An Introduction to Wave Equations and Solitons

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1 Wave Equations

1.1 Introduction

This article is an introduction to wave equations, emphasizing for simplicity the case of a single space dimension. What we mean by a *wave equation* will gradually be made more precise as we proceed, but at first, we will just a mean a certain kind of ordinary differential equation on the space $C^\infty(\mathbb{R}^n, V)$, where V is some finite dimensional vector space, usually $\mathbb{R}$ or $\mathbb{C}$, (and generally we will take $n = 1$). Thus the wave equation will look like:

$$(*) \qquad u_t = f(u),$$

where u signifies a point of $C^\infty(\mathbb{R}^n, V)$, u_t means $\frac{du}{dt}$, and f is a special kind of map of $C^\infty(\mathbb{R}^n, V)$ to itself; namely a "partial differential operator", i.e., $f(u)(x)$ is a smooth function $F(u(x), u_{x_i}(x), u_{x_i x_j}(x), \ldots)$ of the values of u and certain of its partial derivatives at x—in fact, the function F will generally be a polynomial. A solution of $(*)$ is a smooth curve $u(t)$ in $C^\infty(\mathbb{R}^n, V)$ such that, if we write $u(t)(x) = u(x,t)$, then

$$\frac{\partial u}{\partial t}(x,t) = F\left(u(x,t), \frac{\partial u}{\partial x_i}(x,t), \frac{\partial^2 u}{\partial x_i \partial x_j}(x,t), \ldots\right).$$

We will be interested in solving the so-called "Cauchy Problem" for such partial differential equations, i.e., finding a solution, in the above sense, with $u(x,0)$ some given element $u_0(x)$ of $C^\infty(\mathbb{R}^n, V)$. So far, this should more properly be called simply an "evolution equation". In general such equations will describe evolving phenomena which are *not* wave-like in character and, as we said above, it is only after certain additional assumptions are made concerning the function F that it is appropriate to call such an evolution equation a wave equation.

An expert can often tell much about the behavior of solutions to a wave equation from just a cursory look at its structure, and our goal in the next few sections is to disclose some of the secrets that make this possible.

We will be interested of course in the obvious questions of existence, uniqueness, and general properties of solutions of the Cauchy problem, but even more it will be the nature and properties of certain special solutions that will concern us. In particular we will try to understand the mechanism behind the remarkable behavior of what are called *soliton solutions* of certain special wave equations such as the Korteweg de Vries Equation (KdV), the Sine-Gordon Equation (SGE), the Nonlinear Schrödinger Equation (NLS), and other so-called "integrable equations".

As well as first order ODE on $C^\infty(\mathbb{R}^n, V)$ we could also consider second and higher order ODE, but these can be easily reduced to first order ODE by the standard trick of adding more dependent variables. For example, to study the classic wave equation in one space dimension, $u_{tt} = c^2 u_{xx}$, a second order ODE, we can add a new independent variable v and consider instead the first order system $u_t = v$, $v_t = c^2 u_{xx}$ (which we can put in the form (*) by writing $U_t = F(U)$, with $U = (u,v)$, $F(u,v) = (v, c^2 u_{xx})$).

1.2 Travelling Waves and Plane Waves

Before discussing particular model wave equations, we will look at the kind of behavior we expect to see in solutions. There are a number of important simplifications in the description of wave propagation for the case of a single space dimension, and to develop a feeling for many of the important concepts it is best to see them first without the extra complexities that come with higher dimensions, so in what follows we will concentrate almost solely on the case $n = 1$.

Let's recall the basic intuitive idea of what is meant by "wave motion". Suppose that $u(x,t)$ represents the "strength" or "amplitude" of some scalar physical quantity at the spatial point x and time t. If you like, you can think of u as representing the height of water in a canal. Then the graph of $u^0(x) = u(x,t_0)$ gives a snapshot of u at time t_0. It is frequently the case that we can understand the evolution of u in time as representing the propagation of the shape of this graph. In other words, for t_1 close to t_0, the shape of the graph of $u^1(x) = u(x,t_1)$ near x_0 will be related in some simple way to the shape of u^0 near x_0.

Perhaps the purest form of this is exhibited by a so-called *travelling wave*. This is a function u of the form $u(x,t) = f(x - ct)$ where $f : \mathbb{R} \to V$ is a function defining the wave shape, and c is a real number defining the propagation speed of the wave. Let us define the *profile* of the wave at time t to be the graph of the function $x \mapsto u(x,t)$. Then the initial profile (at time $t = 0$) is just the graph of f, and **at any later time t, the profile at time t is obtained by translating each point $(x, f(x))$ of the initial profile ct units to the right to the point $(x + ct, f(x))$.** In other words, the wave profile of a travelling wave just propagates by rigid translation with velocity c.

(We will see below that the general solution of the equation $u_t = cu_x$ is an arbitrary travelling wave moving with velocity c, and that the general solution to the equation $u_{tt} = c^2 u_{xx}$ is the sum (or "superposition") of two arbitrary travelling waves, both moving with speed $|c|$, but in opposite directions.)

There is a special kind of complex-valued travelling wave, called a *plane wave*, that plays a fundamental rôle in the theory of linear wave equations. The general form of a plane wave is $u(x,t) = Ae^{i\phi}e^{i(kx-\omega t)}$, where A is a positive constant called the *amplitude*, $\phi \in [0, 2\pi)$ is called the *initial phase*, and k and ω are two real parameters called the *wave number* and *angular frequency*. (Note that $\frac{k}{2\pi}$ is the number of waves per unit length, while $\frac{\omega}{2\pi}$ is the number of waves per unit time.) Rewriting u in the form $u(x,t) = Ae^{i\phi}e^{ik(x-\frac{\omega}{k}t)}$, we see that it is indeed a travelling wave and that its propagation velocity is $\frac{\omega}{k}$.

In studying a wave equation, one of the first things to look for is the travelling wave solutions (if any) that it admits. For linear wave equations (with constant coefficients) we will see that for each wave number k there is a unique angular frequency $\omega(k)$ for which the equation admits a plane wave solution, and the velocity $\frac{\omega(k)}{k}$ of this plane wave as a function of k (the so-called *dispersion relation* of the equation) not only completely determines the equation, but is crucial to understanding how solutions of the equation disperse as time progresses. Moreover, the fact that there is a unique (up to a multiplicative constant) travelling wave solution $u_k(x,t) = e^{i(kx-\omega(k)t)}$ for each wave number k will allow us to solve the equation easily by representing the general solution as a superposition of these solutions u_k; this is the Fourier method.

Remark 1.2.1. For nonlinear wave equations, the travelling wave solutions are in general severely restricted. Usually only special profiles, characteristic of the particular equation, are possible for travelling wave solutions. In particular they do not normally admit any solutions of the plane wave form $Ae^{i\phi}e^{i(kx-\omega t)}$.

The concepts of travelling wave and plane wave still make sense when the spatial dimension n is greater than one. Given an initial "profile" $f : \mathbb{R}^n \to V$, and a "direction" $\eta \in \mathbf{S}^{n-1}$, we can define the travelling wave $u(x,t)$ with profile f and moving with speed γ in the direction η by $u(x,t) := f(x - \gamma t\eta)$. Note that the graph of the function $x \mapsto u(x,t)$ is just the graph of f translated by $\gamma t\eta$, so it indeed travels in the direction η with speed γ. If we choose a basis v_i for V, then we can write f as a finite sum $f(x) = \sum_{i=1}^{d} f_i(x)v_i$ where $f_i : \mathbb{R}^n \to \mathbb{C}$, thus essentially reducing consideration to the case $V = \mathbb{C}$, so we will assume that f is scalar valued in what follows.

If $\kappa \in \mathbf{S}^{n-1}$ is a direction, then the fibers of the projection $\Pi_\kappa : x \mapsto x \cdot \kappa$ of $\mathbb{R}^n$ onto $\mathbb{R}$ foliates $\mathbb{R}^n$ by the hyperplanes $x \cdot \kappa = a$ orthogonal to κ. A profile, $f : \mathbb{R}^n \to \mathbb{C}$, that is constant on each such hyperplane is called a "plane wave profile", and will be of the form $f(x) = g(x \cdot \kappa)$ where $g : \mathbb{R} \to \mathbb{C}$. If we define $c = \gamma\kappa \cdot \eta$, then the corresponding travelling plane wave is $f(x - \gamma t\eta) = g(x \cdot \kappa - ct)$, i.e., just the travelling wave with profile g and speed c on $\mathbb{R}$, pulled back to $\mathbb{R}^n$ by Π_κ.

The exponential plane wave $u(x,t) = e^{i(kx-\omega t)}$ that we used for the case $n = 1$ has the profile $u(y, 0) = e^{iky}$. If we use this same profile for $n > 1$, i.e., define $g(y) = e^{iky}$, then our travelling waves will have the form $e^{ik(x\cdot\kappa-ct)} = e^{i(kx\cdot\kappa-\omega t)}$ where, as above, $\omega = kc$. If we define $\xi = k\kappa$ and recall that κ was a unit vector, then our travelling wave is $u_{\xi,\omega}(x,t) = e^{i(x\cdot\xi-\omega t)}$, where now the wave number is $\|\xi\|$, and the speed, c, is related to the angular frequency ω by $c = \frac{\omega}{\|\xi\|}$. At any point, x, $u_{\xi,\omega}(x,t)$ is periodic in t with frequency $\frac{\omega}{2\pi}$, and fixing t, $u_{\xi,\omega}(x,t)$ is periodic with period $\frac{\|\xi\|}{2\pi}$ along any line parallel to ξ. We shall see that for $n > 1$ too, there is a dispersion relation associated to any linear wave equation, and the Fourier magic still works; i.e., for each ξ there will be a unique frequency $\omega(\xi)$ such that $u_\xi(x, y) = u_{\xi,\omega(\xi)}(x, t)$ is a solution of the wave equation, and we will again be able to represent the general solution as a superposition of these special travelling wave solutions.

1.3 Some Model Equations

In this section we will introduce some of the more important model wave equations (and classes of wave equations) that will be studied in more detail in later sections

Example 1.3.1. Perhaps the most familiar wave equation is $u_{tt} - c^2\Delta u = 0$, and I will refer to it as "The Classic Wave Equation".Here Δ is the Laplace operator, and the operator $\frac{\partial^2}{\partial t^2} - \Delta$ is called the wave operator, or D'Alembertian operator. As we saw above, we can reduce this to a standard first-order evolution equation by replacing the one-component vector u by a two-component vector (u, v) that satisfies the PDE $(u, v)_t = (v, c^2\Delta u)$, i.e., the wave shape u and velocity v satisfy the system of two linear PDE $u_t = v$ and $v_t = c^2\Delta u$. As we shall see next, in one space dimension it is extremely easy to solve the Cauchy problem for the Classic Wave Equation explicitly. Namely, in one space dimension we can factor the wave operator, $\frac{\partial^2}{\partial t^2} - c^2\frac{\partial^2}{\partial x^2}$, as the product$(\frac{\partial}{\partial t} - c\frac{\partial}{\partial x})(\frac{\partial}{\partial t} + c\frac{\partial}{\partial x})$. This suggests that we transform to so-called "characteristic coordinates", $\xi = x - ct$ and $\eta = x + ct$, in which the wave equation becomes simply $\frac{\partial^2 u}{\partial\xi\partial\eta} = 0$. This clearly has the general solution $u(\xi, \eta) = F(\xi) + G(\eta)$, so transforming back to "laboratory coordinates" x, t, the general solution is $u(x, t) = F(x - ct) + G(x + ct)$. If the initial shape of the wave is $u(x, 0) = u_0(x)$ and its initial velocity is $u_t(x, 0) = v(x, 0) = v_0(x)$, then an easy algebraic computation gives the following very explicit formula:

$$u(x, t) = \frac{1}{2}[u_0(x-ct)+u_0(x+ct)]+\frac{1}{2c}\int_{x-ct}^{x+ct} v_0(\xi)\,d\xi,$$

known as "D'Alembert's Solution" of the Cauchy Problem for the Wave Equation in one space dimension. Note the geometric interpretation in the important "plucked string" case, $v_0 = 0$; the initial profile u_0 breaks up into the sum of two travelling waves, both with the same profile $u_0/2$, and one travels to the right with speed c and the other to the left with speed c. (We shall see later that something similar happens when $n > 1$. One can again decompose the initial shape, but now into a *continuous* superposition of shapes u_κ, one for each "direction" κ on the unit sphere $\mathbf{S}^{n-1}$, and each u_k then moves as a travelling wave with the speed c in the direction ξ.)

