

MANIFOLDS OF SECTIONS OF FIBER BUNDLES AND THE CALCULUS OF VARIATIONS

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Let M be a compact, n -dimensional, C^∞ manifold, possibly with boundary. Denote by $FB(M)$ the category of C^∞ fiber bundles over M . A morphism $f: E \rightarrow F$ of $FB(M)$ is a C^∞ map such that $f_x = f|E_x$ maps E_x into F_x for each $x \in M$, where E_x is the fiber of E at x . If E and F are C^∞ vector bundles and each f_x is linear we call f a vector bundle morphism. We denote by $VB(M)$ the category of C^∞ vector bundles over M and vector bundle morphisms and by $FVB(M)$ the mongrel category of C^∞ vector bundles over M and fiber bundle morphisms.

Our goals in this talk are the following:

(1) To describe the Sobolev Functors L_k^p ($k > n/p$) from $FB(M)$ to the category of C^∞ Banach manifolds and C^∞ maps. (As a set $L_k^p(E)$ is a certain subset of the continuous sections of E , roughly speaking those which in local coordinates and local trivializations have derivatives of order $\leq k$ which are p th power summable. If $f: E \rightarrow F$ is a morphism then $L_k^p(f): L_k^p(E) \rightarrow L_k^p(F)$ is given by $s \mapsto f \circ s$.)

(2) To interpret the Calculus of Variations as the study of the critical points of a certain type of differentiable real valued functions on the manifolds $L_k^p(E)$.

(3) To describe how using the latter point of view the strong global existence theorems provided by Morse Theory and Lusternik-Schnirelman theory, as well as certain classical smoothness theorems, extend naturally from one independent variable problems to certain problems with several independent variables in the calculus of variations.

A considerably more detailed account of the ideas and results presented here will be found in the authors *Foundations of global non-linear analysis* published by W. A. Benjamin Inc. (1968) and in the doctoral dissertation of Mrs. Karen Uhlenbeck (Brandeis, 1968).

1. The L_k^p spaces of a vector bundle. Let ξ be a C^∞ vector bundle over M and $1 \leq p < \infty$. Given a strictly positive smooth measure μ on M and a Riemannian structure $\langle \cdot, \cdot \rangle$ for ξ define $L^p(\xi)$ to be the Banach space of measurable sections s of ξ such that $\|s\|_{L^p}^p = \int \langle s(x), s(x) \rangle^{p/2} d\mu(x)$ is finite. If we change μ or the Riemannian structure we get the same topological vector space with an equivalent norm, i.e., $L^p(\xi)$ is a well-defined topological vector space independent of any

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particular choice of μ or $\langle \cdot, \cdot \rangle$. Let $J^k(\xi)$ denote the k -jet bundle of ξ and $j_k: C^\infty(\xi) \rightarrow C^\infty(J^k(\xi))$ the k -jet extension map. We denote by $L_k^p(\xi)$ (k a nonnegative integer) the completion of $C^\infty(\xi)$ under the topology induced from the injection

$$C^\infty(\xi) \xrightarrow{j_k} C^\infty(J^k(\xi)) \xrightarrow{\subseteq} L^p(J^k(\xi)).$$

We note that we have a continuous inclusion $L_{k+1}^p(\xi) \subseteq L_k^p(\xi)$ which by Rellich's theorem is even completely continuous. This allows us to define $L_k^p(\xi)$ for non-integral positive k by using the "complex method of interpolation" between the integral values. If $f: \xi \rightarrow \eta$ is a vector bundle morphism over M then $s \mapsto f \circ s$ defines a continuous linear map $L_k^p(f): L_k^p(\xi) \rightarrow L_k^p(\eta)$. This establishes each L_k^p as a functor from $VB(M)$ to the category of Banach spaces and continuous linear maps.

If $f: \xi \rightarrow \eta$ is a fiber bundle morphism of vector bundles (i.e. a morphism of $FVB(M)$) then for $pk > n$, $s \mapsto f \circ s$ still defines a continuous (but of course nonlinear) map $L_k^p(f): L_k^p(\xi) \rightarrow L_k^p(\eta)$ and it follows easily that $L_k^p(f)$ must indeed be C^∞ (if $pk \leq n$ then in general $f \circ s$ will not even be in $L_k^p(\eta)$ for all $s \in L_k^p(\xi)$). Thus we have

THEOREM. For $pk > n$, L_k^p extends to a functor from $FVB(M)$ to the category of Banach spaces and C^∞ maps.

The above is a fundamental result for our approach to nonlinear analysis in general and the calculus of variations in particular and we sketch the basic reasons for it.

First of all we have the classical results:

SOBOLEV EMBEDDING THEOREMS. If $k - n/p \geq l - n/q$ and $k \geq l$ then $L_k^p(\xi) \subseteq L_l^q(\eta)$ and the inclusion map is continuous (and even completely continuous if the inequalities are strict). Also if $k - n/p = l + \alpha$, $0 < \alpha < 1$ then $L_k^p(\xi) \subseteq C^{l+\alpha}(\eta)$ and the inclusion is continuous.

