manifold. Thus a cartesian factor of a homotopy manifold is not necessarily a homotopy manifold.

References

1. R. Bing, *A decomposition of E^3 into points and tame arcs such that the decomposition space is topologically different from E^3*, Ann. of Math. vol. 65 (1957).

2. M. Curtis and R. Wilder, *The existence of certain types of manifolds*, Trans. Amer. Math. Soc. vol. 91 (1959) pp. 152–160.

3. H. Griffiths, *A contribution to the theory of manifolds*, Michigan Math. J. vol. 2 (1953) pp. 61–89.

4. R. Roxen, *E^4 is the Cartesian product of a totally non-euclidean space and E^1*, Notices Amer. Math. Soc., Abstr. 563-1, vol. 6 (1959) p. 641.

5. S. Smale, *A Vietoris mapping theorem for homotopy*, Proc. Amer. Math. Soc. vol. 8 (1957) pp. 604–610.

6. R. Wilder, *Monotone mappings of manifolds*, Pacific J. Math. vol. 3 (1957) pp. 1519–1528.

7. ———, *Monotone mappings of manifolds* II, Michigan Math. J. vol. 5 (1958) pp. 19–25.

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LOGARITHMICALLY EXACT DIFFERENTIAL FORMS

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Let M be a connected, differentiable ($=C^\infty$) manifold. Let $C^1(M, \mathbf{C})$ denote the complex vector space of complex valued one-forms on M: an element ω of $C^1(M, \mathbf{C})$ is a function which assigns to each $x\in M$ a linear map ω_x of M_x (the tangent space to M at x, a real vector space) into the complex numbers $\mathbf{C}$, such that if X is a differentiable vector field on M then $x\to\omega_x(X_x)$ is a differentiable complex valued function on M. Each element ω of $C^1(M, \mathbf{C})$ can be written uniquely in the form $\mu+i\nu$ where μ and ν are real valued one-forms on M, and we put $\mu=\operatorname{Re}\omega$ and $i\nu=\operatorname{Im}\omega$. We write $Z^1(M, \mathbf{C})$ for the subspace of $C^1(M, \mathbf{C})$ consisting of closed forms and $B^1(M, \mathbf{C})$ for the subspace of $Z^1(M, \mathbf{C})$ consisting of exact forms.

An element of $C^1(M, \mathbf{C})$ will be called *logarithmically exact* if it is of the form df/f for some nowhere vanishing, differentiable, complex valued function f on M. Since $d(df/f)=(fd^2f-df\wedge df)/f^2=0$ and $df/f-dg/g=d(f/g)/(f/g)$ it is clear that the set $L^1(M, \mathbf{C})$ of logarithmically exact one-forms is a subgroup (but not in general a subspace)

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of $Z^1(M, \mathbf{C})$ and, since $dg = d(\exp g)/(\exp g)$, $B^1(M, \mathbf{C})$ is a subgroup of $L^1(M, \mathbf{C})$. The quotient group $L^1(M, \mathbf{C})/B^1(M, \mathbf{C})$ is thus a subgroup of the complex vector space $H^1(M, \mathbf{C}) = Z^1(M, \mathbf{C})/B^1(M, \mathbf{C})$ (the one-dimensional complex deRham cohomology space of M) and so a torsion free abelian group which we will call the group of *logarithmic cohomology classes* of M.[2]

If ω is a logarithmically exact one-form on a differentiable manifold N and $\Phi: M \to N$ is a differentiable map then $\delta\Phi^*(\omega)$ is a logarithmically exact one-form on M. In fact if $\omega = df/f$ then $\delta\Phi^*(\omega) = d(f \circ \Phi)/(f \circ \Phi)$. Now if I denotes the identity map of the unit circle into the complex plane then $\lambda = dI/I$ is a logarithmically exact one-form on the circle. If θ is a local determination of the argument function on the circle then $\lambda = d(e^{i\theta})/(e^{i\theta}) = id\theta$ so λ is purely imaginary. A logarithmically exact one-form on M will be called *induced* if it is of the form $\delta\Phi^*(\lambda)$ for some differentiable map Φ of M into the unit circle. Clearly such a form is purely imaginary. Since $\delta\Phi^*(\lambda) = \delta\Phi^*(dI/I) = d(I \circ \Phi)/(I \circ \Phi)$ $= d\Phi/\Phi$ a logarithmically exact one-form is induced if and only if it can be written in the form df/f where $|f| \equiv 1$. Now if f is a nowhere vanishing function on M and then $|f|$ is differentiable, $g = f/|f|$ is differentiable, and $|g| \equiv 1$. Since $df/f = d(|f|g)/(|f|g) = d|f|/|f|$ $+ dg/g = d \log |f| + \delta g^*(\lambda)$ and since $d \log |f|$ is clearly real while as remarked above $\delta g^*(\lambda)$ is purely imaginary we see that: *If ω is a logarithmically exact one-form on M then* Re ω *is exact and* Im ω *is an induced logarithmically exact form.* It follows in particular that real logarithmically exact forms are exact, imaginary logarithmically exact forms are induced, and that *every logarithmic cohomology class contains an induced logarithmically exact one-form.*

