NATURAL BUNDLES HAVE FINITE ORDER*

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§0. INTRODUCTION

Let $\mathcal{M}_n$ denote the category of $n$-dimensional smooth (= $C^\infty$) manifolds, with smooth embeddings as morphisms, and let $\mathcal{F}$ denote the category whose objects are smooth fiber bundles $\pi: E \to B$ and whose morphisms are smooth bundle maps. (That is, a morphism of $\pi_1: E_1 \to B_1$ to $\pi_2: E_2 \to B_2$ is a smooth map $H: E_1 \to E_2$ such that for each $x \in B_1$, the fiber $(E_1)_x = \pi_1^{-1}(x)$ is mapped diffeomorphically onto the fiber $(E_2)_y$ over some point $y = h(x)$ in $B_2$. The map $h: B_1 \to B_2$ is automatically smooth and we say that $H$ covers $h$.)

0.1. Definition. A natural bundle over $n$-manifolds is a functor $F: \mathcal{M}_n \to \mathcal{F}$ such that:

1. For each $n$-manifold $M$, $F(M)$ is a bundle over $M$.
2. For each embedding $\phi: M \to N$ of $n$-manifolds, $F(\phi): F(M) \to F(N)$ covers $\phi$.
3. If $U$ is open in $\mathbb{R}^n$ and $f: U \times \mathbb{R}^n \to \mathbb{R}^n$ is a smooth map such that for each $x \in U$ $y = f_x(y) = f(x, y)$ is a diffeomorphism of $\mathbb{R}^n$; then $f: U \times F(\mathbb{R}^n) \to F(\mathbb{R}^n)$ sending $(x, v)$ to $F(f)(v)$ is smooth.

(Added in revision. In [5] it was conjectured that (3) follows from (1) and (2) and it was remarked that to prove this it suffices to show that (1) and (2) implies the following: if for $a \in \mathbb{R}^n$ we denote the translation map $x \mapsto x + a$ by $\tau_a$, then $\tau_a \to \text{id}$ implies that $F(\tau_a) \to \text{id}$. This has now been demonstrated by D. B. A. Epstein and W. Thurston in a forthcoming paper [1]. Hence condition (3) above can be omitted.)

In the remainder of this paper $F$ will always denote a natural bundle over $n$-manifolds.

We understand smooth fiber bundle in the broad sense of a triple $\pi: E \to B$ which is smoothly locally trivial, so that a priori no structural group is assumed beyond the automatic one, the group of all diffeomorphisms of the fiber with its compact open topology. There are well known examples which show that in general such bundles do not admit a reduction to any Lie structural group. However as a corollary of our main theorem it follows that the structure group of $F(M)$ can always be reduced to $O(n)$.

0.2. Proposition. Let $\phi: M \to N$ be a smooth embedding of $n$-manifolds and let $x \in M$. Then $F(\phi)_x: F(M)_x \to F(N)_x$, depends only on the germ of $\phi$ at $x$.

Proof. Let $C$ be any open submanifold of $M$ containing $x$ and suppose $\psi: M \to N$ agrees with $\phi$ in $C$. Let $i_C$ denote the inclusion of $C$ in $M$ so that $\phi \circ i_C = \psi \circ i_C$ and hence by functoriality $F(\phi) \circ F(i_C) = F(\psi) \circ F(i_C)$. By (2) of 0.1 $F(i_C)$ is an isomorphism of $F(C)$ with $F(M) | C$, so in particular there is an inverse $F(i_C)^{-1}: F(M) | C \to F(C)$ and it follows that $F(\phi)$ and $F(\psi)$ agree in $\pi^{-1}(C)$. Hence $F(\phi)_x = F(\psi)_x$.

Our Main Theorem, 0.3 below, is in fact a considerable strengthening of the above proposition. It states that there is an integer $k$, depending on $F$, such that in order for $F(\phi)$ to equal $F(\psi)$ it is enough that $\phi$ and $\psi$ have the same $k$-jet (i.e., Taylor series of order $k$) at $x$.

Let $G^*$ denote the group of germs (at the origin) of origin preserving diffeomorphisms of $\mathbb{R}^n$. By the proposition we have just proved $G^*$ acts on $F(\mathbb{R}^n)_x$, i.e., we have a homomorphism $\hat{F}$ of $G^*$ into the group Diff($F(\mathbb{R}^n)_x$) of diffeomorphisms of $F(\mathbb{R}^n)_x$. Henceforth we regard $F(\mathbb{R}^n)_x$ as a $G^*$-space. If the kernel of $\hat{F}$ includes some normal subgroup $N$ of $G^*$ then we shall regard $\hat{F}$ as defined on $G^*/N$, and so regard $F(\mathbb{R}^n)_x$ as a $G^*/N$-space.