Exercise 1.3.2. Derive D'Alembert's solution. (Hint: $u_0(x) = F(x) + G(x)$, so $u_0'(x) = F'(x) + G'(x)$, while $v_0(x) = u_t(x, 0) = -cF'(x)+cG'(x)$.)

Remark 1.3.3. There are a number of important consequences that follow easily from the form of the D'Alembert solution:

a) The solution is well-defined for initial conditions (u_0, v_0) in the space of distributions, and gives a flow on this much larger space.

b) The quantity $\int_{-\infty}^{\infty} |u_x|^2 + (\frac{1}{c})^2|u_t|^2\,dx$ is a "constant of the motion". More precisely, if this integral is finite at one time for a solution $u(x, t)$, then it is finite and has the same value at any other time.

c) The "domain of dependence" of a point (x, t) of space-time consists of the interval $[x-ct, x+ct]$. That is, the value of any solution u at (x, t) depends only on the values u_0 and v_0 in the interval $[x - ct, x + ct]$. Another way to say this is that the "region of influence" of a point x_0 consists of the interior of the "light-cone" with vertex at x_0, i.e., all points (x, t) satisfying $x_0 - ct < x < x_0 + ct$. (These are the points having x_0 in their domain of dependence.) Still a third way of stating this is that the Classical Wave Equation has signal propagation speed c, meaning that the value of a solution at (x, t) depends only on the values of u_0 and v_0 at points x_0 from which a signal propagating with speed c could reach x in time t (i.e., points inside the sphere of radius ct about x.)

Exercise 1.3.4. Prove b) of the above Remark. (Hint: $|u_x(x, t)|^2 + (\frac{1}{c})^2|u_t(x, t)|^2 = 2(|F'(x - ct)|^2 + |G'(x + ct)|^2)$.)

Example 1.3.5. The "Linear Advection Equation", $u_t - cu_x = 0$. Using again the trick of transforming to the coordinates, $\xi = x - ct$, $\eta = x + ct$, now the equation becomes $\frac{\partial u}{\partial \xi} = 0$, and hence the general solution is $u(\xi) =$ constant, and the solution to the Cauchy Problem is $u(x,t) = u_0(x - ct)$. As before we see that if u_0 is any distribution then $u(t) = u_0(x - ct)$ gives a well-defined curve in the space of distributions that satisfies $u_t - cu_x = 0$, so that we really have a flow on the space of distributions whose generating vector field is $c\frac{\partial}{\partial x}$. Since $c\frac{\partial}{\partial x}$ is a skew-adjoint operator on $L^2(\mathbb{R})$, it follows that this flow restricts to a one-parameter group of isometries of $L^2(\mathbb{R})$, i.e., $\int_{-\infty}^{\infty} u(x,t)^2\,dx$ is a constant of the motion.

Exercise 1.3.6. Prove directly that $\frac{d}{dt}\int_{-\infty}^{\infty} u(x,t)^2\,dx$ is zero. (Hint: It suffices to show this when u_0 is smooth and has compact support, since these are dense in L^2. Now for such functions we can rewrite the integral as $\int_{-\infty}^{\infty} \frac{\partial}{\partial t} u(x,t)^2\,dx$ and the result will follow if we can show that $\frac{\partial}{\partial t} u(x,t)^2$ can be written for each t in the form $\frac{d}{dx}h(x)$, where h is smooth and has compact support.)

Remark 1.3.7. Clearly the domain of dependence of (x,t) is now just the single point $x - ct$, the region of influence of x_0 is the line $x = x_0 + ct$, and the signal propagation speed is again c. The principle difference with Example 1 is that the Linear Advection Equation describes a wave moving in one direction with velocity c, while The Classic Wave Equation describes a pair of waves moving in both directions with velocity c.

Exercise 1.3.8. (Duhamel's Principle.) The homogeneous Linear Advection Equation describes waves moving to the right in a *non-dispersive* and and *non-dissipative* one-dimensional linear elastic medium when there are no external forces acting on it. (The italicised terms will be explained later.) If there is an external force, then the appropriate wave equation will be an inhomogeneous version of the equation, $u_t - cu_x = F(x,t)$. Show that the Cauchy Problem now has the solution $u(x,t) = u_0(x - ct) + \int_0^t F(x - ct + c\xi, \xi)\,d\xi$.

Example 1.3.9. General Linear Evolution Equation, $u_t + P(\frac{\partial}{\partial x})u = 0$. Here $P(\xi)$ is a polynomial with complex coefficients. For example, if $P(\xi) = -c\xi$ then we get back the Linear Advection Equation. We will outline the theory of these equations in a separate subsection below where as we shall see, they can analyzed easily and thoroughly using the Fourier Transform. It will turn out that to qualify as a wave equation, the odd coefficients of the polynomial P should be real and the even coefficients pure imaginary, or more simply, $P(i\xi)$ should be imaginary valued on the real axis. This is the condition for $P(\frac{\partial}{\partial x})$ to be a skew-adoint operator on $L^2(\mathbb{R})$.

Example 1.3.10. The General Conservation Law, $u_t = (F(u))_x$. Here $F(u)$ can any smooth function of u and certain of its partial derivatives with respect to x. For example,if $P(\xi) = a_1\xi + \cdots + a_n\xi^n$, we can get the linear evolution equation $u_t = P(\frac{\partial}{\partial x})u$ by taking $F(u) = a_1 u + \cdots + a_n \frac{\partial^{n-1} u}{\partial x^{n-1}}$, and $F(u) = -(\frac{1}{2}u^2 + \delta^2 u_{xx})$ gives the KdV equation $u_t + uu_x + \delta^2 u_{xxx} = 0$ that we consider just below. Note that if $F(u(x,t))$ vanishes at infinity then integration gives $\frac{d}{dt}\int_{-\infty}^{\infty} u(x,t)\,dx = 0$, i.e., $\int_{-\infty}^{\infty} u(x,t)\,dx$ is a "constant of the motion", and this is where the name "Conservation Law" comes from. We will be concerned mainly with the case that $F(u)$ is a zero-order operator, i.e., $F(u)(x) = F(u(x))$, where F is a smooth function on $\mathbb{R}$. In this case, if we let $f = F'$, then we can write our Conservation Law in the form $u_t = f(u)u_x$. In particular, taking $f(\xi) = c$ (i.e., $F(\xi) = c\xi$) gives the Linear Advection Equation $u_t = cu_x$, while $F(\xi) = -\frac{1}{2}\xi^2$ gives the important Inviscid Burgers Equation, $u_t + uu_x = 0$.

There is a very beautiful and highly developed theory of such Conservation Laws, and again we will devote a separate subsection to outlining some of the basic results from this theory. Recall that for the Linear Advection Equation we have an explicit solution for the Cauchy Problem, namely $u(x,t) = u_0(x - ct)$, which we can also write as $u(x,t) = u_0(x - f(u(x,t))t)$, where $f(\xi) = c$. If we are incredibly optimistic we might hope that we could more generally solve the Cauchy Problem for $u_t = f(u)u_x$ by solving $u(x,t) = u_0(x - f(u(x,t))t)$ as an implicit equation for $u(x,t)$. This would mean that we could generalize our algorithm for finding the profile of u at time t from the initial profile as follows: translate each point $(\xi, u_0(\xi))$ of the graph of u_0 to the right by an amount $f(u_0(\xi))t$

to get the graph of $x \mapsto u(x,t)$. This would of course give us a simple method for solving any such Cauchy Problems, and **the amazing thing is that this bold idea actually works**. However, one must be careful. As we shall see, this algorithm (which goes under the name "the method of characteristics") contains the seeds of its own eventual failure. For a general initial condition u_0 and function f, we shall see that we can predict a positive time T_B (the so-called "breaking time") after which the solution given by the method of characteristics can no longer exist as a smooth, single-valued funtion.

Example 1.3.11. The Korteveg-de Vries (KdV) Equation, $u_t + uu_x + \delta^2 u_{xxx} = 0$. If we re-scale the independent variables by $t \to \beta t$ and $x \to \gamma x$, then the KdV equation becomes:

$$u_t + \left(\frac{\beta}{\gamma}\right) uu_x + \left(\frac{\beta}{\gamma^3}\right) \delta^2 u_{xxx} = 0,$$

so by appropriate choice of β and γ we can obtain any equation of the form $u_t + \lambda uu_x + \mu u_{xxx} = 0$, and any such equation is referred to as "the KdV equation". A common choice, convenient for many purposes, is $u_t + 6uu_x + u_{xxx} = 0$, although the form $u_t - 6uu_x + u_{xxx} = 0$ (obtained by replacing u by $-u$) is equally common and we will use both. This is surely one of the most important and most studied of all evolution equations. It is over a century since it was shown to govern wave motion in a shallow channel, but less than forty years since the remarkable phenomenon of soliton interactions was discovered in studying certain of its solutions. Shortly thereafter the so-called Inverse Scattering Transform (IST) for solving the KdV equation was discovered and the equation was eventually shown to be an infinite dimensional completely integrable Hamiltonian system. This equation, and its remarkable properties will be one of our main objects of study.

Example 1.3.12. The Sine-Gordon Equation (SGE), $u_{tt} - u_{xx} = \sin(u)$. This equation is considerably older than KdV. It was discovered in the late eighteen hundreds to be the master equation for the study of "pseudospherical" surfaces, i.e., surfaces of Gaussian curvature K equal to -1 immersed in $\mathbb{R}^3$, and for that reason it was intensively studied (and its solitons discovered, but not recognized as such) long before KdV was even known. However it was only in the course of trying to find other equations that could be solved by the IST that it was realized that SGE was also a integrable equation.

Example 1.3.13. The Nonlinear Schrödinger Equation, $iu_t + u_{xx} + u|u|^2 = 0$. This is of more recent origin. It was the third evolution equation shown to have soliton behavior and to be integrable, and recently has been intensively studied because it describes the propagation of pulses of laser light in optical fibers. The latter technology that is rapidly becoming the primary means for long-distance, high bandwidth communication, which in turn is the foundation of the Internet and the World Wide Web.

1.4 Linear Wave Equations; Dispersion and Dissipation

Evolution equations that are not only linear but also translation invariant (i.e., have constant coefficients) can be solved explicitly using Fourier methods. They are interesting both for their own sake, and also because they can serve as a tool for studying nonlinear equations.

The general linear evolution equation has the form $u_t + P(\frac{\partial}{\partial x})u = 0$, where to begin with we can assume that the polynomial P has coefficients that are smooth complex-valued functions of x and t: $P(\frac{\partial}{\partial x})u = \sum_{i=1}^{r} a_i(x,t) \frac{\partial^i u}{\partial x^i}$. For each (x_0, t_0), we have a space-time translation operator $T_{(x_0,t_0)}$ acting on smooth functions of x and t by $T_{(x_0,t_0)}u(x,t) = u(x - x_0, t - t_0)$. We say that the operator $P(\frac{\partial}{\partial x})$ is *translation invariant* if it commutes with all the $T_{(x_0,t_0)}$.

Exercise 1.4.1. Show that the necessary and sufficient condition for $P(\frac{\partial}{\partial x})$ to be translation invariant is that the coefficients a_i of P should be constant complex numbers.

