2} u$ .

This suggests that to find a representation of Sine-Gordon as a weakly nonlinear wave equation, we should start with systems of the form  $\frac{\partial u}{\partial t} = \frac{\partial v}{\partial x} + F(u, v)$ ,  $\frac{\partial v}{\partial t} = \frac{\partial u}{\partial x} + G(u, v)$  or  $\frac{\partial}{\partial t}(u, v) = L(u, v) + (F(u, v), G(u, v))$  where  $F$  and  $G$  are entire functions on  $\mathbf{C} \times \mathbf{C}$ , and of course  $F(0, 0) = G(0, 0) = 0$ . We will next show that with appropriate choice of  $F$  and  $G$  we do indeed get Sine-Gordon, and moreover that essentially the only other equations of the form  $u_{tt} = u_{xx} + \Gamma(u)$  that arise

in this way are the Klein-Gordon equation,  $u_{tt} = u_{xx} + u$ , and the Sinh-Gordon equation  $u_{tt} = u_{xx} + \sinh u$ .

Starting out as above,  $\frac{\partial^2 u}{\partial t^2} = \frac{\partial}{\partial t}(\frac{\partial v}{\partial x} + F(u, v)) = \frac{\partial}{\partial x} \frac{\partial v}{\partial t} + F_1 \frac{\partial u}{\partial t} + F_2 \frac{\partial v}{\partial t} = \frac{\partial^2 u}{\partial x^2} + (F_1 F + F_2 G) + \frac{\partial u}{\partial x}(G_1 + F_2) + \frac{\partial v}{\partial x}(G_2 + F_1)$ . For the latter to be of the form  $u_{tt} = u_{xx} + \Gamma(u)$  we must have  $\frac{\partial}{\partial v}(F_1 F + F_2 G) = 0$ ,  $G_1 = -F_2$ , and  $G_2 = -F_1$ , in which case  $u_{tt} = u_{xx} + \Gamma(u)$  with  $\Gamma = F_1 F + F_2 G$ .

Next note that these conditions on  $F$  and  $G$  give  $F_{11} = -G_{21} = -G_{12} = F_{22}$ , or in other words,  $F$  is a solution of the one-dimensional wave equation, and hence a sum of a left moving wave and a right-moving wave:  $F(u, v) = h(u+v) + k(u-v)$ . Then using  $G_1 = -F_2$ , and  $G_2 = -F_1$  it follows that  $G(u, v) = k(u-v) - h(u+v)$ , where  $h(0) = k(0) = 0$  in order to make  $F(0, 0) = G(0, 0) = 0$ . The condition  $\frac{\partial}{\partial v}(F_1 F + F_2 G) = 0$  now gives  $\frac{\partial}{\partial v}(h'(u+v)k(u-v) + h(u+v)k'(u-v)) = 0$  or  $h''(u+v)k(u-v) = h(u+v)k''(u-v)$ , or  $\frac{h''(u+v)}{h(u+v)} = \frac{k''(u-v)}{k(u-v)}$ . Since  $u+v$  and  $u-v$  are coordinates, the only way the last relation can hold identically is for both sides to be a constant  $\lambda$ , i.e.  $h'' = \lambda h$  and  $k'' = \lambda k$ .

If  $\lambda$  is negative, say  $\lambda = -\omega^2$ , then since  $h(0) = k(0) = 0$ , it follows that  $h(u) = A \sin(\omega u)$  and  $k(u) = B \sin(\omega u)$ . If we choose  $\omega = \frac{1}{2}$  and  $A = B = 1$  we get  $F(u, v) = \sin(\frac{u}{2} + \frac{v}{2}) + \sin(\frac{u}{2} - \frac{v}{2}) = 2 \sin \frac{u}{2} \cos \frac{v}{2}$  and similarly  $G(u, v) = -2 \cos \frac{u}{2} \sin \frac{v}{2}$ , and this gives the system of partial differential equations  $\frac{\partial u}{\partial t} = \frac{\partial v}{\partial x} + 2 \sin \frac{u}{2} \cos \frac{v}{2}$ ,  $\frac{\partial v}{\partial t} = \frac{\partial u}{\partial x} - 2 \cos \frac{u}{2} \sin \frac{v}{2}$ , and we will leave it to the reader to check that if  $(u, v)$  is a solution of this system, then  $u$  is a solution of the Sine-Gordon equation. (Other choices of  $A, B$ , and  $\omega$  lead to equations that can be transformed to the Sine-Gordon equation by a simple re-scaling of independent and dependent variables. Similarly taking  $\lambda = 0$  gives the Klein-Gordon equation, and  $\lambda$  positive gives Sinh-Gordon.)