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Since diffeomorphism is a global concept, the notion of a germ of a diffeomorphism seems almost paradoxical. However, there is the following well-known fact which we shall use repeatedly in the sequel without further explicit reference [3, 85]: if \( M \) is a smooth manifold, \( x \in M \), \( \mathcal{O} \) a neighborhood of \( x \) and \( \psi: \mathcal{O} \to M \) is a smooth map such that \( \psi(x) = x \) and \( d\psi: TM_x \to TM_x \) is an orientation preserving isomorphism, then \( \psi \) defines a germ of a diffeomorphism of \( M \) at \( x \), i.e., there is a diffeomorphism of \( M \) which agrees with \( \psi \) in a neighborhood of \( x \). In case \( M = \mathbb{R}^n \) the restriction that \( d\psi \) is orientation preserving is unnecessary.

For \( k \) a non-negative integer let \( G_k^n \) denote the Lie group of \( k \)-jets (at the origin) of origin preserving diffeomorphisms of \( \mathbb{R}^n \). Elements of \( G_k^n \) are just polynomial maps of degree \( k \) or less from \( \mathbb{R}^n \) to \( \mathbb{R}^n \) with constant term zero and (if \( k > 0 \)) linear term in \( GL(n) = G_1^n \). The group law is composition of maps followed by truncation of terms of degree exceeding \( k \).

\((Note \, that \, G_0^n \) is a trivial group.)

There is an obvious short exact sequence of groups:

\[
\varepsilon \rightarrow j_k^n \rightarrow G^n \rightarrow j_{k+1}^n \rightarrow e.
\]

If the kernel of \( \tilde{F} : G^n \to \text{Diff}(F(\mathbb{R}^n)) \) happens to include \( j_k^n \), then we may regard \( \tilde{F} \) as defined on \( G_k^n \), that is, regard \( F(\mathbb{R}^n) \) as a \( G_k^n \) space. It then follows from (3) of Definition 0.1 that the map \( (g, x) \to \tilde{F}(g) x \) of \( G_k^n \times F(\mathbb{R}^n) \to F(\mathbb{R}^n) \) is smooth with respect to the natural Lie group structure of \( G_k^n \), i.e., \( F(\mathbb{R}^n) \) is a smooth \( G_k^n \) space. In these circumstances we say that the natural bundle \( F: \mathcal{M} \rightarrow \mathcal{F} \) has order less than \( k + 1 \), and the order of \( F \) is defined to be the least such \( k \). With these definitions our main result is:

0.3. **Main Theorem.** If \( F \) is a natural bundle over \( n \)-manifolds and \( f \) is its fiber dimension (i.e., \( f = \dim(F(\mathbb{R}^n)) \)) then \( F \) has finite order less than \( 2^n+1 \).

[Added in revision. See remark after Proposition 3.1.]

Below we shall review quickly the classical natural bundles; namely the \( k^{th} \) order frame bundles (which are principal bundles with structure group \( G_k^n \)) and their associated bundles. The significance of the above theorem is that it shows that every natural bundle arises in this way, and so reduces the classification problem for natural bundles to the problem of classifying smooth \( G_k^n \)-spaces (see Proposition 0.5).

0.4. The \( k^{th} \) order frame bundle. For an \( n \)-manifold \( M \) let \( J^k(M, \mathbb{R}^n) \) denote the manifold of \( k \)-jets of smooth maps of \( M \) into \( \mathbb{R}^n \). The set of \( \gamma_k(f) \in J^k(M, \mathbb{R}^n) \) such that \( f(x) = 0 \) is a smooth submanifold and we let \( G_k^n(M) \) denote its open submanifold consisting of \( \gamma_k(f) \) such that \( df, \) maps \( TM_x \) isomorphically onto \( T(\mathbb{R}^n), \mathbb{R}^n \). The projection \( \pi \) of \( G_k^n(M) \) onto \( M \) is defined of course by \( \gamma_k(f) \rightarrow x \). There is an obvious smooth right action of \( G_k^n \) on \( G_k^n(M) \); namely \( \gamma_k(f) \cdot \gamma = \gamma_k(f \circ \gamma) \). Moreover it is clear that the orbits of this action are just the fibers of \( G_k^n(M) \) and that the action is free. To complete the verification that \( G_k^n(M) \) is a smooth principal fibre bundle over \( M \) with structural group \( G_k^n \) it will suffice given \( x_0 \in M \) to find a smooth local section for the action of \( G_k^n \) at \( x_0 \).