Given vector bundles $\xi_1, \dots, \xi_r, \eta$ over M let $L^r(\xi_1, \dots, \xi_r, \eta)$ denote the bundle of r -linear maps of $\xi_1 \oplus \dots \oplus \xi_r$ into η . Given a section T of the latter and sections s_i of ξ_i we get a section $T(s_1, \dots, s_r)$ of η whose value at x is $T(x)(s_1(x), \dots, s_r(x))$. The following is an easy generalization (and consequence) of Hölder's inequality.

THEOREM. The function $(T, s_1, \dots, s_r) \mapsto T(s_1, \dots, s_r)$ defines a continuous multilinear map of $C^0(L^r(\xi_1, \dots, \xi_r, \eta)) \oplus L^{q_1}(\xi_1) \oplus \dots \oplus L^{q_r}(\xi_r)$ into $L^q(\eta)$ provided $1/q \geq \sum_{i=1}^r 1/q_i$.

Putting the latter together with the Sobolev embedding theorem we obtain the following basic result.

MULTIPLICATION THEOREM. The function $(T, s_1, \dots, s_r) \mapsto T(s_1, \dots, s_r)$ is a continuous multilinear map of $C^0(L^r(\xi_1, \dots, \xi_r, \eta)) \oplus L_{k_1}^{p_1}(\xi_1) \oplus \dots \oplus L_{k_r}^{p_r}(\xi_r)$ into $L_l^q(\eta)$ provided $n/q - l \geq \sum_{i \in A} (n/p_i - k_i)$, where A is the set of indices $i = 1, 2, \dots$.

r such that $n/p_i - k_i > 0$ and where the inequality must be strict if, for some i , $n/p_i = k_i$.

It is from the latter result that one derives, rather straightforwardly, that for $pk > n$ a fiber bundle morphism $f: \xi \rightarrow \eta$ defines a C^∞ map $s \mapsto f \circ s$ of $L_k^p(\xi)$ to $L_k^p(\eta)$. The details will be found in §9 of *Foundations of global non-linear analysis*.

2. Vector bundle neighborhoods and the differentiable structure of $L_k^p(E)$. If E and F are C^∞ fiber bundles over M , the F is called a sub-bundle of E if $F \subseteq E$ and the inclusion map is a fiber bundle morphism. If in addition F is open (closed) in E it is called an open (closed) sub-bundle of E . By a vector bundle neighborhood (VBN) in E we shall mean a vector bundle ξ over M which, considered as a C^∞ fiber bundle, is an open sub-bundle of E . We have the following fundamental existence theorem, whose proof is analogous to that of the tubular neighborhood theorem.

EXISTENCE THEOREM FOR VBN. *If E is a C^∞ fiber bundle over M , $s \in C^0(\xi)$, and \mathcal{O} any neighborhood of $s(M)$ in E then there exists a VBN ξ in E with $s(M) \subseteq \xi \subseteq \mathcal{O}$ (and in particular $s \in C^0(\xi)$). If $s \in C^\infty(E)$ we can choose ξ so that s is the zero section of ξ .*

Now let E be a C^∞ fiber bundle over M , $s \in C^0(E)$, and ξ a VBN of s in E (i.e. a VBN in E such that $s \in C^0(\xi)$), the existence of which is assured by the above theorem. If $pk > n$ we say $s \in L_k^p(E)$ if $s \in L_k^p(\xi)$. It is easily seen that this condition is independent of the choice of the VBN of s , i.e. that if $s \in L_k^p(\xi)$ then $s \in L_k^p(\eta)$ for every VBN η of s in E . Thus $L_k^p(E)$ is the union of the $L_k^p(\xi)$ over all VBN ξ of E . Moreover the theorem of the §1, that L_k^p is a functor from $FVB(M)$ to Banach manifolds and C^∞ maps leads easily to the following

L_k^p MANIFOLD STRUCTURE THEOREM. *If $pk > n$ then for each C^∞ fiber bundle E over M there is a unique C^∞ Banach manifold structure for $L_k^p(E)$ such that, for each VBN ξ of E , $L_k^p(\xi)$ is an open submanifold of $L_k^p(E)$. Moreover if $f: E \rightarrow F$ is a C^∞ fiber bundle morphism then $s \mapsto f \circ s$ defines a C^∞ map $L_k^p(f): L_k^p(E) \rightarrow L_k^p(F)$. This establishes L_k^p as a functor from $FVB(M)$ to the category of C^∞ Banach manifolds and C^∞ maps.*

DEFINITION. Let E be a C^∞ fiber bundle over M , $pk > n$ and $\sigma \in L_k^p(E)$. We define a subset $L_k^p(E)_{\partial\sigma}$ of $L_k^p(E)$, called the Dirichlet subspace of $L_k^p(E)$ defined by σ . Namely $L_k^p(E)_{\partial\sigma}$ is the closure in $L_k^p(E)$ of the set of $s \in L_k^p(E)$ which agree with σ in some neighborhood U (depending on s) of ∂M .