While this system of PDE for  $u$  and  $v$  is not in the form (WNWE), if we define  $u^1 = u + v$  and  $u^2 = u - v$ , then  $u^1$  and  $u^2$  satisfy:

$$\begin{aligned} u_t^1 &= +u_x^1 + 2 \sin(\frac{u^1}{2} - \frac{u^2}{2}), \\ u_t^2 &= -u_x^2 + 2 \sin(\frac{u^1}{2} + \frac{u^2}{2}). \end{aligned}$$

which is manifestly in the form (WNWE), with  $L^1(X) = D$ ,  $L^2(X) = -D$ , and  $M^i(X) = 1$ , and moreover we can recover  $u$  from  $u^1$  and  $u^2$  by  $u = \frac{u^1 + u^2}{2}$ .

To simplify the exposition, we will from now on assume we are in the scalar case,  $n = 1$  and that  $G$  is a polynomial. The modifications needed for the general case are obvious.

#### 4. THE GENERALIZED WGMS METHOD (HEURISTICS)

Let us assume that for some particular example of (WNWE) we know that there is a unique solution  $u(t)$  with the initial condition  $u(0) \in V$ . Let  $\Delta t$  be close to zero, and let us look for a time-stepping algorithm that, given a sufficiently good approximation to  $u(t)$  as input will produce an approximation to  $u(t') = u(t + \Delta t)$  as output. If we integrate (WNWE) with respect to  $t$ , from  $t$  to  $t'$ , and use the trapezoidal rule to approximate the integrals on the right hand side, we find:

$$\begin{aligned} u(t') - u(t) &= \frac{\Delta t}{2} L(D)[u(t) + u(t')] \\ &\quad + \frac{\Delta t}{2} M(D)[G(u(t)) + G(u(t'))] \end{aligned}$$

or

$$(I - dL(D))u(t') = (I + dL(D))u(t) + dM(D)[G(u(t)) + G(u(t'))],$$

which we can rewrite as:

$$u(t') = Cu(t) + B[G(u(t)) + G(u(t'))]$$

where  $d = \frac{\Delta t}{2}$ ,  $B = \frac{dM(D)}{I - dL(D)}$ , and  $C = \frac{I + dL(D)}{I - dL(D)}$  is the Cayley transform of the skew-adjoint operator  $dL(D)$ . We note that the skew-adjointness of  $L(D)$  assures that  $I - dL(D)$  is invertible, and that  $C$  is a unitary operator. In fact, as we shall see shortly, on the Fourier transform side, both  $C$  and  $B$  become simple multiplication operators, whose properties are obvious from those of the polynomials  $L(X)$  and  $M(X)$ .

Next, for each  $u$  in  $V$ , we define a map  $H_u : V \rightarrow V$  by

$$H_u(w) := Cu + B[G(u) + G(w)],$$

and we note that the equation above becomes  $H_{u(t)}(u(t')) = u(t')$ , i.e.,  $u(t')$ , which is what we are trying to compute, is a fixed-point of  $H_{u(t)}$ .

Now, we permit ourselves a little optimism—we assume that  $u(t')$  is in fact a *contracting* fixed point of  $H_{u(t)}$ . If this is so then, for  $\Delta t$  small,  $u(t)$  will be close to  $u(t')$ , and we can expect that iterating  $H_{u(t)}$  starting at  $u(t)$ , will produce a sequence that converges to  $u(t')$ . This is the essence of the WGMS time-stepping algorithm (generalized to WNWE).