Let \( \varphi: \mathcal{O} = \mathbb{R}^n \) be a chart for \( M \) with \( x_0 \in \mathcal{O} \). Given \( a \in \mathbb{R}^n \) let \( \tau_a: \mathbb{R}^n \rightarrow \mathbb{R}^n \) denote the translation \( v \mapsto v + a \). Then the set \( S = \{ \gamma_k(\tau_a \circ f) \mid x \in \mathcal{O} \} \) is the desired local section. Now suppose \( \phi \) is a smooth embedding of \( M \) in an \( n \)-manifold \( N \). We define a smooth bundle covering map \( G_k^n(\phi): G_k^n(M) \rightarrow G_k^n(N) \) by \( G_k^n(\phi)(\gamma_k(f)) = \gamma_k(f \circ \phi^{-1}) \mid M \). With this definition it is easy to verify that \( G_k^n(\phi): \mathcal{M} \rightarrow \mathcal{F} \) is a natural bundle of order \( k \). Note moreover that the bundle covering maps \( G_k^n(\phi) \) are equivariant with respect to the actions of \( G_k^n \) on \( G_k^n(M) \) and \( G_k^n(N) \).

\( G_k^n(M) \) is called the \( k^{th} \) order frame bundle of \( M \). If \( k = 0 \) we identify \( G_k^n(M) \) with \( M \). If \( k = 1 \) then \( G_k^n(M) \) is just the well known frame bundle (bundle of bases) of \( M \).

Now suppose \( F \) is any smooth (left) \( G_k^n \) space. Then for each \( n \)-manifold \( M \) we can form the smooth fiber bundle associated to \( G_k^n(M) \) with fiber \( F \). (Namely, we make \( G_k^n(M) \times F \) into a left \( G_k^n \) space with the action \( \gamma(p, f) = (p \gamma^{-1}, \gamma f) \) and form the orbit space of this action.) This is usually denoted by \( G_k^n(M) \times G_k^n(F) \), however, we shall adopt the notation \( F(M) \). If \( \phi: M \rightarrow N \) is a smooth embedding then because \( G_k^n(\phi): G_k^n(M) \rightarrow G_k^n(N) \) is \( G_k^n \) equivariant, so is \( G_k^n(\phi) \times \text{id}_F: G_k^n(M) \times F \rightarrow G_k^n(N) \times F \) and hence it induces a smooth map \( F(\phi): F(M) \rightarrow F(N) \). Again it is easy to see that \( F: \mathcal{M} \rightarrow \mathcal{F} \) is a natural bundle, now of order less than \( k + 1 \).
Note that if \( k = 0 \) then \( F(M) = M \times F \) and \( F(\phi) = \gamma \times \text{id}_F \). If \( k = 1 \) and \( F \) is a finite dimensional representation space of \( GL(n) \) (\( G^* \)), i.e., a \( GL(n) \)-module, and then \( F(M) \) is a tensor bundle over \( M \). In particular if \( F = (\bigotimes R^*) \otimes (R^*) \) we get the tensor bundles \( T_n^* \).

0.5. Proposition. Let \( F_i : M_\alpha \to \mathcal{F} \) be a natural bundle for \( n \)-manifolds of fiber dimension \( f \) and let \( F = F_i(R^*)_{\alpha} \), so that by the Main Theorem (0.3) \( F \) is a smooth \( G^* \)-space where \( k = 2^{n-1} - 1 \). Let \( E : M_\alpha \to \mathcal{F} \) be the natural bundle formed by the construction in 0.4 above. Then \( F_i \) and \( F \) are naturally equivalent functors.

This proposition is a special case of the following more general result.

0.6. Proposition. Let \( F_1 \) and \( F_2 \) be two natural bundles over \( n \)-manifolds and let \( h : F_1(R^*)_{\alpha} \to F_2(R^*)_{\alpha} \) be a smooth \( G^* \)-equivariant map. There is a unique collection \( \{ H(M) : F_1(M) \to F_2(M) \} \) of smooth maps (one for each smooth \( n \)-manifold \( M \)) such that \( H(R^*) \) is a \( G^* \)-space where \( k = 2^{n-1} - 1 \). Let \( E : M_\alpha \to \mathcal{F} \) be the natural bundle formed by the construction in 0.4 above. Then \( F_i \) and \( F \) are naturally equivalent functors.

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(3) \( \phi(x) = t \) for \( x/|x| \) in \( S \) (\( t = 0, 1 \)).

(4) If \( f: \mathbb{R}^n \to \mathbb{R}^n \) is smooth and \( j_0(f) = 0 \) then \( \psi: \mathbb{R}^n \to \mathbb{R}^n \), defined by \( x \to \phi(x)f(x) \), is smooth and \( j_0(\psi) = 0 \).

**Proof.** Let \( \Phi: S^{n-1} \to \mathbb{R} \) be a smooth map such that \( \Phi(x) = t \) for \( x \in S \), and define \( \Phi(0) = 0 \) and \( \phi(x) = \Phi(x/|x|) \) for \( x \neq 0 \). It is easy to verify properties (1) to (3). From Taylor's theorem with remainder it follows that \( |D^r f(x)| = o(|x|^r) \) as \( x \to 0 \) for all \( r \) and \( \alpha \). From this and (2) and the rule for differentiating a product it follows by induction that \( D^r \psi(0) = 0 \) and that \( D^r \psi(x) \to 0 \)

as \( x \to 0 \) for all \( \alpha \).