1.4.2. Invariance Principles

There are at least two good reasons to assume that our equation is translation invariant. First, the eminently practical one that in this case we will be able to use Fourier transform techniques to solve the initial value problem explicitly, and investigate in detail the nature of its solutions.

But there is frequently an even more important philosophical reason for postulating translation invariance. Assume that we are trying to model the dynamics of some fundamental physical field quantity u by an evolution equation of the above type. Thus x will denote the "place where", and t the "time when" the quantity has the value $u(x, t)$. Now, if our proposed physical law is indeed "fundamental", it's validity should not depend on where or when it is applied—it will be the same on Alpha Centauri as on Earth, and the same in a million years as it is today—we can even take that as part of the definition of what we mean by fundamental. The way to give a precise mathematical formulation of this principle of space-time symmetry or homogeneity is to demand that our equation should be invariant under some transitive group acting on space and time.

But, like most philosophical discussions, this only begs a deeper question. How does it happen that the space-time we live in appears to admit a simply-transitive abelian group action under which the physical laws are invariant? On the level of Newtonian physics (or Special Relativity) this is simply taken as axiomatic. General Relativity gives an answer that is both more sophisticated and more satisfying. The basic symmetry principle postulated is the Principle of Equivalence. This demands that the truly Fundamental Laws of Physics should be invariant under the (obviously transitive) group of *all* diffeomorphisms of space-time. Of course there are very few laws that are *that* fundamental—but Einstein's Field Equations for a (pseudo-)Riemannian metric on space-time is one of them, and the physical evidence for its correctness is pretty overwhelming. In a neighborhood of any point of space-time we can then coordinatize space-time by using geodesic coordinates (i.e., by identifying space-time with its tangent space at that point using the Riemannian exponential map). To use Einstein's lovely metaphor, we get into an elevator and cut the rope. In these natural coordinates, the space-time appears flat to second order, and the translation group that comes from the linear structure of the tangent space is an approximate symmetry group.

I will not try here to answer the still far deeper philosophical mystery of *why* our physical world seems to be governed by laws that exhibit such remarkable symmetry. This is closely related to what Eugene Wigner called "The Unreasonable Effectiveness of Mathematics in the Natural Sciences" in a famous article by that name [35]. (For another view of this topic see [15].) But I cannot help wondering if the so-called "Anthropic Principle" is not at least part of the answer. Perhaps only a Universe governed by such symmetry principles manifests the high degree of stability that is conducive to the evolution of the kind of self-cognizant, intelligent life that would worry about this point. In other words: mathematicians, physicists, and philosophers can exist to wonder about why such fundamental laws govern, only in those universes where they do in fact govern.

In any case, we shall henceforth assume that P does in fact have constant complex numbers as coefficients. If we substitute the Ansatz $u(x, t) = e^{i(kx-\omega t)}$ into our linear equation, $u_t + P(\frac{\partial}{\partial x})u = 0$, then we find the relation $-i\omega u + P(ik)u = 0$, or $\omega = \omega(k) := \frac{1}{i}P(ik)$. For $u(x, t)$ to be a plane wave solution, we need the angular frequency, ω, to be real. Thus, we will have a (unique) plane wave solution for each real wave number k just when $\frac{1}{i}P(ik)$ is real (or $P(ik)$ is imaginary) for k on the real axis. This just translates into the condition that the odd coefficients of P should be real and the even coefficients pure imaginary. Let us assume this in what follows. As we shall see, one consequence will be that we can solve the initial value problem for any initial condition u_0 in L^2, and the solution is a superposition of these plane wave solutions—clearly a strong reason to consider this case as describing honest "wave equations", whatever that term should mean. Then we will follow up by taking a look at what happens when we relax this condition.

The relation $\omega(k) := \frac{1}{i}P(ik)$ relating the angular frequency ω and wave number k of a plane wave solution of a linear wave equation is called the *dispersion relation* for the equation. The propagation velocity of the plane wave solution with wave number k is called the *phase velocity* at wave number k, and is given by the formula $\frac{\omega(k)}{k} = \frac{1}{ik}P(ik)$ (which is also sometimes referred to as the dispersion relation for the equation). It is important to observe that the dispersion relation is not only determined by the polynomial P that defines the evolution equation, but it conversely determines P.

Now let u_0 be any initial wave profile in L^2, so that $u_0(x) = \int \hat{u}_0(k)e^{ikx}\,dk$, where $\hat{u}_0(k) = \frac{1}{2\pi}\int u_0(x)e^{-ikx}\,dk$ is the Fourier Transform of u. Defining $\hat{u}(k,t) = e^{-P(ik)t}\hat{u}_0(k)$, we see that $\hat{u}(k,t)e^{ikx} = \hat{u}_0(k)e^{ik(x-\frac{\omega(k)}{k}t)}$ is a plane wave solution to our equation with initial condition $\hat{u}_0(k)e^{ikx}$. We now define $u(x,t)$ (formally) to be the superposition of these plane waves: $u(x,t) \sim \int \hat{u}(k,t)e^{ikx}\,dk$. So far we have not really used the fact that $P(ik)$ is imaginary for k real, and this $u(x,t)$ would still be a formal solution without that assumption. The way we shall use the condition on P is to notice that it implies $|e^{-P(ik)t}| = 1$. Thus, $|\hat{u}(k,t)| = |\hat{u}_0(k)|$, so $\hat{u}(k,t)$ is in L^2 for all t, and in fact it has the same norm as $\hat{u}_0$. It then follows from Plancherel's Theorem that $u(x,t)$ is in L^2 for all t, and has the same norm as u_0. It is now elementary to see that our formal solution $u(x,t)$ is in fact an honest solution of the Cauchy Problem for our evolution equation, and in fact defines a one-parameter group of unitary transformations of L^2. (Another way to see this is to note that since $\frac{\partial}{\partial x}$ is a skew-adjoint operator on L^2, so is any odd power or i times any even power, so that $P(\frac{\partial}{\partial x})$, is skew-adjoint and and hence $\exp(-P(\frac{\partial}{\partial x})t)$ is a one-parameter group of unitary transformations of L^2. But a rigorous proof that $\frac{\partial}{\partial x}$ is a skew-adjoint operator (and not just formally skew-adjopint) involves essentially the same Fourier analysis.)

We next look at what can happen if we drop the condition that the odd coefficients of P are real and the even coefficients pure imaginary.

Consider first the special case of the Heat (or Diffusion) Equation, $u_t - \alpha u_{xx} = 0$, with $\alpha > 0$. Here $P(x) = -\alpha X^2$, so $|e^{-P(ik)t}| = |e^{-k^2 t}|$. Thus, when $t > 0$, $|e^{-P(ik)t}| < 1$, and $|\hat{u}(k,t)| < |\hat{u}_0(k)|$, so again $u(k,t)$ is in L^2 for all t, but now $\|u(x,t)\|_{L^2} < \|u_0(x)\|_{L^2}$. Thus our solution is not a unitary flow on L^2, but rather a contracting, positive semi-group. In fact, it is easy to see that for each initial condition $u_0 \in L^2$, the solution tends to zero in L^2 exponentially fast as $t \to \infty$, and in fact it tends to zero uniformly too. This so-called *dissipative* behavior is clearly not very "wave-like" in nature, and the Heat Equation is not considered to be a wave equation. On the other hand, the fact that the propagator $|e^{-P(ik)t}|$ is so rapidly decreasing implies very strong regularity for the solution $u(x,t)$ as a function of x as soon as $t > 0$.

Exercise 1.4.3. Show that for any initial condition u_0 in L^2, the solution $u(x,t)$ of the Heat Equation is an analytic function of x for any $t > 0$. (Hint: If you know the Paley-Wiener Theorem, this is of course an immediate consequence, but it is easy to prove directly.)

What happens for $t < 0$? In this case $|e^{-P(ik)t}| = |e^{-k^2 t}|$ is not an essentially bounded function of k, and indeed grows more rapidly than any polynomial, so that multiplication by it does not map L^2 into itself. or any of the Sobolev spaces. In fact, it is immediate from the above exercise that if $u_0 \in L^2$ is *not* analytic, then there cannot be an L^2 solution $u(x,t)$ of the Heat Equation with initial condition u_0 on any non-trivial interval $(-T, 0]$.

It is not hard to extend this analysis for the Heat Equation to any monomial P: $P(X) = a_n X^n$, where $a_n = \alpha + i\beta$. Then $|e^{-P(ik)t}| = |e^{i^n \alpha t}||e^{i^{n+1}\beta t}|$. If $n = 2m$ is even, this becomes $|e^{(-1)^m \alpha t}|$, while if $n = 2m+1$ is odd, it becomes $|e^{(-1)^{(m+1)}\beta t}|$. If α (respectively β) is zero, we are back to our earlier case that gives a unitary flow on L^2. If not, then we get essentially back to the dissipative semi-flow behavior of the heat equation. Whether the semi-flow is defined for $t > 0$ or $t < 0$ depends on the parity of m and the sign of α (repectively β).

Exercise 1.4.4. Work out the details.

We will now return to our assumption that $P(D)$ is a skew-adjoint operator, i.e., the odd coefficients of $P(X)$ are real and the even coefficients pure imaginary. We next note that this seemingly ad hoc condition is actually equivalent to a group invariance principle, similar to translation invariance.

1.4.5. Symmetry Principles in General—and CPT in Particular.

One of the most important ways to single out important and interesting model equations for study is to look for equations that satisfy various symmetry or invariance principles. Suppose our equation is of the form $\mathcal{E} = 0$ where $\mathcal{E}$ is some differential operator on a linear space $\mathcal{F}$ of smooth functions, and we have some group G that acts on $\mathcal{F}$. Then we say that the equation is G-invariant (or that G is a symmetry group for the equation) if the operator $\mathcal{E}$ commutes with the elements of G. Of course it follows that if $u \in \mathcal{F}$ is a solution of $\mathcal{E} = 0$, then so is gu for all g in G.

As we have already noted, the evolution equation $u_t + P(D)u = 0$ is clearly invariant under time translations, and is invariant under spatial translations if and only if the coefficiends of the polynomial $P(X)$ are constant. Most of the equations of physical interest have further symmetries, i.e., are invariant under larger groups, reflecting the invariance of the underlyng physics under these groups. For example, the equations of pre-relativistic physics are Gallilean invariant, while those of relativistic physics are Lorentz invariant. We will consider here a certain important discrete symmetry that so far has proved to be universal in physics.

We denote by $\mathbf{T}$ the "time-reversal" map $(x, t) \to (x, -t)$, and by $\mathbf{P}$ the analogous "parity" or spatial reflection map $(x, t) \to (-x, t)$. These involutions act as linear operators on functions on space-time by $u(x, t) \to u(x, -t)$ and $u(x, t) \to u(-x, t)$ respectively. There is a third important involution, that does not act on space-time, but directly on complex-valued functions; namely the conjugation operator $\mathbf{C}$, mapping $u(x, t)$ to its complex conjugate $u(x, t)^*$. Clearly $\mathbf{C}$, $\mathbf{P}$, and $\mathbf{T}$ commute, so their composition $\mathbf{CPT}$ is also an involution $u(x, t) \to u(-x, -t)^*$ acting on complex-valued functions defined on space-time. We note that $\mathbf{CPT}$ maps the function $u(x, t) = e^{i(kx-\omega t)}$ (with real wave number k) to the function $u(x, t) = e^{i(kx-\omega^* t)}$, so it fixes such a u if and only if u is a plane wave.

Exercise 1.4.6. Prove that $u_t + P(D)u = 0$ is $\mathbf{CPT}$-invariant if and only if $P(D)$ is skew-adjoint, i.e., if and only if $P(i\xi)$ is pure imaginary for all real ξ. Check that the KdV, NLS, and Sine-Gordon equation are also $\mathbf{CPT}$-invariant.