For this to work as a numerical method, we must be able to compute  $H_u$  efficiently, and that is where the Fourier Transform comes in. Let us write  $\mathcal{F}$  for the Fourier Transform, mapping  $V$  isomorphically onto  $\hat{V}$ , and  $\mathcal{IF}$  for its inverse. We define operators  $\hat{C} = \mathcal{F}C\mathcal{IF}$  and  $\hat{B} = \mathcal{F}B\mathcal{IF}$  on  $\hat{V}$ . Then  $\mathcal{F}H_u(w) = \mathcal{F}C\mathcal{IF}\mathcal{F}(u) + \mathcal{F}B\mathcal{IF}\mathcal{F}[G(u) + G(w)]$ , so we can rewrite  $H_u$  as:

$$H_u(w) = \mathcal{IF}(\hat{C}\hat{u} + \hat{B}\mathcal{F}[G(u) + G(w)]),$$

where  $\hat{u} = \mathcal{F}(u)$  is the Fourier Transform of  $u$ .

Assuming that we have a good algorithm for computing  $\mathcal{F}$  and  $\mathcal{IF}$  (e.g., the Fast Fourier Transform), it is now clear that it is easy and efficient to calculate  $H_u$ , and hence to carry out the iteration. Indeed, calculating  $G(u)$  and  $G(w)$  at a point  $x$  is just a matter of evaluating the polynomial  $G$  at  $u(x)$  and  $w(x)$ . And since  $M(X)$  and  $L(X)$  are constant coefficient polynomials, the operators  $\hat{C}$  and  $\hat{B}$  are diagonal in the Fourier basis  $e_k(x) = e^{ikx}$ , i.e., they are multiplication operators, by the rational functions  $\frac{1+dL(ik)}{1-dL(ik)}$  and  $\frac{dM(ik)}{1-dL(ik)}$  respectively. Since  $L(D)$  is by assumption skew-adjoint,  $L(ik)$  is pure imaginary, so the denominator  $1 - dL(ik)$  does not vanish. Moreover the function  $\frac{1+dL(ik)}{1-dL(ik)}$  clearly takes its values on the unit circle, and since  $L(X)$  has degree greater than  $M(X)$ , it follows that while the nonlinearity  $G(w)$  may push energy into the high frequency modes of the Fourier Transform, multiplication by  $\frac{dM(ik)}{1-dL(ik)}$  acts as a low-pass filter, attenuating these high frequency modes and giving the WGMS method excellent numerical stability.

5. PROOF THAT  $H_u$  IS A CONTRACTION

In this section we will justify the above optimism by showing that, with a proper choice of the space  $V$ , a suitable restriction of the mapping  $H_u$  does indeed satisfy the hypotheses of the Banach Contraction Theorem provided  $\|u\|$  and  $\Delta t$  are sufficiently small. The space we will choose for  $V$  is the Sobolev Hilbert space  $H^m = H^m(\mathbf{S}^1, V)$ , with  $m > \frac{1}{2}$ . We recall that this is the Hilbert space of all functions  $u$  in  $L^2(\mathbf{S}^1, V)$  such that  $\|u\|_m^2 = \sum_k (1 + k^2)^{\frac{m}{2}} |\hat{u}(k)|^2$  is finite, where as before,  $\hat{u}(k)$  are the Fourier coefficients of  $u$ .