1.2. **Theorem.** If \( \phi: M \to N \) is an embedding of \( n \)-manifolds then \( F(\phi)_*: F(M)_* \to F(N)_* \) depends only on \( j_0(\phi)_* \). In particular the kernel of \( \tilde{F}: \text{Diff}(\mathbb{R}^n)_0 \to \mathbb{R}^n \) includes \( \mathcal{G}_n^0 \).

**Proof.** Let \( \phi_1 \) and \( \phi_2 \) be origin preserving diffeomorphisms of \( \mathbb{R}^n \) such that \( D^\alpha \phi_1(0) = D^\alpha \phi_2(0) \) for all \( \alpha \). It will suffice to show that \( F(\phi_1)_* = F(\phi_2)_* \), and to this end we will construct a germ of a diffeomorphism \( \psi \) of \( \mathbb{R}^n \) (at the origin) such that 0 is in the closure of each of the two open sets \( \mathcal{O}_1 \) and \( \mathcal{O}_2 \), where \( \mathcal{O}_i \) is the interior of the set of \( x \) in \( \mathbb{R}^n \) such that \( \phi_i(x) = \psi(x) \). By 0.2 \( F(\phi_i)_* = F(\psi)_* \) for \( x \in \mathcal{O}_i \), so by continuity of \( F(\phi) \) it will follow that \( F(\phi_1)_* = F(\psi)_* = F(\phi_2)_* \).

To construct \( \psi \) let \( S_0 \) and \( S_1 \) be disjoint closed subjects of \( \mathbb{R}^{n+1} \) with non-empty interiors and let \( \phi_1: \mathbb{R}^n \to \mathbb{R}^n \) be as in Lemma 1.1. Since \( D^\alpha (\phi_1 - \phi_2)(0) = 0 \) for all \( \alpha \), by (4) of Lemma 1.1 \( x \to \phi_1(x) + \phi_2(x)(\phi_1(x) - \phi_2(x)) \) is a smooth map \( \mathbb{R}^n \to \mathbb{R}^n \) whose differential at the origin agrees with that of \( \phi_2 \), so in particular it is non-singular. It follows that there is a diffeomorphism \( \psi \) of \( \mathbb{R}^n \) whose germ at 0 agrees with that of this map. Now for \( x/|x| \) in \( S_0 \), \( \phi(0) = i \) (\( i = 0, 1 \)) so \( \phi_i(x) = \phi(x) (\phi_i(x) - \phi(x)) \) is \( \phi_i(x) \) for \( x/|x| \) in \( S_0 \), and is \( \phi_i(x) \) for \( x/|x| \) in \( S_1 \). It follows that \( \psi \) has the required property.

It is worth noting that the proof of the above Theorem used only properties (1) and (2) ("naturality") of the Definition 0.1 of natural bundle and not property (3) ("continuity").

§2. **The Groups \( G_n^* \) and \( G_\infty^* \)**

In this section we will develop some structure theorems for the groups \( G_n^* \) and \( G_\infty^* \), which are needed to prove the main theorem.

Let \( P_i(\mathbb{R}^n, \mathbb{R}^m) \) denote the space of all homogeneous polynomial maps of degree \( i \) from \( \mathbb{R}^n \) to \( \mathbb{R}^m \). Then \( G_n^* \) can be identified with the open subset \( GL(n) \cap \bigoplus_{i=1}^k P_i(\mathbb{R}^n, \mathbb{R}^n) \) of the vector space \( \bigoplus_{i=1}^k P_i(\mathbb{R}^n, \mathbb{R}^n) \). It follows that the tangent space of \( G_n^* \) at the identity can be naturally identified with the vector space of polynomial maps of \( \mathbb{R}^n \) to itself of degree less than or equal to \( k \) which vanish at the origin, or equivalently with the \( k \)-jets of vector fields on \( \mathbb{R}^n \) at the origin which vanish at the origin.

2.1. **Theorem ([5]).** The Lie algebra \( \mathfrak{g}_n^* \) of \( G_n^* \) is \( \{ j_k(X)_*|X \in C^\infty(\mathbb{R}^n) \} \) and \( X(0) = 0 \), with the bracket \( [j_k(X)_*, j_k(Y)_*] = j_k([X, Y])_o \). Moreover, if \( \exp (tX) \) denotes the flow in \( \mathbb{R}^n \) generated by \( X \), then \( \exp (j_k(X)_*) = j_k(\exp (tX)_0)_* \).

**Proof.** \( \{ j_k(X)_*|X \in C^\infty(\mathbb{R}^n) \} \) and \( X(0) = 0 \) is the tangent space of \( G_n^* \) at the identity. \( t \to j_k(\exp (tX)_0) \) is a group homomorphism from \( \mathbb{R} \) to \( G_n^* \) with \( \frac{d}{dt} j_k(\exp (tX)_0) = j_k(\frac{d}{dt} \exp (tX)_0) = j_k(\frac{d}{dt} \exp (tX)_0) \) as the tangent vector at the identity, hence, \( \exp (j_k(X)_*) = j_k(\exp (tX)_0) \).