1.4.7. Examples of Linear Evolution Equations

Example 1.4.8. Choosing $P(\xi) = c\xi$, gives the Linear Advection Equation $u_t + cu_x = 0$. The dispersion relation is $\frac{\omega(k)}{k} = \frac{P(ik)}{ik} = c$, i.e., all plane wave solutions have the same phase velocity c. For this example we see that $\hat{u}(k, t)e^{ikx} = \hat{u}_0(k)e^{ik(x-ct)}$, and since $\int \hat{u}_0(k)e^{ikx}\, dk = u_0(x)$, it follows that

$$u(x, t) = \int \hat{u}(k, t)e^{ik(x-ct)}\, dk = u_0(x - ct),$$

giving an independent proof of this explicit solution to the Cauchy Problem in this case.

The next obvious case to consider is $P(\xi) = c\xi + d\xi^3$, giving the dispersion relation $\frac{\omega(k)}{k} = \frac{P(ik)}{ik} = c(1 - (d/c)k^2)$, and the wave equation $u_t + cu_x + du_{xxx} = 0$. This is sometimes referred to as the "weak dispersion" wave equation. Note that the phase velocity at wave number k is a constant, c, plus a constant times k^2. It is natural therefore to transform to coordinates moving with velocity c, i.e., make the substitution $x \mapsto x - ct$, and the wave equation becomes $u_t + du_{xxx} = 0$. Moreover, by rescaling the independent variable x we can get rid of the coefficient d. This leads us to our next example.

Example 1.4.9. $P(\xi) = \xi^3$, gives the equation $u_t + u_{xxx} = 0$. Now the dispersion relation is non-trivial; plane wave solutions with wave number k move with phase velocity $\frac{\omega(k)}{k} = \frac{P(ik)}{ik} = -k^2$, so the Fourier components $\hat{u}_0(k)e^{ik(x+k^2 t)}$ of $u(x, t)$ with a large wave number k move faster than those with smaller wave number, causing an initially compact wave profile to gradually disperse as these Fourier modes move apart and start to interfere destructively.

It is not hard in this example to write a formula for $u(x, t)$ explicitly in terms of u_0, instead of $\hat{u}_0$, namely:

$$u(x, t) = \frac{1}{\sqrt{\pi}\sqrt[3]{3t}} \int_{-\infty}^{\infty} \operatorname{Ai}\left(\frac{x - \xi}{\sqrt[3]{3t}}\right) u_0(\xi)\, d\xi.$$

Here Ai is the Airy function, a bounded solution of $w'' - zw = 0$ that can be defined explicitly by:

$$\operatorname{Ai}(z) = \frac{1}{\sqrt{\pi}} \int_0^{\infty} \cos\left(\frac{t^3}{3} + tz\right) dw,$$

and it follows from this from this that $v(x, t) = \frac{1}{\sqrt{\pi}\sqrt[3]{3t}} \operatorname{Ai}\left(\frac{x}{\sqrt[3]{3t}}\right)$ satisfies $v_t + v_{xxx} = 0$, and $\lim_{t \to 0} v(x, t) = \delta(x)$.

Remark 1.4.10. More generally, for a wave equation $u_t + P(\frac{\partial}{\partial x})u = 0$, the solution, $p(x, t)$, of the Cauchy Problem with $p(x, 0) = \delta(x)$ is called the

Fundamental Solution or Propagator for the equation. It follows that the solution to the Cauchy problem for a general initial condition is given by convolution with p, i.e., $u(x,t) = \int_{-\infty}^{\infty} p(x-\xi,t)u_0(\xi)\,d\xi$.

Exercise 1.4.11. (General Duhamel Principle) Suppose p is the fundamental solution for the homogeneous wave equation $u_t + P(\frac{\partial}{\partial x})u = 0$. Show that the solution to the Cauchy Problem for the corresponding inhomogeneous equation $u_t + P(\frac{\partial}{\partial x})u = F(x,t)$ is given by $\int_{-\infty}^{\infty} p(x-\xi,t)u_0(\xi)\,d\xi + \int_0^t d\tau \int_{-\infty}^{\infty} p(x-\xi,t-\tau)F(\xi,\tau)\,d\xi$.

Before leaving linear wave equations we should say something about the important concept of *group velocity*. We consider an initial wave packet, u_0, that is synthesized from a relatively narrow band of wave numbers, k, i.e., $u_0(x) = \int_{k_0-\epsilon}^{k_0+\epsilon} \hat{u}_0(k)e^{ikx}\,dk$. Thus the corresponding frequencies $\omega(k)$ will also be restricted to a narrow band around $\omega(k_0)$, and since all the plane wave Fourier modes are moving at approximately the velocity $\frac{\omega(k_0)}{k_0}$, the solution $u(x,t)$ of the Cauchy Problem will tend to disperse rather slowly and keep an approximately constant profile f, at least for a short initial period. One might expect that the velocity at which this approximate wave profile moves would be $\frac{\omega(k_0)}{k_0}$, the central phase velocity, but as we shall now see, it turns out to be $\omega'(k_0)$. To see this we expand $(kx - \omega(k)t)$ in a Taylor series about k_0:

$$(kx-\omega(k)t) = (k_0x-\omega(k_0)t)+(k-k_0)(x-\omega'(k_0)t)+O((k-k_0)^2),$$

and substitute this in the formula $u(x,t) = \int_{k_0-\epsilon}^{k_0+\epsilon} \hat{u}_0(k)e^{i(kx-\omega(k)t)}\,dk$ for the solution. Assuming ϵ is small enough that the higher order terms in this expansion can be ignored in the interval $[k_0-\epsilon, k_0+\epsilon]$ we get the approximation $u(x,t) \approx f(x-\omega'(k_0)t)e^{i(k_0x-\omega(k_0)t)}$, where $f(x) = \int_{k_0-\epsilon}^{k_0+\epsilon} \hat{u}_0(k)e^{i(k-k_0)x} = u_0(x)e^{-ik_0x}\,dk$. Thus, to this approximation, the solution $u(x,t)$ is just the plane wave solution of the wave equation having wave number k_0, but amplitude modulated by a traveling wave with profile f and moving at the group velocity $\omega'(k_0)$.

Exercise 1.4.12. Consider the solution $u(x,t)$ to a linear wave equation that is the superposition of two plane wave solutions, the first with wave number k_0 and the second with wave number $k_0 + \Delta k$, that is close to k_0. Let $\Delta\omega = \omega(k_0+\Delta k) - \omega(k_0)$. Show that $u(x,t)$ is (exactly!) the first plane wave solution amplitude modulated by a travelling wave of profile $f(x) = 1 + e^{i\Delta kx}$ and velocity $\frac{\Delta\omega}{\Delta k}$. (So that in this case there is no real dispersion.)

Remark 1.4.13. In many important physical applications (e.g., light travelling in a transparent medium such as an optical fiber) $\omega'' < 0$, i.e., the dispersion curve is convex upwards, so that the phase velocity exceeds the group velocity, and high frequency plane waves are slower than low frequency plane waves. Thus, wavelets enter the envelope of a group from the left, and first grow and then diminish in amplitude as they pass through the group and exit to the right. This is called *normal dispersion*, the opposite case $\omega'' > 0$ being referred to as *anomalous dispersion*.

Example 1.4.14. De Broglie Waves.

Schrödinger's Equation for a particle in one dimension, $\psi_t = i\frac{\hbar}{2m}\psi_{xx} + \frac{1}{i\hbar}u\psi$, provides an excellent model for comparing phase and group velocity. Here $h = 6.626 \times 10^{-34}$ Joule seconds is Planck's quantum of action, $\hbar = h/2\pi$, and u is the potential function, i.e., $-u'(x)$ gives the force acting on the particle when its location is x. We will only consider the case of a free particle, i.e., one not acted on by any force, so we take $u = 0$, and Schrödinger's Equation reduces to $\psi_t + P(\frac{\partial}{\partial x})\psi = 0$, where $P(\xi) = \frac{\hbar}{i}\frac{\xi^2}{2m}$. The dispersion relation therefor gives $v_\phi(k) = \frac{\omega(k)}{k} = \frac{P(ik)}{ik} = \frac{\hbar k}{2m}$ as the phase velocity of a plane wave solution of wave number k, (a so-called de Broglie wave), and thus the group velocity is $v_g(k) = \omega'(k) = \frac{\hbar k}{m}$. Now the classical velocity of a particle of momentum p is $\frac{p}{m}$, and this implies the relation $p = \hbar k$ between momentum and wave number. Since the wave-length λ is related to the wave number by $\lambda = \frac{2\pi}{k}$, this gives the formula $\lambda = \frac{h}{p}$ for the so-called de Broglie wave-length of a particle of momentum p. (This was the original de Broglie hypothesis, associating a wave-length to a particle.) Note that the energy E of a particle of momentum p is $\frac{p^2}{2m}$, so $E(k) = \frac{(\hbar k)^2}{2m} = \hbar\omega(k)$, the classic quantum mechanics formula relating energy and frequency.

For this wave equation it is easy and interesting to find explicitly the evolution of a Gaussian wave-packet that is initially centered at x_0 and has wave number centerd at k_0—in fact this is given as an exercise in almost every first text on quantum mechanics. For the Fourier Transform of the initial wave function ψ_0, we take $\hat{\psi}_0(k) = G(k - k_0, \sigma_p)$, where

$$G(k, \sigma) = \frac{1}{(2\pi)^{\frac{1}{4}}\sqrt{\sigma}} \exp\left(-\frac{k^2}{4\sigma^2}\right)$$

is the L^2 normalized Gaussian centered at the origin and having "width" σ. Then, as we saw above, $\psi(x,t)$, the wave function at time t, has Fourier Transform $\hat{\psi}(k,t)$ given by $\hat{\psi}_0(k)e^{-P(ik)t}$, and $\psi(x,t) = \int \hat{\psi}(k,t)e^{ikx}\,dk$. Using the fact that the Fourier Transform of a Gaussian is another Gaussian, we find easily that $\psi(x,t) = A(x,t)e^{i\phi(x,t)}$, where the amplitude A is given by $A(x,t) = G(x - v_g t, \sigma_x(t))$. Here, as above, $v_g = v_g(k_0) = \frac{\hbar k_0}{m}$ is the group velocity, and the spatial width $\sigma_x(t)$ is given by $\sigma_x(t) = \frac{\hbar}{2\sigma_p}(1+\frac{4\sigma_p^4}{\hbar^2}\frac{t^2}{m})$. We recall that the square of the amplitude $A(x,t)$ is just the probability density at time t of finding the particle at x. Thus, we see that this is a Gaussian whose mean (which is the expected position of the particle) moves with the velocity of the classical particle. Note that we have a completely explicit formula for the width $\sigma_x(t)$ of the wave packet as a function of time, so the broadening effect of dispersion is apparent. Also note that the Heisenberg's Uncertainty Principle, $\sigma_x(t)\sigma_p \geq \frac{\hbar}{2}$ is actually an equality at time zero, and it is the broadening of disperion that makes it a strict inequality at later times.

Remark 1.4.15. For a non-free particle (i.e., when the potential u is *not* a constant function) the Schrödinger Equation, $\psi_t = i\frac{\hbar}{2m}\psi_{xx} + \frac{1}{i\hbar}u\psi$, no longer has coefficients that are constant in x, so we don't expect solutions that are exponential in both x and t (i.e., plane waves or de Broglie waves). But the equation is still linear, and it is still invariant under time translations, so do we expect to be able to expand the general solution into a superposition of functions of the form $\psi_E(x,t) = \phi(x)e^{-i\frac{E}{\hbar}t}$. (We have adopted the physics convention, replacing the frequency, ω, by $\frac{E}{\hbar}$, where E is the energy associated to that frequency.) If we substitute this into the Schrödinger equation, then we see that the "energy eigenfunction" (or "stationary wave function") ϕ must satisfy the so-called time-independent Schrödinger Equation, $(-\frac{\hbar^2}{2m}\frac{d^2}{dx^2} + u)\phi = E\phi$. Note that this is just a second-order linear ODE, so for each choice of E it will have a two-dimensional linear space of solutions. This linear equation will show up with a strange twist when we solve the nonlinear KdV equation, $u_t - 6uu_x + u_{xxx} = 0$, by the remarkable Inverse Scattering Method. Namely, we will see that if the one-parameter family of potentials $u(t)(x) = u(x,t)$ evolves so as to satisfy the KdV equation, then the corresponding family of Schrödinger operators, $(-\frac{\hbar^2}{2m}\frac{d^2}{dx^2} + u)$, are unitarily equivalent, a fact that will play a key rôle in the Inverse Scattering Method. (Note that the "time", t, in the time-dependent Schrödinger Equation is not related in any way to the t in the KdV equation.)

