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NATURAL BUNDLES HAVE FINITE ORDER*

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§0. INTRODUCTION

LET $\mathcal{M}_n$ DENOTE the category of n-dimensional smooth ($= C^\infty$) manifolds, with smooth embeddings as morphisms, and let $\mathcal{F}$ denote the category whose objects are smooth fiber bundles $\pi: E \to B$ and whose morphisms are smooth bundle maps. (That is, a morphism of $\pi_1: E_1 \to B_1$ to $\pi_2: E_2 \to B_2$ is a smooth map $H: E_1 \to E_2$ such that for each $x \in B_1$ the fiber $(E_1)_x = \pi_1^{-1}(x)$ is mapped diffeomorphically onto the fiber $(E_2)_y$ over some point $y = h(x)$ in B_2. The map $h: B_1 \to B_2$ is automatically smooth and we say that H *covers* h.)

0.1. *Definition.* A *natural bundle over* n-manifolds is a functor $F: \mathcal{M}_n \to \mathcal{F}$ such that:

(1) For each n-manifold M, $F(M)$ is a bundle over M.

(2) For each embedding $\phi: M \to N$ of n-manifolds, $F(\phi): F(M) \to F(N)$ covers ϕ.

(3) If U is open in $\mathbf{R}^m$ and $f: U \times \mathbf{R}^n \to \mathbf{R}^n$ is a smooth map such that for each $x \in U$ $y \mapsto f_x(y) = f(x, y)$ is a diffeomorphism of $\mathbf{R}^n$; then $\tilde{f}: U \times F(\mathbf{R}^n) \to F(\mathbf{R}^n)$ sending (x, v) to $F(f_x)(v)$ is smooth.

(Added in revision. In [5] it was conjectured that (3) follows from (1) and (2) and it was remarked that to prove this it suffices to show that (1) and (2) imply the following: if for $a \in \mathbf{R}^n$ we denote the translation map $x \mapsto x + a$ by τ_a, then $a_n \to 0$ implies that $F(\tau_{a_n}) \to \mathrm{id}$. This has now been demonstrated by D. B. A. Epstein and W. Thurston in a forthcoming paper[1]. Hence condition (3) above can be omitted.)

In the remainder of this paper F will always denote a natural bundle over n-manifolds.

We understand smooth fiber bundle in the broad sense of a triple $\pi: E \to B$ which is smoothly locally trivial, so that *a priori* no structural group is assumed beyond the automatic one, the group of all diffeomorphisms of the fiber with its compact open topology. There are well known examples which show that in general such bundles do not admit a reduction to any Lie structural group. However as a corollary of our main theorem it follows that the structure group of $F(M)$ can always be reduced to $\mathbf{O}(n)$.

0.2. PROPOSITION. *Let $\phi: M \to N$ be a smooth embedding of n-manifolds and let $x \in M$. Then $F(\phi)_x: F(M)_x \to F(N)_{\phi(x)}$ depends only on the germ of ϕ at x.*

Proof. Let $\mathcal{O}$ be any open submanifold of M containing x and suppose $\psi: M \to N$ agrees with ϕ in $\mathcal{O}$. Let $i_{\mathcal{O}}$ denote the inclusion of $\mathcal{O}$ in M so that

$$\phi \circ i_{\mathcal{O}} = \phi|\mathcal{O} = \psi|\mathcal{O} = \psi \circ i_{\mathcal{O}}$$

and hence by functoriality $F(\phi) \circ F(i_{\mathcal{O}}) = F(\psi) \circ F(i_{\mathcal{O}})$. By (2) of 0.1 $F(i_{\mathcal{O}})$ is an isomorphism of $F(\mathcal{O})$ with $F(M)|\mathcal{O}$, so in particular there is an inverse $F(i_{\mathcal{O}})^{-1}: F(M)|\mathcal{O} \to F(\mathcal{O})$ and it follows that $F(\phi)$ and $F(\psi)$ agree in $\pi^{-1}(\mathcal{O})$. Hence $F(\phi)_x = F(\psi)_x$.