And

\[
[j_k(X)_*, j_k(Y)_*] = \left. \frac{d^2}{dt^2} \right|_{t=0} j_k(\exp (-tX)_0) \cdot j_k(\exp (-tY)_0) \cdot j_k(\exp (tY)_0) \cdot j_k(\exp (tX)_0) \\
= \left. \frac{d}{dt} \right|_{t=0} j_k(\exp (-tX) \circ \exp (-tY) \circ \exp (tY) \circ \exp (tX))_0 \\
= \frac{1}{2} j_k(\left. \frac{d^2}{dt^2} \right|_{t=0} \exp (-tX) \circ \exp (-tY) \circ \exp (tX) \circ \exp (tY))_0 \\
= \frac{1}{2} j_k(\exp (-tX) \circ \exp (-tY) \circ \exp (tX) \circ \exp (tY))_0 \\
= j_k([X, Y])_0.
\]
2.2. COROLLARY ([15]). Let $X, Y$ be two vector fields on $\mathbb{R}^n$ such that $X(0) = Y(0) = 0$ and $j_k(X)_0 = j_k(Y)_0$. Then $j_k(\exp tX)_0 = j_k(\exp tY)_0$.

Let $\phi_k: G^*_n \to G^*_n$ be the projection sending $j_k(\phi)$ to $j_k(\phi)$ for $k \gg l$. Then $\{G^*_n, \phi_k\}$ is an inverse system of Lie groups. By the well-known extension lemma of E. Borel, given any system $\{a_k \in \mathbb{R} | a \in Z^n, a_i \geq 0\}$ there exists a smooth map $f: \mathbb{R}^n \to \mathbb{R}^n$ such that $D^i f(0) = a_i$.

It follows that $G^*_n$ is the inverse limit of $\{G^*_n, \phi_k\}$ with $j_k: G^*_n \to G^*_n$ sending $j_k(\phi)_0$ to $j_k(\phi)_0$ as projections. $B^*_n, \phi_k$ will denote the normal subgroup $\ker j_k$ of $G^*_n$, and $N^*_n, \phi_k$ the normal subgroup $\ker j_k$ of $G^*_n$. Since the differential of $\phi_k$ also sends $j_k(X)_0$ to $j_k(Y)_0$, we denote it by $\phi_k: G^*_n \to G^*_n$. So $\{G^*_n, \phi_k\}$ also forms an inverse system of Lie algebras and the inverse limit is the infinite dimensional Lie algebra $\mathfrak{g}^*_n = \{j_k(X)_0 | X \in C^n(\mathbb{R}^n)\}$ and $(X(0) = 0)$ with bracket $[j_k(X)_0, j_k(Y)_0] = j_k([X, Y])_0$ and $j_k: \mathfrak{g}^*_n \to \mathfrak{g}^*_n$ as projections. Let $N^*_n, \phi_k$ denote the ideal ker $j_k$ of $\mathfrak{g}^*_n$; then the following proposition tells us that $N^*_n, \phi_k$ is "the Lie algebra of $\mathbb{R}^n$" and that the exponential map is bijective.

2.3. PROPOSITION. Given $j_k(\phi)_0 \in N^*_n, \phi_k$ there exists a vector field $X \in C^n(\mathbb{R}^n)$ such that $j_k(X)_0 = j_k(\phi)_0$. Moreover $j_k(X)_0$ is uniquely determined.

Proof Since $B^*_n, \phi_k$ is a simply connected nilpotent subgroup of $G^*_n$ ([15]) for $m > k$, the exponential map is a diffeomorphism of its Lie algebra $\mathfrak{b}^*_n, \phi_k$ onto $B^*_n, \phi_k$. Let $j_m(X)_0 \in \mathfrak{m}^*_n, \phi_k$ be the unique element such that $\exp (j_m(X)_0) = j_m(\phi)_0$. Then $j_k(\exp (j_m(X)_0)) = j_k(\exp (j_m(X)_0)) = j_m(j_k(j_m(\phi)_0)) = j_m(j_k(\phi)_0) = j_k(\phi)_0$, so $j_k(X)_0 = j_k(X)_0$ for all $m > k$ and certainly $j_k(X)_0 = j_k(X)_0 = 0$. By Corollary 2.2 $j_k(\exp (X)_0) = j_k(\exp (X)_0)$. But $j_k(\exp (X)_0) = \exp (j_m(X)_0)$ by Theorem 2.1, and we have $\exp (j_m(X)_0) = j_m(\phi)_0$, so $j_m(\exp (X)_0) = j_m(\phi)_0$, for all $m > k$, i.e., $j_m(\exp (X)_0) = j_m(\phi)_0$. □

2.4. Definition. For a diffeomorphism $f$ of $\mathbb{R}$ we define a diffeomorphism $\phi$ of $\mathbb{R}^n$ by sending $(x_1, \ldots, x_n)$ to $(\phi(x_1), \ldots, \phi(x_n))$. This induces a canonical embedding $i: G^*_n \to G^*_n$.