Remark 1.4.16. We next explain how to generalize the Fourier methods above for solving linear PDE to the case $n > 1$. For simplicity, we will only consider the case of scalar equations—i.e., we will assume that u is a complex-valued function (rather than one taking values in some complex vector space V), but the more general vector-valued case can be handled similarly, (see the exercise below). As we saw earlier, the analog of plane waves in more space dimensions are travelling waves of the form $u_{\xi,\omega}(x,t) = e^{i(x\cdot\xi - \omega t)}$, where $\xi \in \mathbb{R}^n$. Now $\frac{\xi}{\|\xi\|} \in \mathbf{S}^{n-1}$ is the direction of the plane wave motion, the wave number is $\|\xi\|$, and the speed, c, is related to the angular frequency ω (which must be real) by $c = \frac{\omega}{\|\xi\|}$.

Suppose we have a constant coefficient linear wave equation, $u_t + P(D)u = 0$. Here $P(X) = \sum_{|\alpha|\leq r} A_\alpha X^\alpha$ is a complex polynomial in $X = (X_1, \ldots, X_n)$, and we are using standard "multi-index notation". Thus, α denotes an n-tuple of non-negative integers, $|\alpha| = \alpha_1 + \cdots + \alpha_n$, $X^\alpha = X_1^{\alpha_1} \ldots X_n^{\alpha_n}$, $D = (\frac{\partial}{\partial x_1}, \ldots \frac{\partial}{\partial x_n})$, and $D^\alpha = \frac{\partial^{|\alpha|}}{\partial x_1^{\alpha_1}\ldots\partial x_n^{\alpha_n}}$. Note that $D^\alpha u_{\xi,\omega} = (i\xi)^\alpha u_{\xi,\omega}$, and hence $P(D)u_{\xi,\omega} = P(i\xi)u_{\xi,\omega}$, where $P(i\xi)$ is the so-called total symbol of $P(D)$, i.e., $\sum_{|\alpha|\leq r} i^{|\alpha|}\xi^\alpha A_\alpha$. On the other hand, $\frac{\partial}{\partial t}u_{\xi,\omega} = -i\omega u_{\xi,\omega}$, so $u_{\xi,\omega}$ is a solution of $u_t + P(D)u = 0$ if and only if $\omega = \omega(\xi) = \frac{1}{i}P(i\xi)$, and this is now the dispersion relation. Since ω must be real, if we want to have a plane wave solution for each ξ, the condition as before is that $P(i\xi)$ must be pure imaginary for all $\xi \in \mathbb{R}^n$. This clearly is the case if and only if A_α is real for $|\alpha|$ odd, and imaginary for $|\alpha|$ even, and this is also equivalent to requiring that $P(D)$ be a skew-adjoint operator on $L^2(\mathbb{R}^n, \mathbb{C})$. If $u_0(x) = \int \hat{u}(\xi)e^{ix\cdot\xi}\, d\xi$ in $L^2(\mathbb{R})$ is given, then $u(x,t) = \int \hat{u}(\xi)u_{\xi,\omega(\xi)}(x,t)\, d\xi$ solves the given wave equation and $u(x,0) = u_0(x)$. As in the case $n = 1$, the transformation $U(t)$ mapping u_0 to $u(t) = u(x,t)$ defines a one-parameter group of unitary transformations acting on $L^2(\mathbb{R}^n, \mathbb{C})$ with $P(D)$ as its infinitesimal generator, i.e., $U(t) = \exp(tP(D))$.

Exercise 1.4.17. Analyze the vector-valued wave equation $u_t + P(D)u = 0$, with u now a $\mathbb{C}^d$-valued function. Again, $P(X) = \sum_{|\alpha|\le r} A_\alpha X^\alpha$, the coefficients A_α now lying in the space of linear operators on $\mathbb{C}^d$ (or $d \times d$ complex matrices). Show that for $P(D)$ to be a skew-adjoint operator on $L^2(\mathbb{R}^n, \mathbb{C}^d)$ generating a one-parameter group of unitary operators $U(t)$, the total symbol $P(i\xi) = \sum_{|\alpha|\le r} i^{|\alpha|}\xi^\alpha A_\alpha$ must be skew-adjoint operator on $\mathbb{C}^d$ for all ξ in $\mathbb{R}^n$, and this in turn means that A_α is self-adjoint for $|\alpha|$ odd and skew-adjoint for $|\alpha|$ even. Check this is equivalent to the **CPT**-invariance of $u_t + P(D)u = 0$. Write an explicit formula for $U(t)u_0$ using vector-valued Fourier transforms.

Example 1.4.18. Finally we take a quick look at the classic linear wave equation, $u_{tt} - c^2\Delta u = 0$ with more spatial dimensions. If we substitute the plane wave Ansatz $u(x,t) = u_{\xi,\omega}(x,t) = e^{i(x\cdot\xi - \omega t)}$ into this equation, we find that $(\omega^2 - c^2\|\xi\|^2)u = 0$, so $\omega(k) = c\|\xi\|$, or $\frac{\omega(k)}{\|\xi\|} = c$. Thus, all the plane wave solutions travel at the same speed, c, but now they can travel with this speed in infinitely many different directions ξ, instead of just the two possible directions ("right" and "left") when $n = 1$. If we now take a Gaussian initial condition $u(x,0)$ (and say assume that $u_t(x,0) = 0$) and analyze it into its Fourier components, we see that because the various components all move with speed c but in different directions, the original Gaussian wave packet will spread out and become dispersed.

Both a plucked piano string and the waves from a pebble dropped in a pond satisfy the classic wave equation. But in the first case we observe two travelling waves race off in opposite directions, maintaining their original shape as they go, while in the second we see a circular wave pattern moving out from the central source, gradually losing amplitude as the energy spreads out over a larger and larger circle.

Exercise 1.4.19. Show that $u_{tt} - c^2\Delta u = 0$ is Lorentz invariant, conformally invariant, and **CPT**-invariant.

1.5 Conservation Laws

Let $u(x,t)$ denote the density of some physical quantity at the point x and time t, and $\overrightarrow{\phi}(x,t)$ its flux, i.e., $\overrightarrow{\phi}(x,t)\cdot\overrightarrow{n}\,dA$ is the rate of flow of the quantity across a surface element dA with unit normal $\overrightarrow{n}$. Finally, let $g(x,t)$ denote the rate at which the quantity is being created at x at time t. Then, essentially by the meanings of these definitions, given any region V with smooth boundary ∂V,

$$\frac{d}{dt}\int_V u(x,t)\,dV = \int_V g(x,t)\,dV - \int_{\partial V} \overrightarrow{\phi}(x,t)\cdot\overrightarrow{n}\,dA,$$

which is the general form of a conservation law in integrated form. (Note that in the one space dimensional case this becomes $\int_a^b u(x,t)\,dx = \int_a^b g(x,t)\,dx - [\phi(b,t) - \phi(a,t)]$.) If u is C^1, then by Gauss's Theorem,

$$\int_V \left(\frac{\partial u(x,t)}{\partial t} + \nabla\cdot\overrightarrow{\phi}(x,t)\right) dV = \int_V g(x,t)\,dV,$$

so, dividing by the volume of V, and letting V shrink down on a point x we get the corresponding differential form of the conservation law,

$$\frac{\partial u(x,t)}{\partial t} + \nabla\cdot\overrightarrow{\phi}(x,t) = g(x,t),$$

or in one space dimension, $u_t + \phi_x = g$. We will be mainly concerned with the case $g = 0$, and we note that in this case it follows that if ϕ vanishes at infinity, then $\int_{-\infty}^{\infty} u(x,t)\,dt$ is independent of t (which explains why this is called a conservation law).

As it stands, this is a single equation for two "unknown" functions u and ϕ, and is underdetermined. Usually however we have some so-called "constitutive equation" expressing ϕ in terms of u. In the most general case, $\phi(x,t)$ will be a function not only of $u(x,t)$ but also of certain partial derivatives of u with respect to x, however we will only consider the case of constitutive equations of the form $\phi(x,t) = F(u(x,t))$, where F is a smooth function on $\mathbb{R}$ whose derivative F' will be denoted by f. Thus our conservation law finally takes the form:

$$(CL) \qquad u_t + f(u)u_x = 0.$$

We will usually assume that $f'(u) \geq 0$, so that f is a non-decreasing function. This is satisfied in most of the important applications.

Example 1.5.1. Take $F(u) = cu$, so $f(u) = c$ and we get once again the Linear Advection Equation $u_t - cu_x = 0$. The Method of Characteristics below will give yet another proof that the solution to the Cauchy Problem is $u(x,t) = u_0(x - ct)$.

Example 1.5.2. Take $F(u) = \frac{1}{2}u^2$, so $f(u) = u$ and we get the important Inviscid Burgers Equation, $u_t + uu_x = 0$.

We will next explain how to solve the Cauchy Problem for such a Conservation Law using the so-called Method of Characteristics. We look for smooth curves $(x(s), t(s))$ in the (x,t)-plane along which the solution to the Cauchy Problem is constant. Suppose that $(x(s_0), t(s_0)) = (x_0, 0)$, so that the constant value of $u(x,t)$ along this so-called characteristic curve is $u_0(x_0)$. Then $0 = \frac{d}{ds}u((x(s), t(s)) = u_x x' + u_t t'$, and hence

$$\frac{dx}{dt} = \frac{x'(s)}{t'(s)} = -\frac{u_t}{u_x} = f(u(x(s), t(s)) = f(u_0(x_0)),$$

so the characteristic curve is a straight line of slope $f(u_0(x_0))$, i.e., u has the constant value $u_0(x_0)$ along the line $\Gamma_{x_0} : x = x_0 + f(u_0(x_0))t$. Note the following geometric interpretation of this last result: **to find the wave profile at time t (i.e., the graph of the map $x \mapsto u(x,t)$), translate each point $(x_0, u_0(x_0))$ of the initial profile to the right by the amount $f(u_0(x_0))t$.** (This is what we promised to show in Example 1.3.4.) The analytic statement of this geometric fact is that the solution $u(x,t)$ to our Cauchy Problem must satisfy the implicit equation $u(x,t) = u_0(x - tf(u(x,t)))$. Of course the above is heuristic—how do we know that a solution exists?—but it isn't hard to work backwards and make the argument rigorous.

The idea is to first define "characteristic coordinates" (ξ, τ) in a suitable strip $0 \leq t < T_B$ of the (x,t)-plane. We define $\tau(x,t) = t$ and $\xi(x,t) = x_0$ along the characteristic Γ_{x_0}, so $t(\xi, \tau) = \tau$ and $x(\xi, \tau) = \xi + f(u_0(\xi))\tau$. But of course, for this to make sense, we must show that there is a unique Γ_{x_0} passing through each point (x,t) in the strip $t < T_B$.

The easiest case is $f' = 0$, say $f = c$, giving the Linear Advection Equation, $u_t + cu_x = 0$. In this case, all characteristics have the same slope, $1/c$, so that no two characteristics intersect, and there is clearly exactly one characteristic through each point, and we can define $T_B = \infty$.