Our Main Theorem, 0.3 below, is in fact a considerable strengthening of the above proposition. It states that there is an integer k, depending on F, such that in order for $F(\phi)_x$ to equal $F(\psi)_x$ it is enough that ϕ and ψ have the same k-jet (i.e., Taylor series of order k) at x.

Let G^n denote the group of germs (at the origin) of origin preserving diffeomorphisms of $\mathbf{R}^n$. By the proposition we have just proved G^n acts on $F(\mathbf{R}^n)_0$, i.e., we have a homomorphism $\tilde{F}$ of G^n into the group $\mathrm{Diff}(F(\mathbf{R}^n)_0)$ of diffeomorphisms of $F(\mathbf{R}^n)_0$. Henceforth we regard $F(\mathbf{R})_0$ as a G^n-space. If the kernel of $\tilde{F}$ includes some normal subgroup N of G^n then we shall regard $\tilde{F}$ as defined on G^n/N, and so regard $F(\mathbf{R}^n)_0$ as a G^n/N-space.

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(Since diffeomorphism is a global concept, the notion of a germ of a diffeomorphism seems almost paradoxical. However, there is the following well-known fact which we shall use repeatedly in the sequel without further explicit reference[3, §5]: if M is a smooth manifold, $x \in M$, $\mathcal{O}$ a neighborhood of x and $\psi: \mathcal{O} \to M$ is a smooth map such that $\psi(x) = x$ and $d\psi_x: TM_x \to TM_x$ is an orientation preserving isomorphism, then ψ defines a germ of a diffeomorphism of M at x, i.e., there is a diffeomorphism of M which agrees with ψ in a neighborhood of x. In case $M = \mathbf{R}^n$ the restriction that $d\psi_x$ is orientation preserving is unnecessary.)

For k a non-negative integer let $G_k{}^n$ denote the Lie group of k-jets (at the origin) of origin preserving diffeomorphisms of $\mathbf{R}^n$. Elements of $G_k{}^n$ are just polynomial maps of degree k or less from $\mathbf{R}^n$ to $\mathbf{R}^n$ with constant term zero and (if $k > 0$) linear term in $GL(n) = G_1{}^n$. The group law is composition of maps followed by truncation of terms of degree exceeding k. (Note that $G_0{}^n$ is a trivial group.)

There is an obvious short exact sequence of groups:

$$e \to \mathcal{J}^n_{k+1} \to G^n \xrightarrow{j_k} G_k{}^n \to e.$$

If the kernel of $\tilde{F}: G^n \to \mathrm{Diff}(F(\mathbf{R}^n))$ happens to include $\mathcal{J}^n_{k+1}$ then we may regard $\tilde{F}$ as defined on $G_k{}^n$, that is, regard $F(\mathbf{R}^n)_0$ as a $G_k{}^n$ space. It then follows from (3) of Definition 0.1 that the map $(g, x) \to \tilde{F}(g)x$ of $G_k{}^n \times F(\mathbf{R}^n)_0 \to F(\mathbf{R}^n)_0$ is smooth with respect to the natural Lie group structure of $G_k{}^n$, i.e., $F(\mathbf{R}^n)_0$ is a smooth $G_k{}^n$ space. In these circumstances we say that the natural bundle $F: \mathcal{M}_n \to \mathcal{F}$ has order less than $k+1$, and the order of F is defined to be the least such k. With these definitions our main result is:

0.3. Main Theorem. *If F is a natural bundle over n-manifolds and f is its fiber dimension (i.e., $f = \dim(F(\mathbf{R}^n)_0)$) then F has finite order less than 2^{f+1}.*

[Added in revision. See remark after Proposition 3.1.]

Below we shall review quickly the classical natural bundles; namely the k^{th} order frame bundles (which are principal bundles with structure group $G_k{}^n$) and their associated bundles. The significance of the above theorem is that it shows that every natural bundle arises in this way, and so reduces the classification problem for natural bundles to the problem of classifying smooth $G_k{}^n$-spaces (see Proposition 0.5).