THEOREM. *If $pk > n$ and E is a C^∞ fiber bundle over M then for each $\sigma \in L_k^p(E)$, $L_k^p(E)_{\partial\sigma}$ is a closed C^∞ submanifold of $L_k^p(E)$.*

REMARK. Of course if $\partial M = \emptyset$ then $L_k^p(E)_{\partial\sigma} = L_k^p(E)$.

3. The calculus of variations. In what follows we suppose that there is given a strictly positive smooth measure μ on M and we let E denote a C^∞ fiber bundle over M . The space of k -jets of local sections of E , regarded as a C^∞ fiber bundle over E , will be denoted by $J_0^k(E)$. It can also be regarded as a C^∞ fiber bundle over E in

which case we denote it by $J^k(E)$. As usual $j_k: C^\infty(E) \rightarrow C^\infty(J^k(E))$ denotes the k -jet extension map.

If F is a C^∞ real valued function on $J^k(E)$ then for each $s \in C^\infty(E)$ we get a real valued function $L(s)$ on M by $L(s)(x) = F(j_k(s)_x)$. Such a function $L: C^\infty(E) \rightarrow C^\infty(\mathbf{R}_M)$ (where $\mathbf{R}_M = M \times \mathbf{R}$ is the product line bundle over M) is called a k th order Lagrangian for E . The set of all k th order Lagrangians for E is a vector space $\text{Lgn}_k(E)$.

DEFINITION. If $pk > n$ then we shall call an element $L \in \text{Lgn}_k(E)$ L_k^p -smooth if $L: C^\infty(E) \rightarrow C^\infty(\mathbf{R}_M)$ extends to a C^∞ map (clearly unique) $L: L_k^p(E) \rightarrow L^1(\mathbf{R}_M)$. Since the linear map $f \mapsto \int f(x) d\mu(x)$ of $L^1(\mathbf{R}_M) \rightarrow \mathbf{R}$ is continuous it follows that such a Lagrangian defines a C^∞ map $J^L = J: L_k^p(E) \rightarrow \mathbf{R}$ by $J(s) = \int L(s)(x) d\mu(x)$ and then by restriction we have a C^∞ function $J: L_k^p(E)_{\partial\sigma} \rightarrow \mathbf{R}$ on each Dirichlet subspace of $L_k^p(E)$.

Roughly speaking the "Dirichlet Problem" associated to a given L_k^p -smooth Lagrangian L and "boundary conditions" $\sigma \in L_k^p(E)$ is to describe the critical locus of $J: L_k^p(E)_{\partial\sigma} \rightarrow \mathbf{R}$. In particular one wants to find criteria for the following:

(a) LUSTERNIK-SCHNIRELMAN EXISTENCE THEOREM. This means that on each component of $L_k^p(E)_{\partial\sigma}$ there should be at least as many critical points as the Lusternik-Schnirelman category of that component (i.e. the smallest number of closed sets, each contractible in the component, needed to cover the component).

(b) MORSE EXISTENCE THEOREMS (in case $p = 2$). This means the critical points should all be nondegenerate and of finite index and the type numbers should satisfy the Morse relations.

(c) SMOOTHNESS THEOREMS. This means that certain smoothness hypotheses on σ should imply corresponding smoothness conclusions for critical points (e.g. if $\sigma \in L_{k+r}^p(E)$) we would like all critical points to be in $L_{k+r}^p(E)$ so in particular if $\partial M = \emptyset$ or $\sigma \in C^\infty(E)$ then we should have the generalized "Weyl Lemma," that all critical points are automatically C^∞ .

Before going on to see what can be said in this respect we first consider the basic question of finding criteria for when a Lagrangian is L_k^p smooth. It turns out that the Lagrangians that occur naturally in geometric and physical applications have a certain polynomial-like character which we shall now explain, and that the degree (or rather the "weight") of this polynomial determines the p for which the Lagrangian is L_k^p -smooth. Thus the nature of the Lagrangian itself picks out for us the manifolds $L_k^p(E)$ on which it is natural to consider the corresponding Dirichlet problem.