From now on we will assume that the equation is "truly nonlinear", in the sense that $f'(u) > d > 0$, so that f is a strictly increasing function.

If u_0' is everywhere positive, then $u_0(x)$ is strictly increasing, and hence so is $f(u_0(x))$. In this case we can again take $T_B = \infty$. For, since the slope of the characteristic Γ_{x_0} issuing from $(x_0, 0)$ is $\frac{1}{f(u_0(x))}$, it follows that if $x_0 < x_1$ then Γ_{x_1} has smaller slope than Γ_{x_0}, and hence these two characteristics cannot intersect for $t > 0$, so again every point (x,t) in the upper half-plane lies on at most one characteristic Γ_{x_0}.

Finally the interesting general case: suppose u_0' is somewhere negative. In this case we define T_B to be the infimum of $[-u_0'(x)f'(u_0(x))]^{-1}$, where the inf is taken over all x with $u_0'(x) < 0$. For reasons that will appear shortly, we call T_B the *breaking time*. As we shall see, T_B is the largest T for which the Cauchy Problem for (CL) has a solution with $u(x,0) = u_0(x)$ in the strip $0 \leq t < T$ of the (x,t)-plane. It is easy to construct examples for which $T_B = 0$; this will happen if and only if there exists a sequence $\{x_n\}$ with $u_0'(x_n) \to -\infty$. In the following we will assume that T_B is positive, and that in fact there is a point x_0 where $T_B = \frac{-1}{u_0'(x_0)f'(u_0(x_0))}$. In this case, we will call Γ_{x_0} a *breaking characteristic*.

Now choose any point x_0 where $u_0'(x_0)$ is negative. For x_1 slightly greater than x_0, the slope of Γ_{x_1} will be greater than the slope of Γ_{x_0}, and it follows that these two characteristics will meet at the point (x,t) where $x_1 + f(u_0(x_1))t = x = x_0 +$

$f(u_0(x_0))t$, namely when $t = -\frac{x_1 - x_0}{f(u_0(x_1)) - f(u_0(x_0))}$.

Exercise 1.5.3. Show that T_B is the least t for which any two characteristics intersect at some point (x, t) with $t \geq 0$.

Exercise 1.5.4. Show that there is always at least one characteristic curve passing through any point (x, t) in the strip $0 \leq t < T_B$ (and give a counterexample if u_0' is not required to be continuous).

Thus the characteristic coordinates (ξ, τ) are well-defined in the strip $0 \leq t < T_B$ of the (x, t)-plane. Note that since $x = \xi + f(u_0(\xi))\tau$, $\frac{\partial x}{\partial \xi} = 1 + f'(u_0(\xi))u_0'(\xi)\tau$, and $\frac{\partial x}{\partial \tau} = f(u_0(\xi))$, while $\frac{\partial t}{\partial \xi} = 0$ and $\frac{\partial t}{\partial \tau} = 1$. It follows that the Jacobian of (x, t) with respect to (ξ, τ) is $\frac{\partial x}{\partial \xi} = 1 + f'(u_0(\xi))u_0'(\xi)$, which is positive in $0 \leq t < T_B$, so that (ξ, τ) are smooth coordinates in this strip. On the other hand, if Γ_{x_0} is a breaking characteristic, then then the Jacobian aproaches zero along Γ_{x_0} as t approaches T_B, confirming that the characteristic coordinates cannot be extended to any larger strip.

By our heuristics above, we know that the solution of the Cauchy Problem for (CL) with initial value u_0 should be given in characteristic coordinates by the explicit formula $u(\xi, \tau) = u_0(\xi)$, and so we define a smooth function u in $0 \leq t < T_B$ by this formula. Since the map from (x, t) to (ξ, τ) is a diffeomophism, this also defines u as a smooth function of x and t, but it will be simpler to do most calculations in characteristic coordinates. In any case, since a point (x, t) on the characteristic Γ_ξ satisfies $x = \xi + f(u_0(\xi))t$, we see that $u = u(x, t)$ is the solution of the implicit equation $u = u_0(x - tf(u))$. It is obvious that $u(x, 0) = u_0(x)$, and we shall see next that $u(x, t)$ satisfies (CL).

Exercise 1.5.5. Use the chain-rule: $u_x = u_\xi \frac{\partial \xi}{\partial x}$ and $u_t = u_\xi \frac{\partial \xi}{\partial t}$ to compute the partial derivatives u_x and u_t as functions of ξ and τ:

$$u_t(\xi, \tau) = -\frac{u_0'(\xi) f(u_0(\xi))}{1 + u_0'(\xi) f'(u_0(\xi))\tau}$$

and

$$u_x(\xi, \tau) = \frac{u_0'(\xi)}{1 + u_0'(\xi) f'(u_0(\xi))\tau}$$

and deduce from this that u actually satisfies the equation (CL) in $0 \leq t < T_B$.

Exercise 1.5.6. Show that, along a breaking characteristic Γ_{x_0}, the value of u_x at the point $x = x_0 + f(u_0(x_0))t$ is given by $\frac{u_0'(x_0)T_B}{T_B - t}$. (Note that this is just the slope of the wave profile at time t over the point x.)

We can now get a qualitative but very precise picture of how u develops a singularity as t approaches the breaking time T_B, a process usually referred to as *shock formation* or *steepening and breaking of the wave profile*.

Namely, let Γ_{x_0} be a breaking characteristic and consider an interval I around x_0 where u_0 is decreasing. Let's follow the evolution of that part of the wave profile that is originally over I. Recall our algorithm for evolving the wave profile: each point $(x, u_0(x))$ of the initial profile moves to the right with a constant velocity $f(u_0(x))$, so at time t it is at $(x + f(u_0(x))t, u_0(x))$. Thus, the higher part of the wave profile, to the left, will move faster than the lower part to the right, so the profile will bunch up and become steeper, until it eventually becomes vertical or "breaks" at time T_B when the slope of the profile actually becomes infinite over the point $x_0 + f(u_0(x_0))T_B$. (In fact, the above exercise shows that the slope goes to $-\infty$ like a constant times $\frac{1}{t - T_B}$.) Note that the values of u remain bounded as t approaches T_B. In fact, it is clearly possible to continue the wave profile past $t = T_B$, using the same algorithm. However, for $t > T_B$ there will be values x^* where the vertical line $x = x^*$ meets the wave profile at time t in two distinct points (corresponding to two characteristics intersecting at the point (x^*, t)), so the profile is no longer the graph of a single-valued function.

Remark 1.5.7. Despite the fact that u_x blows up along a breaking characteristic as $t \to T_B$, surprisingly the total variation of the function $x \mapsto u(x, t)$ not only doesn't blow up as t approaches T_B, it is actually a constant of the motion, i.e., $\int |u_x(x, t)|\, dx = \int |u_0'(\xi)|\, d\xi$. To see this, note that $\frac{\partial x}{\partial \xi} = 1 + f'(u_0(\xi))u_0'(\xi)t$ is clearly positive for $t < T_B$, so that $|u_x(x, t)|\, dx = |u_x(\xi, \tau)|\, \frac{\partial x}{\partial \xi}\, d\xi = |u_x(\xi, \tau)|\, (1 + f'(u_0(\xi))u_0'(\xi)\tau)\, d\xi$ and use the above formula for $u_x(\xi, \tau)$. Thus, although $|u_x|$ gets very large as t approaches T_B, it is only large on a set of small measure.

For certain purposes it is interesting to know how higher derivatives u_{xx}, $u_{xxx}, \ldots$ behave as t

approaches T_B along a breaking characteristic, (in particular, in the next section we will want to compare u_{xxx} with uu_x). These higher partial derivatives can be estimated in terms of powers of u_x using $\frac{\partial}{\partial x} = \frac{\partial}{\partial \xi}(\frac{\partial x}{\partial \xi})^{-1}$, and $\frac{\partial x}{\partial \xi} = 1 + f'(u_0(\xi))u_0'(\xi)\tau$.

Exercise 1.5.8. Show that along a breaking characteristic Γ_{x_0}, as $t \to T_B$, $u_{xx} = O(u_x^3) = O((t - T_B)^{-3})$. Similarly, show that $u_{xxx} = O(u_x^5) = O((t - T_B)^{-5})$.

For more details on the formation of singularities in conservation laws and other PDEs see [16, 23]. Below we will consider what happens after the breaking time T_B. Although we can no longer have a smooth solution, it turns out that there may still be physically meaningful solutions of the integrated form of the conservation law. But first we consider an interesting example.

Example 1.5.9. Traffic Flow on a Highway.

We imagine an ideal, straight, infinitely long highway, modeled by the real line. To simplify the analysis, we assume that there are no entrance or exit ramps, and we will smooth out the discrete nature of the cars and model the traffic density and flux by approximating continuous statistical averages. We choose an arbitrary origin, and we let $u(x,t)$ denote the density of cars at the point x at time t (in units of cars per kilometer), and we let $\phi(x,t)$ denote the flux of cars, i.e., the rate at which cars are passing the point x at time t (in units of cars per second). We will also want to consider the speed of the traffic at x at time t, which we will denote by $v(x,t)$ and measure it in kilometers per second. We have the obvious relation $\phi = vu$. If we choose any two points a and b along the highway, then clearly we have the conservation law in integrated form, $\frac{d}{dt}\int_a^b u(x,t)\,dx + \phi(b,t) - \phi(a,t) = 0$; i.e., the rate of change of the total number of cars between a and b plus the rate at which cars are coming in at b minus the rate at which they are leaving at a must be zero (since no cars are leaving between a and b). Assuming u is smooth, and letting a and b approach x we get the differential form of the conservation law, $u_t + \phi_x = 0$.

To proceed further we will need a constitutive relation, relating u and ϕ. It is natural to try to model this using the intuitively observed "law" of traffic flow that the denser the traffic, the slower drivers will travel. For simplicity, we assume that there is a maximum velocity v_{max} (the "speed limit") and a maximum density u_{max} and we assume that the speed at which drivers travel is v_{max} on an empty road (i.e., when $u = 0$), 0 when traffic is bumper to bumper, (i.e., when $u = u_{max}$) and linear in between. This leads to the relation $v(u) = v_{max}(1 - u/u_{max})$, and then using $\phi = vu$ we derive the constitutive relation, $\phi(u) = F(u) = uv(u)$. The conservation law then takes the form $u_t + F'(u)u_x = u_t + (v(u) + uv'(u))u_x = 0$, or $u_t + v_{max}(1 - 2u/u_{max})u_x = 0$.