0.4. *The k^{th} order frame bundle.* For an n-manifold M let $J^k(M, \mathbf{R}^n)$ denote the manifold of k-jets of smooth maps of M into $\mathbf{R}^n$. The set of $j_k(f)_x$ in $J^k(M, \mathbf{R}^n)$ such that $f(x) = 0$ is a smooth submanifold and we let $\underset{\sim}{G}_k{}^n(M)$ denote its open submanifold consisting of $j_k(f)_x$ such that df_x maps TM_x isomorphically onto $T(\mathbf{R}^n)_0 = \mathbf{R}^n$. The projection π of $\underset{\sim}{G}_k{}^n(M)$ onto M is defined of course by $j_k(f)_x \mapsto x$. There is an obvious smooth right action of $G_k{}^n$ on $\underset{\sim}{G}_k{}^n(M)$; namely $j_k(f)_x \circ \gamma = j_k(\gamma^{-1} \circ f)_x$. Moreover it is clear that the orbits of this action are just the fibers of $\underset{\sim}{G}_k{}^n(M)$ and that the action is free. To complete the verification that $\underset{\sim}{G}_k{}^n(M)$ is a smooth principal fibre bundle over M with structural group $G_k{}^n$ it will suffice given $x_0 \in M$, to find a smooth local section for the action of $G_k{}^n$ at x_0.

Let $\phi: \mathcal{O} \approx \mathbf{R}^n$ be a chart for M with $x_0 \in \mathcal{O}$. Given $a \in \mathbf{R}^n$ let $\tau_a: \mathbf{R}^n \to \mathbf{R}^n$ denote the translation $v \to v - a$. Then the set $S = \{j_k(\tau_{\phi(x)} \circ \phi)_x | x \in \mathcal{O}\}$ is the desired local section. Now suppose ϕ is a smooth embedding of M in an n-manifold N. We define a smooth bundle covering map $\underset{\sim}{G}_k{}^n(\phi): \underset{\sim}{G}_k{}^n(M) \to \underset{\sim}{G}_k{}^n(N)$ by $\underset{\sim}{G}_k{}^n(\phi)(j_k(f)_x) = j_k(f \circ \phi^{-1})_{\phi(x)}$. With this definition it is easy to verify that $\underset{\sim}{G}_k{}^n: \mathcal{M}_n \to \mathcal{F}$ is a natural bundle of order k. Note moreover that the bundle covering maps $\underset{\sim}{G}_k{}^n(\phi)$ are equivariant with respect to the actions of $G_k{}^n$ on $\underset{\sim}{G}_k{}^n(M)$ and $\underset{\sim}{G}_k{}^n(N)$.

$\underset{\sim}{G}_k{}^n(M)$ is called the k^{th} order frame bundle of M. If $k = 0$ we identify $\underset{\sim}{G}_k{}^n(M)$ with M. If $k = 1$ then $\underset{\sim}{G}_k{}^n(M)$ is just the well known frame bundle (bundle of bases) of M.

Now suppose F is any smooth (left) $G_k{}^n$ space. Then for each n-manifold M we can form the smooth fiber bundle associated to $\underset{\sim}{G}_k{}^n(M)$ with fiber F. (Namely, we make $G_k{}^n(M) \times F$ into a left $G_k{}^n$ space with the action $\gamma(p, f) = (p\gamma^{-1}, \gamma f)$ and form the orbit space of this action.) This is usually denoted by $G_k{}^n(M) \times G_k{}^n F$, however, we shall adopt the notation $\underset{\sim}{F}(M)$. If $\phi: M \to N$ is a smooth embedding then because $\underset{\sim}{G}_k{}^n(\phi)$: $\underset{\sim}{G}_k{}^n(M) \to \underset{\sim}{G}_k{}^n(N)$ is $G_k{}^n$ equivariant, so is $\underset{\sim}{G}_k{}^n(\phi) \times \mathrm{id}_F: \underset{\sim}{G}_k{}^n(M) \times F \to \underset{\sim}{G}_k{}^n(N) \times F$ and hence it induces a smooth map $\underset{\sim}{F}(\phi): \underset{\sim}{F}(M) \to \underset{\sim}{F}(N)$. Again it is easy to see that $\underset{\sim}{F}: \mathcal{M}_n \to \mathcal{F}$ is a natural bundle, now of order less than $k+1$.