Consider first the case when M is the n -disc D^n and E is the product vector bundle $E = D^n \times V$. Then $J^k(E) = (D^n \times V) \times \bigoplus_{|\alpha| \leq k} V$, the sum being over all n -multi-indices $\alpha = (\alpha_1, \dots, \alpha_n)$ with $|\alpha| = \alpha_1 + \dots + \alpha_n \leq k$. If $s \in C^\infty(E)$ then s is given by a C^∞ map $x \mapsto (x, \sigma(x))$ of D^n into $D^n \times V$ and its k -jet at x is given by

$$j_k(s)_x = ((x, \sigma(x)), D^\alpha \sigma(x))$$

where $D^\alpha = \partial^{|\alpha|}/\partial x_1^{\alpha_1} \cdots \partial x_n^{\alpha_n}$. Let $((x, p_0), p^z)$ denote the natural "coordinate projections" of $J^k(E)$. A C^∞ function $F(x, p_0, p^z)$ on $J^k(E)$ will be called a "monomial function of weight $\leq \omega$ " if it is of the form

$$F(x, p_0, p^z) = L(x, p_0)(p^{z_1}, \dots, p^{z_j})$$

where L is a C^∞ map of $D^n \times V$ into the vector space $L^j(V; \mathbf{R})$ of j -linear maps of V into \mathbf{R} and $|\alpha_1| + \cdots + |\alpha_j| \leq \omega$. A finite sum of monomials of weight $\leq \omega$ will be called a *polynomial function of weight $\leq \omega$* .

Let $L \in \text{Lgn}_k(E)$ be a k th order Lagrangian for E , say $L(s)(x) = F(j_k(s)_x)$. Then L will be said to be polynomial of weight $\leq \omega$ if F is a polynomial function of weight $\leq \omega$. The polynomial k th order Lagrangians of weight $\leq \omega$ clearly form a vector subspace of the vector space of all k order Lagrangians for E which we denote by $\text{Lgn}_k^\omega(E)$. Now at first glance the space $\text{Lgn}_k^\omega(E)$ seems to depend on the choice of coordinates in D^n and, even more, on the vector space structure of V , so the following may be a little surprising.

THEOREM ON INVARIANCE OF POLYNOMIAL LAGRANGIANS. *Let $q: D^n \rightarrow D^n$ be a C^∞ map and let $f: E \rightarrow E$ be a C^∞ fiber bundle morphism (i.e. a C^∞ map $(x, v) \rightarrow (x, h(x, v))$ of $D^n \times V$ into $D^n \times V$), and define $T: C^\infty(E) \rightarrow C^\infty(E)$ as follows: if $s: x \rightarrow (x, \sigma(x))$ is in $C^\infty(E)$ then $Ts(x) = (x, h(q(x), s(q(x))))$. Then if $L \in \text{Lgn}_k^\omega(E)$ so also is $L \circ T$.*

Of course the proof is just a simple inductive application of the "chain rule." The above theorem justifies the following definition.

DEFINITION. Let E be a C^∞ fiber bundle over M and $L \in \text{Lgn}_k(E)$. We say $L \in \text{Lgn}_k^\omega(E)$ if given any chart $q: D^n \rightarrow M$, any VBN ξ of $E|_q(D^n)$ and any trivialization $f: D^n \times V \approx q^*\xi$ the map $L^*: C^\infty(D^n, V) \rightarrow \mathbf{R}$ defined by $L^*(s)(x) = L(f \circ s \circ q^{-1}) \times (q(x))$, is in $\text{Lgn}_k^\omega(D^n \times V)$. Equivalently it suffices that given any $e \in E$ the latter hold for at least one such choice of q , ξ , and f with $e \in \xi$.

The importance of the classes $\text{Lgn}_k^\omega(E)$ for the calculus of variations, aside from the fact that they include the usual geometric and physically interesting examples, lies in the following:

SMOOTHNESS THEOREMS FOR POLYNOMIAL LAGRANGIANS. *Let $1 \leq p < \infty$, and $pk > n$. Then $L \in \text{Lgn}_k^\omega(E)$ is L_k^p -smooth provided $\omega \leq pk$. In particular, for p integral, all $L \in \text{Lgn}_k^{pk}(E)$ are L_k^p -smooth.*

Now for some examples. Let W be a C^∞ manifold and let E denote the product bundle $M \times W$ over M . We identify $C^\infty(E)$ with the C^∞ maps of M into W . If $s \in C^\infty(E)$ then ds_x is a linear map $T(M)_x \rightarrow T(W)_{s(x)}$. Now suppose both M and W are Riemannian. Then the space of linear maps of $T(M)_x$ into $T(W)_{s(x)}$ has a natural quadratic "Hilbert-Schmidt" norm (given by $\|T\|^2 = \text{tr} T^*T$) so in particular we can form $\|ds_x\|^p$ for $1 \leq p < \infty$, and if p is an even integer then it is easily checked that $L: C^\infty(E) \rightarrow C^\infty(M, \mathbf{R})$ defined by $L(s)(x) = \|ds_x\|^p$ belongs to $\text{Lgn}_1^p(E)$. We therefore have a natural C^∞ real valued function J on $L_1^p(E)$, called the energy function of degree p , defined by $J(s) = \int \|ds_x\|^p d\mu(x)$ where for μ we