2.5. PROPOSITION. The normal subgroup $N$ of $G^*_n$ generated by $i(N^*_n)$ is $N^*_n$.

Proof By proposition 2.3 it will suffice to show that the ideal $\mathcal{N}$ generated by $N^*_n$ in $\mathfrak{g}^*_n$ is $\mathcal{N}^*_n$ and since the latter is an ideal which includes $\mathcal{N}^*_n$ it will be enough to show that $\mathcal{N}^*_n \subseteq \mathcal{N}$, i.e., that elements of $\mathfrak{g}^*_n$ of the form $\sum a_n x^n(\partial/\partial x_i)$ are in $\mathcal{N}$. If $q(x)(\partial/\partial x_i)$ is in $\mathcal{N}$ then so is $[q(x)(\partial/\partial x_i), x_i(\partial/\partial x_i)] = (\partial q/\partial x_i) x_i(\partial/\partial x_i)$. Using this remark recursively, starting from $q(x) = x_i$ with $l \geq k$ we see that $p(x)(\partial/\partial x_i) \in \mathcal{N}$ for any homogeneous polynomial $p(x)$ of degree $\geq l$. Hence more generally $\sum a_n x^n(\partial/\partial x_i)$ is in $\mathcal{N}$. Since $\mathcal{N}$ is an ideal containing $x_i(\partial/\partial x_i)$ and $x_i(\partial/\partial x_i)$ it also contains the elements

$$\sum_{|\alpha| = k} a_n x^n = -\left[ x_i \frac{\partial}{\partial x_i} \sum_{|\alpha| = k} a_n x_1^{\alpha} x_2^{\alpha_1} \cdots x_n^{\alpha_n} \frac{\partial}{\partial x_1} \right]$$

and

$$\sum_{|\alpha| = k} a_n x^n = -\left[ x_i \frac{\partial}{\partial x_i} \sum_{|\alpha| = k} a_n x_1^{\alpha} x_2^{\alpha_1} \cdots x_n^{\alpha_n} \frac{\partial}{\partial x_1} \right]$$

where in the latter $i \geq 2$. It follows in particular that $\mathcal{N}$ contains any element of the form $\sum a_n x^n(\partial/\partial x_i)$ and so by a preceding remark it also contains all elements of the form

$$\sum_{|\alpha| = k} a_n x^n(\partial/\partial x_i)$$

Then

$$\left[ \sum_{|\alpha| = k} a_n x^n, x_i, \frac{\partial}{\partial x_i} \right] = \sum_{|\alpha| = k} a_n \frac{\partial(x^n)}{\partial x_i} x_i \frac{\partial}{\partial x_i} - \sum_{|\alpha| = k} a_n x^n \frac{\partial}{\partial x_i}$$

is in $\mathcal{N}$ and hence so is $\sum_{|\alpha| = k} a_n x^n(\partial/\partial x_i)$. □
In this section we shall show that the order of natural bundles is at least "pointwise finite", i.e., we shall prove that:

3.1. THEOREM. Let $F: \mathcal{M} \to \mathcal{F}$ be a natural bundle. Then for each $p$ in $F(R^n)$ there is an integer $k(p)$ such that $(G_\ast)^n_0$, the isotropy subgroup of the action of $G_\ast^n$ on $F(R^n)$, includes $N_\ast^n_k(p)$.

First we prove "infinitesimal pointwise finite order" and for this we need the following:

3.2. LEMMA. Let $\mathcal{A}$ be a subalgebra of $\mathcal{G}_\ast^n$ of finite codimension and suppose $\mathcal{A} = \lim \lceil ij_\ast(\mathcal{A}) \rceil$. Then there is an integer $k$ such that $\mathcal{A} + N_\ast^n_k \subseteq \mathcal{A}$.

Proof. We have a decreasing sequence of subalgebras of $\mathcal{G}_\ast^n$:

$$\mathcal{A} = \mathcal{A} + N_\ast^n_1 \supseteq \mathcal{A} + N_\ast^n_2 \supseteq \cdots \supseteq \mathcal{A} + N_\ast^n_l \cdots \supseteq \mathcal{A}.$$

Since $\mathcal{A}$ has finite codimension there is an integer $k$ for which $\mathcal{A} + N_\ast^n_k = \mathcal{A} + N_\ast^n_l$ for all $l \geq k$, and we claim $N_\ast^n_k \subseteq \mathcal{A}$. For suppose $j_\ast(f)_0 \in N_\ast^n_k \subseteq \mathcal{A} + N_\ast^n_l$ for $l \geq k$. Then there exists $j_\ast(f)_0 \in \mathcal{A}$ such that $j_\ast(f)_0 = j_\ast(f)_0$, i.e., $j_\ast(f)_0 \lim \lceil ij_\ast(\mathcal{A}) \rceil = \mathcal{A}$. ⬜