Note that if $k=0$ then $\underset{\sim}{F}(M)=M\times F$ and $\underset{\sim}{F}(\phi)=\gamma\times \mathrm{id}_F$. If $k=1$ and F is a finite dimensional representation space of $GL(n)$ $(=G_1{}^n)$, i.e., a $GL(n)$-module, and then $\underset{\sim}{F}(M)$ is a tensor bundle over M. In particular if $F=(\overset{p}{\otimes}\mathbf{R}^n)\otimes\overset{q}{(}\mathbf{R}^{n^*})$ we get the tensor bundles $\underset{\sim}{\mathcal{T}_q{}^p}$.

0.5. Proposition. *Let $F_1\colon \mathcal{M}_n\to\mathcal{F}$ be a natural bundle for n-manifolds of fiber dimension f and let $F=F_1(\mathbf{R}^n)_0$, so that by the Main Theorem* (0.3) *F is a smooth $G_k{}^n$-space where $k=2^{f+1}-1$. Let $\underset{\sim}{F}\colon M_n\to\mathcal{F}$ be the natural bundle formed by the construction in* 0.4 *above. Then F_1 and $\underset{\sim}{F}$ are naturally equivalent functors.*

This proposition is a special case of the following more general result.

0.6. Proposition. *Let F_1 and F_2 be two natural bundles over n-manifolds and let $h\colon F_1(\mathbf{R}^n)_0\to F_2(\mathbf{R}^n)_0$ be a smooth G^n-equivariant map. There is a unique collection $\{H(M)\colon F_1(M)\to F_2(M)\}$ of smooth maps (one for each smooth n-manifold M) such that $H(\mathbf{R}^n)\colon F_1(\mathbf{R}^n)\to F_2(\mathbf{R}^n)$ restricted to $F_1(\mathbf{R}^n)_0$ agrees with h, and which are natural in the sense that if $\phi\colon M\to N$ is any embedding then $H(N)\circ F_1(\phi)=F_2(\phi)\circ H(M)$.*

Proof. Given $x_0\in M$ let ϕ be an embedding of $\mathbf{R}^n$ in M with $\phi(0)=x_0$. Then clearly the only possible choice of $H(M)_{x_0}\colon F_1(M)_{x_0}\to F_2(M)_{x_0}$ is $F_2(\phi)_0\circ h\circ F_1(\phi)_0{}^{-1}$. If ψ is a second such embedding of $\mathbf{R}^n$ in M then the germ γ of $\psi^{-1}\circ\phi$ at 0 is an element of G^n so that $h\gamma=\gamma h$, i.e., $hF_1(\psi^{-1}\circ\phi)_0=F_2(\psi^{-1}\circ\phi)_0h$ and by the functoriality of F_1 and F_2 we get $F_2(\phi)_0\circ h\circ F_1(\phi)^{-1}=F_2(\psi)_0\circ h\circ F_1(\psi)^{-1}$, so $H(M)_{x_0}$ is well defined. For $x\in\phi(\mathbf{R}^n)$ let $\tau_x\colon\mathbf{R}^n\to\mathbf{R}^n$ denote $v\to v+\phi^{-1}(x)$ and $\phi^x=\phi\circ\tau_x$. Then $H(M)_x\colon F_1(M)_x\to F_2(M)_x$ is given by $F_2(\phi^x)_0\circ h\circ F_1(\phi^x)_0{}^{-1}$ and it follows easily from (3) of definition 0.1 that $H(M)$ is smooth in $\mathcal{O}=\phi(\mathbf{R}^n)$. It is now elementary that the collection $\{H(M)\}$ so defined has all the desired properties. ■