take the natural Riemannian measure on M . If $n = 1$, so that M is (up to isometry) either the interval $[0, l]$ or the circle of length l , then we can take $p = 2$ (so that $L_1^p(E)$ is a Hilbert manifold) and we get for J the classical energy function $J: L_1^2(E) \rightarrow \mathbf{R}$, given by $J(s) = \int_0^l \|s'(t)\|^2 dt$. The generalized Dirichlet problem mentioned above reduces in this case to the classical theory of geodesics of W (with given end points or closed depending on whether M is an interval or a circle) and in particular the classical smoothness theorems and Lusternik-Schnirelman and Morse existence theorems for geodesics become special cases of the more general theorems of this sort for the Dirichlet Problems associated to Lagrangians of the "coercive linearly-embedded" type which we consider in the next section.

We now give a very general method for constructing polynomial Lagrangians (which in particular includes the above example). We assume our bundle E is embedded as a closed sub-bundle of a C^∞ vector bundle ξ over M . This in itself is no restriction; for example we can find a proper C^∞ embedding f of E , considered as a C^∞ manifold, into a vector space V and then if $\pi: E \rightarrow M$ is the projection, $e \mapsto (\pi(e), f(e))$ is a C^∞ proper fiber bundle isomorphism of E onto a closed C^∞ sub-bundle of the product vector bundle $M \times V$. We note that $L_k^p(E)$ ($pk > n$) is a C^∞ submanifold of the Banach space $L_k^p(\xi)$. If $L: C^\infty(\xi) \rightarrow C^\infty(M, \mathbf{R})$ is an element of $\text{Lgn}_k^\omega(\xi)$ then it is easily checked that $L|C^\infty(E): C^\infty(E) \rightarrow C^\infty(M, \mathbf{R})$ is an element of $\text{Lgn}_k^\omega(E)$. From this we see easily the following.

LINEARLY EMBEDDED LAGRANGIAN THEOREM. *Let the C^∞ fiber bundle E over M be a closed sub-bundle of a C^∞ vector bundle ξ . Let η_1, \dots, η_r be C^∞ vector bundles over M and let $A_i: C^\infty(\xi) \rightarrow C^\infty(\eta_i)$ be C^∞ linear differential operators of order $k_i \leq k$. Let T be a C^∞ section of the bundle $L^r(\eta_1, \dots, \eta_r; \mathbf{R}_M)$ of r -linear functionals on $\eta_1 \oplus \dots \oplus \eta_r$. Then $L: C^\infty(E) \rightarrow C^\infty(M, \mathbf{R})$ defined by $L(s)(x) = T(x)(A_1 s(x), \dots, A_r s(x))$ is an element of $\text{Lgn}_k^\omega(E)$ where $\omega = k_1 + \dots + k_r$. In particular if $\|\cdot\|_i$ is a C^∞ Riemannian structure for η_i and p is an even positive integer then $L: C^\infty(E) \rightarrow C^\infty(M, \mathbf{R})$ defined by $Ls(x) = \sum_{i=1}^r \|A_i s(x)\|_i^p$ is in $\text{Lgn}_k^{pk}(E)$.*

Note that if W is a Riemannian manifold and we embed W isometrically in an orthogonal vector space V (so $E = M \times W$ is a closed sub-bundle of $\xi = M \times V$) then $d: C^\infty(\xi) \rightarrow C^\infty(T^*(M) \otimes \xi)$ is a linear differential operator of order one so by the above $Ls(x) = \|ds_x\|^p$ is an element of $\text{Lgn}_1^p(E)$, giving our previous example.

In the next section we will give some conditions on the linear operators A_i of the above theorem that lead to Lusternik-Schnirelman, and Morse type existence theorem and also smoothness theorems for the corresponding calculus of variations problem.

4. Coercive, linearly embedded, Dirichlet problems. In this section E is a C^∞ fiber bundle which is a closed sub-bundle of a C^∞ vector bundle ξ over M , μ is a strictly positive smooth measure on M , k is a positive integer, and p an even integer with $pk > n$. Let $\sigma_0 \in L_k^p(E)$ and let $X = L_k^p(E)_{\partial\sigma_0}$. Let $L \in \text{Lgn}_k^{pk}(E)$ and define a C^∞ function $J: X \rightarrow \mathbf{R}$ by $J(s) = \int L(s)(x) d\mu(x)$. We shall describe below conditions on L which guarantee that this Dirichlet problem satisfies

Lusternik-Schnirelman and Morse existence theorems and also various smoothness theorems.

Let $\|\cdot\|_{L_k^p}$ be an admissible norm for the Banach space $L_k^p(\xi)$. Since X is a closed C^∞ submanifold of $L_k^p(\xi)$, this choice of norm induces on X the structure of a complete Finsler manifold. In particular if $s \in X$ then $\|dJ_s\| = \sup \{|dJ_s(v)| | \|v\|_{L_k^p} = 1, v \in T(X)_s\}$ is well defined.