The principal property of these spaces that we shall need is that  $H^m(\mathbf{S}^1, \mathbf{R})$  is a commutative Banach algebra under pointwise multiplication when  $m > \frac{1}{2}$  (cf. [A], Theorem 5.23, or [P]). As a first consequence, it follows that if  $P : V \rightarrow V$  is a polynomial mapping, then  $u \mapsto P(u)$  is a map of  $H^m$  to itself, and moreover  $\|P(u)\|_m < C \|u\|_m^r$ , where  $r$  is the degree of  $P$ . We will permit ourselves the abuse of notation of denoting this latter map by  $P$ , and it is now elementary to see that it is Frechet differentiable, and in fact that  $DP_u(v) = P'(u)v$ , where  $P'$  is the derivative of  $P$ . (This will follow if we can show that there is an algebraic identity of the form  $P(X + Y) = P(X) + P'(X)Y + Q(X, Y)Y^2$ , for some polynomial  $Q$  in  $X$  and  $Y$ . But it is clearly enough to check this for monomial  $P$ , in which case it is immediate from the binomial theorem.)

Let us denote by  $B_R$  the ball of radius  $R$  in  $H^m$ . Then as an immediate consequence of the preceding remarks we have:

**Proposition 1.** *For any  $R > 0$  there exist positive constants  $C_1$  and  $C_2$  such that  $\|G(u)\|_m < C_1$  and  $\|DG_u\| < C_2$  for all  $u$  in  $B_R$ .*

It will be important for us to have a good estimate of how the norm of  $B$  depends on  $\Delta t$ .

**Proposition 2.** *Given  $T > 0$ , there is a positive constant  $C_3$  such that the norm of the operator  $B$  on  $H^m$  satisfies  $\|B\| < C_3 \Delta t^{\frac{q}{2}}$ , for all  $\Delta t < T$ , where  $\ell = \text{degree}(L(X))$  and  $q = \text{degree}(L(X)) - \text{degree}(M(X))$ . Thus  $\lim_{\Delta t \rightarrow 0} \|B\| = 0$ .*

*Proof.* It is clear that the Fourier basis  $e_k(x) = e^{ikx}$  is orthogonal with respect to the  $H^m$  inner-product (though not orthonormal, except for the case  $H^0 = L^2$ ). Thus, since all constant coefficient differential operators are diagonalized in this basis, we can compute their norms on  $H^m$  by taking the maximum absolute values of their eigenvalues on the  $e_k$ . In the case of  $B$ , we have already seen that these eigenvalues are  $\frac{dM(ik)}{1-dL(ik)}$ . Since  $d = \frac{\Delta T}{2}$ , to prove the proposition it will suffice to show that  $\frac{dM(ik)}{1-dL(ik)} < C_3 d^{\frac{q}{2}}$  for all real  $k$  and all  $d < 2T$ .

Writing  $L(X) = \sum_{j=0}^{\ell} b_j X^j$  and  $M(X) = \sum_{j=0}^{\ell-q} a_j X^j$ , let us define parametric families of polynomials  $L_c$  and  $M_c$  for  $c \geq 0$  by  $L_c(X) = \sum_{j=0}^{\ell} (c^{\ell-j} b_j) X^j$  and  $M_c(X) = \sum_{j=0}^{\ell-q} (c^{\ell-q-j} a_j) X^j$ . Now note that if we define  $\delta = d^{\frac{1}{2}}$  then (since  $\delta^{\ell-j} (\delta X)^j = dX^j$ ) clearly  $L_{\delta}(\delta X) = dL(X)$ , and similarly  $M_{\delta}(\delta X) = \delta^q dM(X)$ , so  $\frac{dM(ik)}{1-dL(ik)} = \delta^q \frac{M_{\delta}(i\delta k)}{1-L_{\delta}(i\delta k)}$ , and to complete the proof it will suffice to show that the family of rational functions  $R_c(x) = \frac{M_c(ix)}{1-L_c(ix)}$  is uniformly bounded for  $0 \leq c \leq (\Delta T/2)^{\frac{1}{2}}$  and  $x$  real. If  $\tilde{\mathbf{R}}$  is the one-point compactification of  $\mathbf{R}$  and we define

$R_c(\infty) = 0$ , then since the denominator of  $R_c(X)$  never vanishes and has degree greater than the numerator, it follows that  $(c, x) \mapsto R(c, x)$  is continuous and hence bounded on the compact space  $[0, (\Delta T/2)^{\frac{1}{\ell}}] \times \tilde{\mathbf{R}}$ . ■