We shall now examine how logarithmically exact forms can be characterized by their periods. A *path* on M is a piecewise differentiable map γ of a closed interval $[a, b]$ into M and is called a *cycle* if $\gamma(a)$ $= \gamma(b)$. If ω is a one-form on M we write $\omega(\gamma)$ for the integral of ω over γ. If ω is closed and γ a cycle then $\omega(\gamma)$ depends only on the cohomology class $[\omega]$ of ω in $H^1(M, \mathbf{C})$ and the integral homology class $[\gamma]$ of γ in $H_1(M, \mathbf{Z})$ and we write it as $([\gamma], [\omega])$. The set of numbers of the form $([\gamma], [\omega])$ as $[\gamma]$ varies over $H_1(M, \mathbf{Z})$ is a sub-

[2] The author was introduced to the notion of logarithmically exact forms in a conversation with Professor I. E. Segal. They arose naturally while Segal was investigating certain generalizations of quantum field theory that occur when Euclidean three-space is replaced by more general manifolds. In a certain sense $L^1(M, \mathbf{C})/B^1(M, \mathbf{C})$, the group of logarithmic cohomology classes, classifies the unitary equivalence classes of certain natural representations of the generalized "Heisenberg commutator relations." It was Segal's conjecture that this group had a rank equal to the first Betti number of M (Theorem B) that motivated the author's work on this paper.

group of C called the group of periods of $[\omega]$ (or of ω) and denoted by $\Omega([\omega])$ (or by $\Omega(\omega)$). We now come to our main result. Let $2\pi i\boldsymbol{Z}$ denote the subgroup of C consisting of integral multiples of $2\pi i$. Then

THEOREM A. *A closed one-form ω on M is logarithmically exact if and only if its group $\Omega(\omega)$ of periods is a subgroup of $2\pi i\boldsymbol{Z}$. Equivalently a one-dimensional cohomology class of M is logarithmic if and only if its group of periods is a subgroup of $2\pi i\boldsymbol{Z}$.*

PROOF. If ω is logarithmically exact then by a previous remark it is cohomologous to an induced logarithmically exact form $\delta\Phi^*(\lambda)$. If γ is a cycle in M then $\Phi \circ \gamma$ is a cycle in the circle which is then an integral multiple $n\gamma_0$ of the fundamental cycle γ_0 of the circle. Then $\omega(\gamma)=\delta\Phi^*(\lambda)(\gamma)=\lambda(\Phi\circ\gamma)=n\lambda(\gamma_0)$. Since $\lambda(\gamma_0)=\int_{\theta=0}^{\theta=2\pi}d(e^{i\theta})/e^{i\theta}=2\pi i$ it follows that $\Omega(\omega)\subseteq 2\pi i\boldsymbol{Z}$. Conversely suppose $\Omega(\omega)\subseteq 2\pi i\boldsymbol{Z}$. Let p_0 be a fixed point of M and define a complex valued function f on M by $f(p)=\exp(\omega(\gamma))$ where γ is a path joining p_0 to p. If γ_1 and γ_2 are two paths joining p_0 to p then $\gamma_1-\gamma_2$ is a cycle so $\omega(\gamma_1)-\omega(\gamma_2)$ $=\omega(\gamma_1-\gamma_2)$ is an integral multiple of $2\pi i$ and hence $\exp(\omega(\gamma_1))$ $=\exp(\omega(\gamma_2))$ so f is well defined. In any simply connected region we can write ω locally as dg and then clearly in this region we have $f=ce^g$ for some constant c, and hence $df/f=dg=\omega$. Since this holds in a neighborhood of each point we have in fact $\omega=df/f$ so ω is logarithmically exact. q.e.d.

We now make the assumption that the first Betti number b_1 of M is finite, i.e. that $H_1(M, \boldsymbol{Z})$/Torsion is finitely generated. Let $\gamma_1, \cdots, \gamma_{b_1}$ be a basis for $H_1(M, \boldsymbol{Z})$/Torsion. By deRham's theorem we can choose closed one-forms $\omega_1, \cdots, \omega_{b_1}$ on M such that $\omega_j(\gamma_k)$ $=\delta_{jk}2\pi i$. Then clearly $\Omega(\omega_i)\subseteq 2\pi i\boldsymbol{Z}$ so by Theorem A each ω_i is a logarithmically exact form, and changing ω_i within its cohomology class if necessary we can assume ω_i is induced. Moreover it follows from Theorem A that a linear combination $\sum_{i=1}^{b_1} C_i\omega_i$ of the ω_i will be logarithmically exact if and only if each C_i is integral. Now since a closed one-form has its cohomology class completely determined by its periods over $\gamma_1, \cdots, \gamma_{b_1}$ it follows that the linear combinations of the ω_i form a linear complement to $B^1(M, \boldsymbol{C})$ in $Z^1(M, \boldsymbol{C})$. Hence:

THEOREM B. *If the first Betti number b_1 of M is finite then the group of logarithmic cohomology classes of M is a free abelian group with b_1 generators. More precisely there exist b_1 induced logarithmically exact forms $\omega_1, \cdots, \omega_{b_1}$ such that their cohomology classes $[\omega_i]$ form an integral basis for $L^1(M, \boldsymbol{C})/B^1(M, \boldsymbol{C})$ and a complex basis for $H^1(M, \boldsymbol{C})$.*

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