Now let $X$ be a smooth vector field on $R^n$ vanishing at the origin. By the continuity assumption 0.1(3) the smooth local flow $\exp tX$ on $R^n$ induces a smooth flow $F(\exp tX)(p)$ on $F(R^n)$, and therefore defines a generating smooth vector field on $F(R^n)$, which by 1.2 and 2.2 depends only on $j_\ast(X)_0$. We denote this vector field by $dF(j_\ast(X)_0)$. It also follows from the continuity assumption that the map $dF: \mathcal{G}_\ast^n \to C^1(T(F(R^n)))$ is a Lie algebra homomorphism. Then for $p \in F(R^n)$, the map $j_\ast(X)_0 \to dF[j_\ast(X)_0](p)$ is a linear map from $\mathcal{G}_\ast^n$ to the finite dimensional vector space $T(F(R^n))$ and its kernel $(\mathcal{G}_\ast^n)_p$ is a subalgebra of $\mathcal{G}_\ast^n$ with finite codimension, (consisting of the $j_\ast(X)_0 \in \mathcal{G}_\ast^n$ such that the flow $F(\exp tX)(p)$ fixes $p$, i.e., the isotropy subalgebra of $\mathcal{G}_\ast^n$ at $p$). Moreover, by 3.4 below $(\mathcal{G}_\ast^n)_p \subseteq N_\ast^n_k(p)$. This proves infinitesimal point-wise finite order which coupled with Proposition 2.3 implies Theorem 3.1 on pointwise finite order.

3.3. PROPOSITION [5, Lemma 1.5]. Let $\phi$ and $\psi_m$ be origin preserving diffeomorphisms of $R^n$ and suppose $j_\ast(\phi_m)_0 = j_\ast(\psi_m)_0$ for all $m$. Then there exists a sequence $\{\psi_m\}$ of origin preserving diffeomorphisms of $R^n$ such that $\phi_m$ and $\psi_m$ have the same germ at the origin and $\psi_m \to \phi$ in the $C^n$ topology.

3.4. COROLLARY. Let $\mathcal{A} = (\mathcal{G}_\ast^n)_p$. Then $\mathcal{A} = \lim \lceil ij_\ast(\mathcal{A}) \rceil$.

Proof. Let $j_\ast(X)_0 \in (\mathcal{G}_\ast^n)_p$ and $j_\ast(X)_0 = j_\ast(X)_0$. We want to show that $j_\ast(X)_0 \in (\mathcal{G}_\ast^n)_p$, i.e., that $F(\exp tX)(p) = p$ for all $t$. Now by 2.2 $j_\ast(\exp tX)(p) = j_\ast(\exp tX)_0$ and so by 3.3 for each $t$ there exists $\psi_m^t$ with $j_\ast(\psi_m^t)_0 = j_\ast(\exp tX)_0$ and $\psi_m^t \to \exp tX$ in the $C^n$ topology. By the continuity assumption $F(\psi_m^t) \to F(\exp tX)$ in the compact open topology. But by 1.2 $F(\psi_m^t)_0 = F(\exp tX)_0$, and $F(\exp tX)_0(p) = p$ by assumption. If follows that $F(\exp tX)(p) = p$. ⬜

§4. PROOF OF THE MAIN THEOREM

We have shown in the last section that every point of the $G_\ast^n$-space $F(R^n)$ has finite order, which may vary from point to point. However, there is an upper bound, namely $2^{k+1}$, where $f = \dim(F(R^n))_0$. We now prove this.

Recall we identify $\mathcal{G}_1$ with the polynomials in one variable of degree less than $k + 1$ without constant term. $(p(x) - j_k(p(x))(\partial / \partial x))_0$. We now prove this.

4.1. PROPOSITION. Let $H_0$ be a Lie subalgebra of $\mathcal{G}_1$, $x^t \not\in H_0$, and $\dim(\mathcal{G}_1/H_0) = m$. Then $k < 2^{m+1}$.