DEFINITION. J satisfies condition (C) if for each subset S of X such that J is bounded on S and $\|dJ\|$ is not bounded away from zero on S , there exists a critical point of J adherent to S .

The following theorem represents a combination of results due to the author S. Smale, and J. T. Schwartz.

THEOREM. If J satisfies condition (C) and is bounded below, then for each component X_0 of X there exists $x_0 \in X_0$ such that $J(x_0) = \inf \{J(x) | x \in X_0\}$ and moreover J has at least as many critical points on X_0 as $\text{cat}(X_0)$ the Lusternik-Schnirelman category of X_0 . Assuming J is not anywhere locally constant (a condition always satisfied by "reasonable" calculus of variations problems) there exists $X_0 \in X$ such that $J(x_0) = \inf \{J(x) | x \in X\}$. Finally assuming X is Riemannian (i.e. $p = 2$) and that the critical points of J are nondegenerate, then the Morse inequalities are satisfied.

For full details on the above the reader is referred to the author's *Morse theory on Hilbert manifolds* (Topology **2** (1963), 299-340), *Lusternik-Schnirelman theory on Banach manifolds* (Topology **5** (1966), 115-132), *S. Smale's Morse theory and a nonlinear generalization of the Dirichlet problem* (Ann. of Math. **80** (1964), 382-396) and J. T. Schwartz' *Generalizing the Lusternik-Schnirelman theory of critical points* (Comm. Pure Appl. Math. **17** (1964), 307-315).

We shall now define a class of Lagrangians L , which we call p -coercive, and state theorems to the effect that the functions J defined by such Lagrangians satisfy condition (C) and are bounded below and hence satisfy the conclusions of the above theorem. These definitions and theorems represent joint work by the author and Mrs. Karen Uhlenbeck.

Let η_1, \dots, η_r be C^∞ Riemannian vector bundles over M and let $\eta = \eta_1 \oplus \dots \oplus \eta_r$. Let $A_i: C^\infty(\xi) \rightarrow C^\infty(\eta_i)$ denote k th order C^∞ linear differential operators and let $A = A_1 \oplus \dots \oplus A_r: C^\infty(\xi) \rightarrow C^\infty(\eta)$. We define explicit norms $\|\cdot\|_{L^p}$ for the Banach spaces $L^p(\eta_i)$ (and $L^p(\eta)$) by $\|s\|_{L^p}^p = \int \|s(x)\|_{\eta}^p d\mu(x)$, where $\|\cdot\|_{\eta}$ is the Riemannian norm in the appropriate fiber.

DEFINITION. We say $\{A_i\}$ is an ample set of k th order linear differential operators for ξ if $\|s\|_{L^p} + \sum_{i=1}^r \|A_i s\|_{L^p}$ is an admissible norm for the space $L_k^p(\xi)$. If $\sum_{i=1}^r \|A_i s\|_{L^p}$ is an admissible norm for $L_k^p(\xi)$ then we shall say that $\{A_i\}$ is a strongly ample set of k th order linear differential operators for ξ .

REMARK. It is easily seen that $\{A_i\}$ is (strongly) ample if and only if $\{A\}$ is (strongly) ample.

Now *a priori* it would seem that the definition of ample depends on p . However, this is in fact not the case, and indeed it is a classical result of the theory of linear

differential operators that $\{A\}$ is ample if and only if it is "over determined elliptic" i.e. if and only if for each nonzero cotangent vector (v, x) of M the symbol of A at (v, x) , $\sigma_k(A)(v, x): \xi_x \rightarrow \eta_x$, is injective, and moreover that in this case A is strongly ample if and only if in addition the equation $Au = 0$ has only the zero solution in $L_k^p(\xi)_0 = \text{closure in } L_k^p(\xi) \text{ of } C^\infty \text{ sections of } \xi \text{ with support disjoint from } \partial M$. Thus

THEOREM. *A necessary and sufficient condition that $\{A_i\}$ be ample is that for each nonzero cotangent vector (v, x) of M the intersection of the kernels of the linear maps $\sigma_k(A_i)(v, x): \xi_x \rightarrow \eta_{ix}$ be zero. If in addition there is no $u \in L_k^p(\xi)_0$ such that $A_i u = 0$, $i = 1, \dots, r$ then $\{A_i\}$ is strongly ample.*

EXAMPLE. Let ξ be a product bundle $M \times V$, η the bundle $L(T(M), \xi) = T^*(M) \otimes \xi$ and $d: C^\infty(\xi) \rightarrow C^\infty(\eta)$ the usual differential. Then d is a first order linear differential operator and $\sigma_1(d)(v, x)e = v \otimes e$, so clearly $\{d\}$ is ample. Moreover $du = 0$ if and only if u is constant on each component of M . Since elements of $L_k^p(\xi)_0$ vanish on ∂M it follows that $\ker(d|_{L_k^p(\xi)_0}) = 0$ (and hence d is strongly ample) if and only if each component of M has a nonempty boundary.