Of course, traffic engineers use much more realistic models that take into account on and off ramps, and use more sophisticated contitutive relations, but already with this model one can see interesting phenomena such as the development of a "shock wave" as oncoming traffic meets traffic stopped at a red light. We illustrate this below, after first introducing the simplest kind of non-smooth solutions of conservation laws

1.5.10. Shock Waves and the Rankine-Hugoniot Jump Condition

Let $x_s(t)$ denote a smooth curve C in the closed upper-half (x,t)-plane, defined for all $t \geq 0$, and so dividing the upper-half plane into two open regions, R^- to the left and R^+ to the right. Let $u(x,t)$ be a smooth solution of the conservation law $u_t + \phi_x$ in the union of these two regions. We assume that the restrictions $u|R^-$ and $u|R^+$ each extend continuously to the boundary curve C, although these two extensions do not necessarily agree. Given a point $p = (x_s(t), t)$ on C, the difference between the limits $u(x_s^+, t)$ of u at p from the right and the limit $u(x_s^-, t)$ from the left defines the "jump" $[u](x_s, t) = u(x_s^+, t) - u(x_s^-, t)$ across C at this point. Since ϕ is given by a constitutive equation $\phi(x,t) = F(u(x,t))$, we also have a corresponding jump $[\phi](x_s, t) = \phi(x_s^+, t) - \phi(x_s^-, t)$ in ϕ as we cross C. We will call such a piecewise smooth solution u of the conservation law a *shock wave* solution of the conservation law with *shock path* C if in addition to satisfying the equation in each of R^- and R^+, it also satisfies the integrated form of the conservation law, i.e., for all $a < b$, $\frac{d}{dt}\int_a^b u(x,t)\,dx + \phi(b,t) - \phi(a,t) = 0$. By choosing $a < x_s(t) < b$ and breaking the above integral into a sum corresponding to the sub-intervals $[a, x_s(t)]$ and $[x_s(t), b]$ (and then letting a and b approach $x_s(t)$), we can easily derive the following:

Rankine-Hugoniot Jump Condition. Let u be a shock wave solution of the conservation law $u_t + \phi_x$ with shock path C given by $(x_s(t), t)$. Then $x_s(t)$ satisfies the following ordinary differential equation, known as the Rankine-Hugoniot Jump Condition:

$$\frac{dx_s(t)}{dt} = \frac{[\phi](x_s, t)}{[u](x_s, t)}.$$

Exercise 1.5.11. A Shock Wave Solution of the Inviscid Burgers Equation. Let's try to solve the Inviscid Burgers Equation, $u_t + uu_x = 0$, with the initial condition $u_0(x) = 1$ for $x < 0$ and $u_0(x) = 1$ for $x \geq 0$. It is easy to see that there are pairs of characterisics that meet after an arbitrarily short time, so $T_B = 0$, and this can have no smooth solution. Show that $u(x, t) = 1$ for $x < t/2$ and $u(x, t) = 0$ for $x > t/2$ is a shock wave solution with shock path $x_s(t) = t/2$.

Exercise 1.5.12. A Shock Wave at a Red Light. Consider highway traffic that is backing up as it runs into a red light. Assume that the oncoming traffic has a constant density u_1, and at time $t = 0$ it runs into the stopped traffic which has density u_{max} beginning at $x = 0$ and extending to the right. Show that the shock curve is given by $x_s(t) = -v_{max}(\frac{u_1}{u_{max}})t$ and the density is u_1 to the left, and u_{max} to the right. In other words, traffic is backing up at the speed $v_{max}(\frac{u_1}{u_{max}})$.

1.6 Split-Stepping

We now return to the KdV equation, say in the form $u_t = -uu_x - u_{xxx}$. If we drop the nonlinear term, we have left the dispersive wave equation $u_t = -u_{xxx}$, that we considered in the section on linear wave equations. Recall that we can solve its Cauchy Problem, either by using the Fourier Transform or by convolution with an explicit fundamental solution that we wrote in terms of the Airy function.

On the other hand, if we drop the linear term, we are left with the inviscid Burgers Equation, $u_t = -uu_x$, which as we know exhibits steepening and breaking of the wave profile, causing a shock singularity to develop in finite time T_B for any non-trivial initial condition u_0 that vanishes at infinity. Up to this breaking time, T_B, we can again solve the Cauchy Problem, either by the method of characteristics, or by solving the implicit equation $u = u_0(x - ut)$ for u as a function of x and t.

Now, in [8] it is proved that KdV defines a global flow on the Sobolev space $H^4(\mathbb{R})$ of functions $u : \mathbb{R} \to \mathbb{R}$ having derivatives of order up to four in L^2, so it is clear that dispersion from the linear u_{xxx} term must be counteracting the peaking from the nonlinear uu_x term, preventing the development of a shock singularity.

In order to understand this balancing act better, it would be useful to have a method for taking the two flows defined by $u_t = -u_{xxx}$ and $u_t = -uu_x$ and combining them to define the flow for the full KdV equation. (In addition, this would give us a method for solving the KdV Cauchy Problem numerically.)

In fact there is a very general technique that applies in such situations. In the pure mathematics community it is usually referred to as the Trotter Product Formula, while in the applied mathematics and numerical analysis communities it is called split-stepping. Let me state it in the context of ordinary differential equations. Suppose that Y and Z are two smooth vector fields on $\mathbb{R}^n$, and we know how to solve each of the differential equations $dx/dt = Y(x)$ and $dx/dt = Z(x)$, meaning that we know both of the flows ϕ_t and ψ_t on $\mathbb{R}^n$ generated by X and Y respectively. The Trotter Product Formula is a method for constructing the flow θ_t generated by $Y + Z$ out of ϕ and ψ; namely, letting $\Delta t = \frac{t}{n}$, $\theta_t = \lim_{n\to\infty}(\phi_{\Delta t}\psi_{\Delta t})^n$. The intuition behind the formula is simple. Think of approximating the solution of $dx/dt = Y(x) + Z(x)$ by Euler's Method. If we are currently at a point p_0, to propagate one more time step Δt we go to the point $p_0 + \Delta t\,(Y(p_0) + Z(p_0))$. Using the split-step approach on the other hand, we first take an Euler step in the $Y(p_0)$ direction, going to $p_1 = p_0 + \Delta t\, Y(p_0)$, then take a second Euler step, but now from p_1 and in the $Z(p_1)$ direction, going to $p_2 = p_1 + \Delta t\, Z(p_1)$. If Y and Z are constant vector fields, then this gives exactly the same final result as the simple full Euler step with $Y + Z$, while for continuous Y and Z and small time step Δt it is a good enough approximation that the above limit is valid. The situation is more delicate for flows on infinite dimensional manifolds, nevertheless it was shown by F. Tappert in [33] that the

Cauchy Problem for KdV can be solved numerically by using split-stepping to combine methods for $u_t = -uu_x$ and $u_t = -u_{xxx}$.

[Tappert actually uses an interesting variant, known as Strang splitting, which was first suggested in [31] to solve multi-dimensional hyperbolic problems by split-stepping one-dimensional problems. One advantage of splitting in numerical analysis comes from the greatly reduced effort required to solve the smaller bandwidth linear systems that arise when implicit schemes are necessary to maintain stability, but in addition, Strang demonstrated that second-order accuracy of the component methods need not be compromised by the asymmetry of the splitting, as long as the pattern $\phi_{\frac{\Delta t}{2}}\psi_{\frac{\Delta t}{2}}\psi_{\frac{\Delta t}{2}}\phi_{\frac{\Delta t}{2}}$ is used, to account for possible non-commutativity of Y and Z. (This may be seen by multiplying the respective exponential series.) Serendipitously, when output is not required, several steps of Strang splitting require only marginal additional effort: $(\phi_{\frac{\Delta t}{2}}\psi_{\frac{\Delta t}{2}}\psi_{\frac{\Delta t}{2}}\phi_{\frac{\Delta t}{2}})^n = (\phi_{\frac{\Delta t}{2}}\psi_{\Delta t}(\phi_{\Delta t}\psi_{\Delta t})^{n-1}\phi_{\frac{\Delta t}{2}}.]$

Aside from such numerical considerations, split-stepping suggests a way to understand the mechanism by which dispersion from u_{xxx} balances shock formation from uu_x in KdV. Namely, if we consider wave profile evolution under KdV as made up of a succession of pairs of small steps (one for $u_t = -uu_x$ and the one for $u_t = -u_{xxx}$), then when u, u_x, and u_{xxx} are not too large, the steepening mechanism will dominate. But recall that as the time t approaches the breaking time T_B, u remains bounded, and along a breaking characteristic u_x only blows up like $(T_B - t)^{-1}$ while u_{xxx} blows up like $(T_B - t)^{-5}$. Thus, near breaking in time and space, the u_{xxx} term will dwarf the nonlinearity and will disperse the incipient shock. In fact, computer simulations do show just such a scenario playing out.

2 The Korteweg-de Vries Equation

We have seen that in the Korteweg-de Vries equation:

$$\text{(KdV)} \qquad u_t + 6uu_x + u_{xxx} = 0,$$

expresses a balance between dispersion from its third-derivative term and the shock-forming tendency of its nonlinear term, and in fact many models of one-dimensional physical systems that exhibit mild dispersion and weak nonlinearity lead to KdV as the controlling equation at some level of approximation.

2.1 Early History, Exact Solutions, and Solitons

We will give here only a very abbreviated version of the historical origins of KdV. For more details see [27] and further references given there.

As already mentioned, KdV first arose as the modelling equation for solitary gravity waves in a shallow canal. Such waves are rare and not easy to produce, and they were apparently only first noticed in 1834 (by the naval architect, John Scott Russell). Early attempts by Stokes and Airy to model them mathematically seemed to indicate that such waves could not be stable—and their very existence was at first a matter of debate. Later work by Boussinesq and Rayleigh corrected errors in this earlier theory, and finally a paper in 1894 by Korteweg and de Vries [21] settled the matter by giving a convincing mathematical argument that wave motion in a shallow canal is governed by KdV, and showing by explicit computation that their equation admitted travelling-wave solutions that had exactly the properties described by Russell, including the relation of height to speed that Russell had determined experimentally in a wave tank he had constructed.

But it was only much later that the truly remarkable properties of the KdV equation became evident. In 1954, Fermi, Pasta and Ulam (FPU) used one of the very first digital computers to perform numerical experiments on a one-dimensional, anharmonic lattice model, and their results contradicted the then current expectations of how energy should distribute itself among the normal modes of such a system [12]. A decade later, Zabusky and Kruskal re-examined the FPU results in a famous paper [38]. They showed that, in a certain continuum limit, the FPU lattice was approximated by the KdV equation. They then did their own computer experiments, solving the Cauchy Problem for KdV with initial conditions corresponding to those used in the FPU experiments. In the results of these simulations they observed the first example of a "soliton", a term that they coined to describe a remarkable particle-like behavior (elastic scattering) exhibited by certain KdV solutions. Zabusky and Kruskal showed how the coherence of solitons explained the anomalous results observed by Fermi, Pasta, and Ulam. But in solving that mystery, they had uncovered a larger one; KdV solitons were unlike anything that had been seen before, and the search for an explanation of their remarkable behavior led to a series of discoveries that changed the course of applied mathematics for the next thirty years.

We next fill in some of the mathematical details behind the above sketch, beginning with a discussion of explicit solutions to the KdV equation.

Finding the travelling wave solutions of KdV is straightforward; if we substitute the Ansatz $u(x,t) = f(x - ct)$ into KdV we obtain the ODE $-cf' + 6ff' + f'''$, and adding as boundary condition that f should vanish at infinity, a routine computation leads to the two-parameter family of travelling-wave solutions $u(x,t) = 2a^2 \operatorname{sech}^2(a(x - 4a^2 t + d))$.

Exercise 2.1.1. Carry out the details of this computation. (Hint: Get a first integral by writing $6ff' = (3f^2)'$.)

These are the solitary waves seen by Russell, and they are now usually referred to as the 1-soliton solutions of KdV. Note that their amplitude, $2a^2$, is just half their speed, $4a^2$, while their "width" is proportional to a^{-1}; i.e., taller solitary waves are thinner and move faster.

These solutions were found by Korteweg and de Vries, who also carried out the more complicated calculations that arise when one assumes periodicity instead of decay as a boundary condition. The profile of the periodic travelling wave is given in terms of the Jacobi elliptic function cn,

$$u(x,t) = 2a^2k^2 \operatorname{cn}^2(a(x - 4(2k^2 - 1)a^2 t)),$$

and following Korteweg and de Vries they are called cnoidal waves. Here $0 \le k \le 1$ is the modulus of the elliptic function cn. Note that as the modulus $k \to 1$, cn converges to sech, and so the cnoidal waves have the solitary wave as a limiting case.

Next, following M . Toda [34], we will "derive" the n-soliton solutions of KdV. We first rewrite the 1-soliton solution as $u(x,t) = 2\frac{\partial^2}{\partial x^2} \log\cosh(a(x - 4a^2 t + \delta)$, or $u(x,t) = 2\frac{\partial^2}{\partial x^2} \log K(x,t)$ where $K(x,t) = (1 + e^{2a(x-4a^2t+\delta)})$.