Proof. By induction on $m$. For $m = 1$, we have $H_0 \oplus R x^t = \mathcal{G}_1$. So there exist real numbers $a$, $b$, $c$ such that $x + ax^t$, $x^2 + bx^t$, $x^3 + cx^t \in H_0$. Suppose $k \geq 4$. Since $[x + ax^t, x^2 + bx^t] + (x^2 + bx^t) = -b(k - 2)x^t \in H_0$ and $x^t \not\in H_0$, $b = 0$. Similarly, $[x + ax^t, x^t + cx^t] + 2(x^t + cx^t) = -c(k - 3)x^t \in H_0$, and $x^t \not\in H_0$ imply that $c = 0$. Hence $x^2, x^3 \in H_0$ which imply that $x^t \in H_0$ for
all $i \geq 2$, contradicting the fact that $x^i \notin H_0$. So $k < 4 = 2^{i+1}$. Now suppose it is true for integers less than or equal to $m$. Let $H_k$ be a Lie subalgebra of $\mathfrak{g}_k$ such that $x^k \notin H_k$ and $\dim(\mathfrak{g}_k/H_k) = m + 1$, $m \geq 1$. Consider $\tilde{H}_k = H_k + Rx^k = J^{-1}_{k-l}(j_k(H_k))$, a Lie subalgebra of $\mathfrak{g}_k$; then $\dim(\mathfrak{g}_k/\tilde{H}_k) = m$. We claim that if $l$ is the smallest positive integer such that $B_{l+1} \subset \tilde{H}_k$ then $k - 1 \geq l \geq m$ and $(k-1)/2 \leq l$. The first part of the claim is clear, for $Rx^k = B_{k+1} \subset \tilde{H}_k$ and $\dim(\mathfrak{g}_k/B_{k+1}) \geq \dim(\mathfrak{g}_k/\tilde{H}_k)$. For the second part of the claim, suppose on the contrary that $(k-1)/2 > l$, i.e. $k-l > 1$. Since $B_{l+1} \subset \tilde{H}_k$ and $\tilde{H}_k = H_k + Rx^k$, there exist real numbers $a, b$ such that $x^{l+1} + ax^k, x^{k-l} + bx^k \in H_k$. Hence $[x^{l+1} + ax^k, x^{k-l} + bx^k] = -(k-1-2l)x^k \in H_k$. But $k-1-2l > 0$, so $x^k \in H_k$ a contradiction. Let $H_0 = \mathfrak{h}_k(H_k)$, then $x^i \notin H_0$. For if $x^i \in H_0$ then there is an element $h$ in $B_{l+1} \subset \tilde{H}_k$ such that $x^i + h \in H$, which implies $x^i \in H_k$. So $B_i \subset \tilde{H}_k$, contradicting the choice of $l$. It is clear that $p - \dim(\mathfrak{g}_k/\tilde{H}_k) = \dim(\mathfrak{g}_k/H_k) = m$. Therefore we can use the induction hypothesis $l < 2^{l+1}$. But $p - m$ and $(k-1)/2 \leq l$, so $k < 2^{l+1} - 1 = 2^{m+1} - 1$. 

[Added in revision. In [1] Epstein and Thurston obtain the much better bound $k \leq 2m + 1$ (which they show is sharp for $n = 1$) as follows: from the relation $[x^i, x^j] = (i - j)x^{i+j-1}$ and the assumption $x^i \notin H_0$ we see that $H_k$ cannot contain both terms of the form $(x^i + \text{higher order})$ and $(x^{i+j} + \text{higher order})$ unless $2l = k + 1$, and from this we conclude the codimension of $H_k$ in $\mathfrak{g}_k$ cannot be less than $[k/2]$. This of course implies that in our main Theorem, 0.3, we can also replace the bound $2^{l+1}$ by $2^{m+1}$.]

4.2. Theorem. Every natural bundle $F: M \to \mathfrak{g}$ is of finite order less than $r = 2^{l+1}$, where $f = \dim(F(R))$.

Proof. By Proposition 2.3, it suffices to prove that $\mathcal{N}_F$ is included in $(\mathfrak{g}_p)_{\mu}$, for all $p \in F(R)_{\nu}$. By Theorem 3.1 there is a least integer $k(p)$ with $(\mathfrak{g}_p)_{\mu}$ including $(\mathfrak{g}_k)_{\mu}$. Then $x^k \notin H_p$ and $m - \dim((\mathfrak{g}_k)_{\mu}/H_p) \leq \dim(\mathfrak{g}_k/H_p) = f$. By 4.1 $k(p) < 2^{l+1}$, so a fortiori $k(p) < 2^{l+1} = r$, i.e. $\mathcal{N}_F \subset (\mathfrak{g}_p)_{\mu}$.

4.3. Proof of the Main Theorem. Recall the canonical embedding $i: G \to G$ defined in 2.4. $\tilde{F} \circ i: G \to \text{Diff}(F(R))$ corresponds to a natural bundle $F: M \to \mathfrak{g}(F(R)) = F(M \times R^l)|M \times 0$ and $F(\varphi) = (F(\varphi \times \text{id}_{(R^l)}))|M \times 0$. Then by 4.2 $F$, has an order $l < 2^{l+1}$, i.e. $(\mathcal{N}_F) \subset \ker \tilde{F}$. But ker $\tilde{F}$ is a normal subgroup of $G$, so by proposition 2.5 $\mathcal{N}_F \subset \ker \tilde{F}$. Therefore $\tilde{F}$ factors through $G$, i.e. $F$ has order less than $2^{l+1}$.

REFERENCES


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