Now let $F: J^k(\xi) \rightarrow \mathbf{R}$ be a C^∞ real valued function. Then for each $s \in M$, $F|_{J^k(\xi)_x}$ is a C^∞ real valued function on the vector space $J^k(\xi)_x$, so if $s \in C^\infty(\xi)$ we can form the second differential of $F|_{J^k(\xi)_x}$ at $j_k(s)_x$, a bilinear functional on $J^k(\xi)_x$ which we denote by $\delta^2 F_{j_k(s)_x}$. Then if $u \in C^\infty(\xi)$ we get a C^∞ real valued function $\delta^2 F_{j_k(s)}(j_k(u), j_k(u))$ on M whose value at x is $\delta^2 F_{j_k(s)_x}(j_k(u)_x, j_k(u)_x)$.

DEFINITION. Let $L \in \text{Lgn}_k^{pk}(\xi)$ be defined by $L(s)(x) = F(j_k(s)_x)$ where F is a C^∞ function on $J^k(\xi)$. We say that L is (strictly) p -coercive if there exists a (strongly) ample family $\{A_i\}$ of k th order linear differential operators for ξ such that given $s, u \in C^\infty(\xi)$

$$\delta^2 F_{j_k(s)}(j_k(u), j_k(u)) \geq \sum_{i=1}^r \|A_i s\|^{p-2} \|A_i u\|^2.$$

The following is easily verified.

THEOREM. *If $\{A_i\}$ is a (strongly) ample set of k th order linear differential operators for ξ then $L \in \text{Lgn}_k^{pk}(\xi)$ defined by*

$$L(u) = \sum_{i=1}^r \|A_i u\|^p$$

is (strictly) p -coercive.

COROLLARY. *If $\xi = M \times V$ is a product bundle and $d: C^\infty(\xi) \rightarrow C^\infty(T^*(M) \otimes \xi)$ is the usual differential, then $L \in \text{Lgn}_1^p(\xi)$ defined by $L(s) = \|ds\|^p$ is p -coercive and is strictly p -coercive provided each component of M has nonempty boundary.*

There is a generalization of the above theorem in case $p = 2$, which reduces to the preceding theorem if we take $T = A^*A$.

THEOREM. *Let ξ be a Riemannian bundle with inner product $\langle \cdot, \cdot \rangle_x$ on ξ_x and let $T: C^\infty(\xi) \rightarrow C^\infty(\xi)$ be a selfadjoint linear differential operator of order $2k$ on ξ which*

is strongly elliptic (i.e. $\sigma_{2k}(T)(v, x): \xi_x \rightarrow \xi_x$ is positive definite for each nonzero cotangent vector (v, x) of M). Then the function $J: L_k^2(E)_{\partial\sigma} \rightarrow \mathbf{R}$ defined by $J(s) = \int \langle Ts(x), s(x) \rangle_x d\mu(x)$ for $s \in C^\infty(\xi)$ can be written in the form $J(s) = \int L(s)(x) d\mu(x)$ where $L \in \text{Lgn}_k^{2k}(\xi)$ is 2-coercive, and is strictly 2-coercive if T is strictly positive on $L_k^2(\xi)_0$.

An interesting application of the above theorem is obtained by taking $\xi = M \times V$ and letting T be the k th power of the Laplace Beltrami operator for M (perhaps adding a constant larger than the smallest eigenvalue on $L_k^2(\xi)_0$ so as to make it positive). This gives rise to the k th order energy function J whose extremals are called polyharmonic maps. See Eells and Sampson, *Energie et déformations en géométrie différentielle*, Ann. Inst. Fourier **14** (1964), 61–69.

The following is our basic theorem in the direction of giving sufficient conditions on a linearly embedded Lagrangian in order that the associated Dirichlet problems should satisfy condition (C) and be bounded below, and hence satisfy the conclusions of Lusternik-Schnirelman theory as stated in the first theorem of this section.

THEOREM. Let $L \in \text{Lgn}_k^{pk}(\xi)$ be p -coercive, and if the fiber of the bundle E is not compact assume L is even strictly p -coercive. Then $J: L_k^p(E)_{\partial\sigma} \rightarrow \mathbf{R}$ defined by $J(s) = \int L(s)(x) d\mu(x)$ is bounded below and satisfies condition (C).

The details of the proof of this theorem will be found in §19 of *Foundations of global non-linear analysis*.