We now try to generalize, looking for solutions of the form

$$u(x,t) = 2\frac{\partial^2}{\partial x^2} \log K(x,t),$$

with K of the form $K(x,t) = 1 + A_1 e^{2\eta_1} + A_2 e^{2\eta_2} + A_3 e^{2(\eta_1+\eta_2)}$, where $\eta_i = a_i(x - 4a_i^2 t + d_i)$, and we are to choose the A_i and d_i by substituting in KdV and seeing what works.

Exercise 2.1.2. Show that KdV is satisfied for $u(x,t)$ of this form and for arbitrary choices of $A_1, A_2, a_1, a_2, d_1, d_2$, provided only that we define

$$A_3 = \left(\frac{a_2 - a_1}{a_1 + a_2}\right)^2 A_1 A_2.$$

The solutions of KdV that arise this way are called the 2-soliton solutions of KdV.

Exercise 2.1.3. Show that if we take $A_i = \frac{1}{2a_i}$ then $K(x,t) = \det B(x,t)$, where $B(x,t)$ is the 2×2 matrix, $B_{ij}(x,t) = \delta_{ij} + \frac{1}{a_i + a_j} e^{\eta_i + \eta_j}$.

Yes, you guessed it, this generalizes in the obvious way. If we define an $n \times n$ matrix $B(x,t)$ with the matrix elements defined in the same way, then $u(x,t) = 2\frac{\partial^2}{\partial x^2} \log\det B(x,t)$ is a solution of KdV for all choices of a_i and d_i, and the resulting solutions of this form (a $2n$ parameter family) are called n-soliton solutions of KdV.

Of course this is a complete swindle! Only knowing the answer in advance allowed us to make the correct choice of Ansatz for K. Later we shall see how to get the n-soliton family of solutions for KdV in a completely straightforward way using the Inverse Scattering Method.

But, for now, we want to look more closely at the 2-soliton solutions, and more specifically their asymptotic behavior as t approaches $\pm\infty$. We could do this for an arbitrary 2-soliton, but for simplicity let us take $a_1 = a_2 = 3$.

Exercise 2.1.4. Show that for these choices of a_1 and a_2,

$$u(x,t) = 12\frac{3 + 4\cosh(2x - 8t) + \cosh(4x - 64t)}{[\cosh(3x - 36t) + 3\cosh(x - 28t)]^2},$$

so in particular $u(x,0) = 6\operatorname{sech}^2(x)$.

Exercise 2.1.5. Show that for t large and negative, $u(x,t)$ is asymptotically equal to $2\operatorname{sech}^2(x - 4t - \phi) + 8\operatorname{sech}^2(x - 16t + \frac{\phi}{2})$, while for t large and positive, $u(x,t)$ is asymptotically equal to $2\operatorname{sech}^2(x - 4t + \phi) + 8\operatorname{sech}^2(x - 16t - \frac{\phi}{2})$, where $\phi = \log(3)/3$. (This is hard. For the solution see [34], Chapter 6.)

Note what this says. If we follow the evolution from $-T$ to T (where T is large and positive), we first see the superposition of two 1-solitons; a larger and thinner one to the left of and overtaking a shorter, fatter, and slower-moving one to the right. Around $t = 0$ they merge into a single lump (with the shape $6\operatorname{sech}^2(x)$), and then they separate again, with their original shapes restored, but now the taller and thinner one is to the right. It is almost as if they had passed right through each other—the only effect of their interaction is the pair of phase shifts—the slower one is retarded slightly from where it would have been, and the faster one is slightly ahead of where it would have been. Except for these phase shifts, the final result is what we might expect from a linear interaction. It is only if we see the interaction as the two solitons meet that we can detect its highly nonlinear nature. (Note that at time $t = 0$, the maximum amplitude, 6, of the combined wave is actually less than the maximum amplitude, 8, of the taller wave when they are separated.) But of course the really striking fact is the resilience of the two individual solitons—their ability to put themselves back together after the collision. Not only is no energy radiated away, but their actual shapes are preserved.

(Remarkably, on page 384 of Russell's 1844 paper, there is a sketch of a 2-soliton interaction experiment that Russell had carried out in his wave tank!)

We shall see later that every initial profile u_0 for the KdV equation can be thought of as made up of two parts: an n-soliton solution for some n, and a dispersive "tail". The tail is transient, that is, it rapidly tends to zero in the sup norm (although its L^2 norm is preserved), while the n-soliton part behaves in the robust way that is the obvious generalization of the 2-soliton behavior we have just analyzed.

Now back to the computer experiment of Zabusky and Kruskal. For numerical reasons, they chose to deal with the case of periodic boundary conditions—in effect studying the KdV equation $u_t + uu_x + \delta^2 u_{xxx} = 0$ (which they label (1)) on the circle instead of on the line. For their published report, they chose $\delta = 0.022$ and used the initial condition $u(x,0) = \cos(\pi x)$. With the above background, it is interesting to read the following extract from their 1965 report, containing the first use of the term "soliton":

> (I) Initially the first two terms of Eq. (1) dominate and the classical overtaking phenomenon occurs; that is u steepens in regions where it has negative slope. (II) Second, after u has steepened sufficiently, the third term becomes important and serves to prevent the formation of a discontinuity. Instead, oscillations of small wavelength (of order δ) develop on the left of the front. The amplitudes of the oscillations grow, and finally *each* oscillation achieves an almost steady amplitude (that increases linearly from left to right) and has the shape of an individual solitary-wave of (1). (III) Finally, each "solitary wave pulse" or *soliton* begins to move uniformly at a rate (relative to the background value of u from which the pulse rises) which is linearly proportional to its amplitude. Thus, the solitons spread apart. Because of the periodicity, two or more solitons eventually overlap spatially and interact nonlinearly. Shortly after the interaction they reappear virtually unaffected in size or shape. In other words, solitons "pass through" one another without losing their identity. *Here we have a nonlinear physical process in which interacting localized pulses do not scatter irreversibly.*

Bibliography

1. Ablowitz, M.J.,Clarkson, P.A.,*Solitons, nonlinear evolution equations and inverse scattering*, Cambridge Univ. Press (1991)
2. Ablowitz, M.J., Kaup, D.j., Newell, A.C. and Segur, H., *Method for solving the Sine-Gordon equation*, Phy. Rev. Lett. **30** (1973),1262–1264
3. Ablowitz, M.J., Kaup, D.j., Newell, A.C. and Segur, H., *The inverse scattering transform—Fourier analysis for nonlinear problems*, Stud. Appl. Math. **53** (1974), 249–315
4. Abraham, R., Marsden, J.E., *Foundations of Mechanics*, Benjamin/Cummings (1978)
5. Adams,R.A., *Sobolev Spaces*, Academic Press (1975)
6. Adler, M., *On a trace functional for formal pseudo-differential operators and the symplectic structure of the Korteweg-de Vries equation*, Invent. Math. **50** (1979), 219–248
7. Arnold, V.I., *Mathematical Methods of Classical Mechanics*, Springer-Verlag (1978)
8. Bona, J.L. and Smith, R., *The Initial-Value Problem for the Korteveg-de Vries Equation*, Philos. Trans. Royal Soc. London, Series A **278** (1975), 555–604
9. Drazin, P.G., Johnson, R.S., *Solitons: an introduction*, Cambridge Univ. Press (1989)
10. Forneberg,B. and Whitham, G.B., *A Numerical and Theoretical Study of Certain Nonlinear Wave Phenomena*, Proc. R. Soc. Lond. A **289** (1978), 373–403
11. Faddeev, L.D., Takhtajan, L.A., *Hamiltonian methods in the theory of solitons*, (1987), Springer-Verlag
12. Fermi, E., Pasta, J., Ulam, S., *Studies of Nonlinear Problems. I*, Lectures in Applied Math. **15** (1974), 143–155
13. Gardner, C.S., Greene, J.M., Kruskal, M.D., Miura, R.M., *Method for solving the Korteweg-de Vries equation*, Physics Rev. Lett. **19** (1967), 1095-1097
14. Gel'fand, I.M., Levitan, B. M., *On the determination of a differential equation from its spectral function*, Izv. Akad. Nauk SSSR Ser. Mat. **15** (1951), 309–366
15. Hamming, R.W., *The Unreasonable Effectiveness of Mathematics*, Amer. Math. Monthly **87** (1980)
16. John, F., *Nonlinear Wave Equations, Formation of Singularities*, American Math. Soc., ULS (1990)
17. Kato, T., *On the Cauchy Problem for the (Generalized) Korteweg-de Vries Equation*, Studies in Applied Math., Adv. in Math. Supp. Stud. **8** (1983), 93–128
18. Kato, T., *Quasi-linear equations of evolution, with applications to partial differential equations*, Lecture Notes in Math., vol. **448**, Springer-Verlag, Berlin and New York (1988), 27–50
19. Kay, B., Moses, H.E., *The determination of the scattering potential from the spectral measure function, III*, Nuovo Cim. **3** (1956), 276–304
20. Kostant, B., *The solution to a generalized Toda lattice and representation theory*, Adv. Math. **34** (1979), 195-338
21. Korteweg, D.J., de Vries, G., *ON the change of form of long waves advancing in a rectangular canal, and on a new type of long stationary waves*, Philos. Mag. Ser. 5 **39** (1895), 422–443
22. Lax, P.D., *Integrals of nonlinear equations of evolution and solitary waves*, Comm. Pure. Appl. Math. **21** (1968), 467–490
23. Lax, P.D., *Hyperbolic Systems of Conservation Laws and the Mathematical Theory of Shock Waves*, SIAM, CBMS **11** (1973)
24. Lax, P.D., *Outline of a theory of the KdV equation in Recent Mathematical Methods in Nonlinear Wave Propagation*, Lecture Notes in Math., vol. **1640**, Springer-Verlag, Berlin and New York 1996, 70–102
25. Marchenko, V.A., *On the reconstruction of the potential energy from phases of the scattered waves*, Dokl. Akad. Nauk SSSR **104** (1955), 695–698
26. Wineberg, S.B., Gabl,E.F., Scott, L.R. and Southwell, C.E.*Implicit Spectral Methods for Wave Propogation Problems*, J. Comp. Physics **97** (1991), 311–336
27. Palais,R.S., *Foundations of Global Non-linear Analysis*, Benjamin and Co. 1968
28. Palais, R.S., *The symmetries of solitons*, Bulletin AMS, **34** (1997), 339-403
29. Russell, J.S., *Report on Waves//14th Mtg. of the British Assoc. for the Advance. of Science*, John Murray, London, pp. 311–390 + 57 plates (1844)
30. Sattinger, D.H., *Weak and Strong Nonlinear Waves*, preprint
31. Strang, G., *On the Construction and Comparison of Difference Schemes*, SIAM J. Numerical Analysis **5** (1968),
32. Symes, W.W., *Systems of Toda type, Inverse spectral problems, and representation theory*, Inventiones Math. **59** (1980), 13-51
33. Tappert, F., *Numerical Solutions of the Korteweg-de Vries Equations and its Generalizations by the Split-Step Fourier Method*, in Nonlinear Wave Motion, Lecture Notes in Applied Math. (AMS) v. **15** (1974), 215–216
34. Toda, M., *Nonlinear Waves and Solitons*, Kluwer 1989
35. Wigner, E., *The Unreasonable Effectiveness of Mathematics in the Natural Sciences*, Comm. in Pure and Appl. Math. **13** (1960)

36. Zakharov, V.E., Faddeev, L.D., *Korteweg-de Vries equation, a completely integrable Hamiltonian system*, Func. Anal. Appl. **5** (1971), 280-287
37. Zakharov, V.E., Shabat, A.B., *Exact theory of two-dimensional self-focusing and one-dimensional of waves in nonlinear media*, Sov. Phys. JETP **34** (1972), 62-69
38. Zabusky, N.J., Kruskal, M.D., *Interaction of solitons in a collisionless plasma and the recurrence of initial states*, Physics Rev. Lett. **15** (1965), 240–243