**Theorem.** *Given  $R > 0$  there exist positive  $r$  and  $T$  such that  $H_u$  is a contraction mapping of  $B_R$  into itself provided that  $u$  is in  $B_r$  and  $\Delta t < T$ . Moreover there is a uniform contraction constant  $K < 1$  for all such  $u$  and  $\Delta t$ .*

*Proof.* From Proposition 1 and the definition of  $H_u$  it follows that  $H_u$  is differentiable on  $H^m$  and that  $D(H_u)_v = B \circ DG_v$ . Then, again by Proposition 1,  $\|D(H_u)_v\| < C_2 \|B\|$  for all  $u$  in  $B_R$ , and so by Proposition 2,  $\|D(H_u)_v\| < C_2 C_3 \Delta t^{\frac{\ell}{\ell}}$ . Given  $K < 1$ , if we choose  $T < \left(\frac{K}{C_2 C_3}\right)^{\frac{\ell}{\ell}}$  then  $\|D(H_u)_v\| < K$  on the convex set  $B_R$  and hence  $K$  is a contraction constant for  $H_u$  on  $B_R$ , and it remains only to show that if we choose  $r$  sufficiently small, and perhaps a smaller  $T$  then  $H_u$  also maps  $B_R$  into itself for  $u$  in  $B_r$ .

But using the definition of  $H_u$  again, it follows that

$$\|H_u(w)\|_m < \|Cu\|_m + \|B\| (\|G(u)\|_m + \|G(w)\|_m),$$

and recalling that  $C$  is unitary on  $H^m$ , it follows from Propositions 1 and 2 that  $\|H_u(w)\|_m < r + 2C_1 C_3 T^{\frac{\ell}{\ell}}$ . Thus  $H_u$  will map  $B_R$  into itself provided  $r + 2C_1 C_3 T^{\frac{\ell}{\ell}} < R$ , i.e., provided  $r < R$  and  $T < \left(\frac{R-r}{2C_2 C_3}\right)^{\frac{\ell}{\ell}}$ . ■

This completes a constructive proof of short-time existence for equations of the WNWE type. We note that for the standard examples, KdV and KP, it is well-known that the solutions exist for all time, and it would be interesting to know if this is true in general, and if not to have a specific counter-example.

## 6. NUMERICS

There are several types of numerical errors inherent in the WGMS algorithm. The first and most obvious is the error in approximating the integral of the right hand side of the equation using the trapezoidal rule. A second “truncation” error occurs when we stop the fixed point iteration after a finite number of steps.

In actually implementing the WGMS algorithm to solve an initial value program numerically, one usually chooses an integer  $N$  of the form  $2^e$ , and works in the space  $V_N$  of “band-limited” functions  $u$  whose Fourier coefficients  $\hat{u}(k)$  vanish for  $|k| > N/2$ . Of course,  $V_N$  is in all the Sobolev spaces. If we start with an initial condition  $u_0$  not actually in  $V_N$  then there will be an aliasing error when the initial Fast Fourier Transform projects it into  $V_N$ . Also, since the WGMS method does not rigorously preserve  $V_N$ , there will be further such errors at each time step. It would be interesting to analyze these local errors and how they propagate in order to obtain a bound for the global error.

## REFERENCES

- [A] Adams,R.A., *Sobolev Spaces*, Academic Press, 1975.
- [FW] Forneberg,B. and Whitham, G.B., *A Numerical and Theoretical Study of Certain Non-linear Wave Phenomena*, Proc. R. Soc. Lond. A **289** (1978), 373–403.
- [LS] Li, Y. and Sattinger, D.H., *Soliton Collisions in the Ion Acoustic Plasma Equations*, J. math. fluid mech. **1** (1999), 117–130.
- [P] Palais,R.S., *Foundations of Global Non-linear Analysis*, Benjamin and Co., 1968.
- [WGSS] Wineberg, S.B., Gabl,E.F., Scott, L.R. and Southwell, C.E., *Implicit Spectral Methods for Wave Propagation Problems*, J. Comp. Physics **97** (1991), 311–336.