An interesting problem, which seems not to have been attacked in any generality, is to find conditions in the case $p = 2$ that insure the nondegeneracy of the critical points of a Dirichlet problem of the above type. Generalizing from a well-known theorem of M. Morse in the geodesic case one would be led to conjecture that for “almost all” choices of Dirichlet boundary conditions σ_0 (in some appropriate sense) all the critical points of $J: L_k^2(E)_{\partial\sigma_0} \rightarrow \mathbf{R}$ are nondegenerate. Presumably this might follow from some transversality argument. If $J(s) = \int \langle Ts(x), s(x) \rangle_x d\mu(x)$ as above then S. Smale has given a beautiful generalization of the Morse Index Theorem that allows one to compute the index of a nondegenerate² critical point of J . (*On the Morse index theorem*, J. Math. Mech. **14** (1965), 1049–1055 and Corrigenda **16** (1967), 1069–1070).

We now comment briefly on smoothness theorems for critical points in the present context. We first consider the case $p = 2$ and $J: L_k^2(E)_{\partial\sigma} \rightarrow \mathbf{R}$ defined by $J(s) = \int \langle Ts(x), s(x) \rangle_x d\mu(x)$ where $T: C^\infty(\xi) \rightarrow C^\infty(\xi)$ is a selfadjoint strongly elliptic linear differential operator of order $2k$. In this case Mrs. Karen Uhlenbeck has proved in her thesis that if $\sigma \in L_{k-\tau}^2(E)$ then every critical point of J is in $L_{k+\tau}^2(E)$, so in particular we have the generalized Weyl Lemma, that if $\partial M = \emptyset$ or if $\sigma \in C^\infty(E)$ then all critical points of J are C^∞ . The proof will be found in §19 of *Foundations of global analysis* for the case that T is “quasi scalar” i.e. where $\sigma_{2k}(T)(v, x): \xi_x \rightarrow \xi_x$ is always multiplication by a scalar (e.g. where T is the k th

power of the Laplace Beltrami operator on M , in which case $\sigma_{2k}(T)(v, x)$ is multiplication by $\|v\|^{2k}$). In this generality the theorem was first proved by Dr. John Saber in his thesis (Brandeis 1965). Saber also proved that condition (C) was satisfied in this case.

In her thesis Karen Uhlenbeck has obtained interesting smoothness theorems for the case $p > 2$. For simplicity we state them in something less than their full generality. We assume $\partial M = \emptyset$ and that $J: L_k^p(E) \rightarrow \mathbf{R}$ is of the form $J(s) = \int \|As(x)\|^p d\mu(x)$ where A is an elliptic k th order linear differential operator $A: C^\infty(\xi) \rightarrow C^\infty(\xi)$ having scalar symbol. Then every critical point of J lies in the Hölder space $C^{k+\alpha}(E)$ where $\alpha = 1/p - 1$. In particular this covers the interesting geometric case where M is Riemannian, $E = M \times W$, $\xi = M \times W$ (W an orthogonal vector space) and A is a power of the Laplace-Beltrami operator of M .

5. Beyond the linearly embedded case. The problems we have been considering are of a basically nonlinear character, and it is somewhat less than satisfying to have to treat them by the "linear embedding" technique (i.e. embedding E in a vector bundle ξ and describing admissible Lagrangians on E in terms of linear differential operators on ξ).

The situation is in many ways parallel to that which obtains in the study of Riemannian geometry. One can develop the whole theory of Riemannian manifolds either intrinsically, or alternatively one can consider Riemannian manifolds M isometrically embedded in an orthogonal vector space V (by Nash's embedding theorem this is no loss of generality) and use the linear structure of V to define such concepts as curvature, parallel translation etc. in M . There are even certain technical advantages in this second approach, in that one can avoid developing all the complicated machinery of connections in principal bundles etc. that usually go along with the intrinsic approach. Yet there seems to be virtually unanimous agreement that both for aesthetic reasons and in order to get a deeper understanding of what is going on, the intrinsic approach is preferable and indeed that the technical machinery that one develops in the intrinsic approach is worth studying for its own sake.

In the present approach to global nonlinear analysis and the calculus of variations we appear to be in a stage of development analogous to that of Riemannian geometry just before E. Cartan. We can formulate the foundations of the theory and the questions we would like to attack intrinsically, but we simply do not have the machinery necessary to carry out the details of the theory intrinsically, so to prove theorems we are forced to fall back on embedding our problems in a linear situation.

Needless to say the next and most exciting stage of the theory lies before us; gaining the insights, the techniques, and the machinery necessary to handle global nonlinear problems intrinsically. It is already becoming clear that this will involve a study of the intrinsic Finsler geometry of the manifolds $L_k^p(E)$ that are induced by certain differential-geometric structures on E that come from Lagrangians on $J^k(E)$.

Both W. Klingenberg and H. Eliasson, for example, have handled the geodesic problem intrinsically from this point of view. And in her thesis Karen Uhlenbeck has made considerable progress in handling quite general classes of calculus of variations problems in several independent variables intrinsically. As one would hope, there are definite indications that the machinery involved is independently interesting and suggests new and interesting global questions in nonlinear analysis.

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