## 7. APPENDIX—A MATLAB IMPLEMENTATION

The following is a MATLAB M-file that implements the WGMS algorithm to solve both KdV and NLS. It should be clear how to add further case statements to the definitions of the functions Bhatfcn, G, and mfcn to extend this to handle other weakly nonlinear equations.

```
function xxx = kdv_nls;
%KdV_nls.m:Kortevge de Vries & Nonlinear Schrodinger IVP
%Pseudo-spectral integration of
%u_t + uu_x + u_xxx = 0 and
%iu_t+ u_xx + nu |u|^2 u =0
clear;
global KDVEqn NLSeqn TheEqn
global nu disp a b m k dt;
global Cu B_hat G_u;
KDVEqn = 1; NLSeqn = 2;
TheEqn = input('Enter either 1 for kdv or 2 for nls ');
%initialization of variables for the two cases.
    switch TheEqn;
case NLSeqn
    N=1024; period=2*pi; nu = 2; b = 3;
    Top = b*sqrt(2/nu)*1.1; Bottom = - Top;
    Left = - period/2; Right = period/2;
    TheEquationName = 'Non-Linear Schrodinger';
case KDVEqn
    N=512; period = 20; disp = 0.05;
    Top = 1.75; Bottom = -0.1;
    Left = - period/2; Right = period/2;
    TheEquationName = 'Korteweg de Vries';
end;
```



```

h = 2*pi/N;          % spatial increment (before scaling).
x = (-pi:h:pi - h); % unscaled space lattice
k = -i*[(0:N/2) (1-N/2:-1)]; % Fourier transform of d/dx.
a=period/(2*pi);    % spatial scale-factor
t=0;                % initial time
dt=0.01;            % time step
y=a*x;              % scaled space lattice
u = initial_condition(x); % initial condition
plohandle = plot(y,a*real(u));
set(plohandle,'erasemode','background');
axis([ Left Right Bottom Top]);
title(TheEquationName);
m = mfcn;
B_hat=bhatfcn; % Linear factor of nonlinear part of RHS
C_hat=(1+m)./(1-m); % Cayley transform of linear part of RHS
while 1;
Cu = C_hat.*fft(u);
G_u = G(u);          %The nonlinearity of the RHS applied to u
w = u;
for n=1:3;
w = Iterator(w); % =ifft(Cu + B_hat.*fft(G_u + G(w)));
end
u=w;
set(plohandle,'ydata',a*real(u)); drawnow;
t=t+dt;
end

%Subsidiary functions used in this program

function nlnr = G(q);
global KDVEqn NLSeqn TheEqn;
switch TheEqn
case NLSeqn
nlnr = q.*abs(q).^2;
case KDVEqn
nlnr = q.^2;
end;

```

```

function ic = initial_condition(x);
global KDVEqn NLSeqn TheEqn nu a b;
    switch TheEqn
case NLSeqn
    ic = sqrt(2/nu)*b*sech(b*x);
case KDVEqn
    ic = exp(-1.2*a*x.^2)/a;
end;

function bhat = bhatfcn;
global KDVEqn NLSeqn TheEqn nu disp a b m k dt;
    switch TheEqn
case NLSeqn
    bhat = 0.5*i*nu*dt./(1-m);
case KDVEqn
    bhat = 0.5*dt*k./(1 - m);
end;

function mfn = mfcn;
global KDVEqn NLSeqn TheEqn nu disp a k dt;
    switch TheEqn
case NLSeqn

    mfn = 0.5*i*dt*k.^2;
case KDVEqn
    mfn = disp*a^(-3)*0.5*dt*k.^3;
end;

function iter = Iterator(w);
global Cu B_hat G_u TheEqn;
    iter = ifft(Cu + B_hat.*fft(G_u + G(w)));

```

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