0.7. *Historical remarks.* The classical concept corresponding to natural bundles is "geometric object." An historical survey and description of the gradual evolution of the bundle and functorial approach to the subject will be found in A. Nijenhuis' paper[2] and will not be repeated here. Some of the remarks above will be found in slightly different form in Nijenhuis' paper. In particular the fact that natural bundles of order less than $k+1$ can all be constructed (up to natural equivalence) by the method of 0.4 above is mentioned in Theorem 5.17 of [2]. In paragraph 4.14 Nijenhuis points out that classically when discussing geometric objects, finiteness of order was nearly always assumed *a priori* yet there was little discussion of the necessity for doing so. He does not commit himself as to whether he believes that natural bundles are *necessarily* of finite order, but it seems to be here that the question was first explicitly raised in print. A student of Nijenhuis, S. Salvioli in her thesis[6] proved that natural bundles had "infinitesimal pointwise finite order". Her proof used Peetre's theorem on the characterization of differential operators[4]. We give a more elementary proof of this in §3.

In [5] one of the authors proved that, for natural vector bundles, the dimension of the fiber always bounded the order (in fact even the number of primary components of the fiber, regarded as a $GL(n)$ module, was a bound). It was this phenomenon that suggested to us that some function of the fiber dimension might also be a bound for the order in the non-linear case.

§1. $G_\infty{}^n$ ACTS ON $F(\mathbf{R}^n)_0$

Let $G_\infty{}^n$ denote the group of infinite jets (at the origin) of origin preserving diffeomorphisms of $\mathbf{R}^n$. There is an exact sequence of groups

$$e\to\mathcal{J}^n\to G^n\xrightarrow{i_\infty}G_\infty{}^n\to e.$$

In this section, using a classical theorem of E. Borel and a technique of J. Peetre[3], we shall show that the kernel of $\tilde{F}\colon G^n\to\mathrm{Diff}(F(\mathbf{R}^n)_0)$ includes $\mathcal{J}^n$, so that $\tilde{F}$ may be regarded as defined on $G_\infty{}^n$ and $F(\mathbf{R}^n)$ may be considered a $G_\infty{}^n$ space.

1.1. Lemma (E. Borel). *If S_0 and S_1 are disjoint closed subsets of the unit sphere S^{n-1} of $\mathbf{R}^n$ then there is a function $\phi\colon\mathbf{R}^n\to\mathbf{R}$ such that:*

(1) *ϕ is smooth except at* 0.

(2) $D^\alpha\phi(x)=\mathcal{O}(|x|^{-|\alpha|})$ *for all α as $x\to 0$.*

(3) $\phi(x)=t$ *for* $x/|x|$ *in* S_t $(t=0,1)$.

(4) *If* $f: \mathbf{R}^n \to \mathbf{R}^m$ *is smooth and* $j_\infty(f)_0=0$ *then* $\psi: \mathbf{R}^n \to \mathbf{R}^m$, *defined by* $x \to \phi(x)f(x)$, *is smooth and* $j_\infty(\psi)_0=0$.

Proof. Let $\Phi: S^{n-1} \to \mathbf{R}$ be a smooth map such that $\Phi(x)=t$ for $x \in S_t$ and define $\phi(0)=0$ and $\phi(x)=\Phi(x/|x|)$ for $x \neq 0$. It is easy to verify properties (1) to (3). From Taylor's theorem with remainder it follows that $|D^\alpha f(x)| = \mathcal{O}(|x|^r)$ as $x \to 0$ for all α and r. From this and (2) and the rule for differentiating a product it follows by induction that $D^\alpha\psi(0)=0$ and that $D^\alpha\psi(x)\to 0$ as $x \to 0$ for all α. ■

1.2. THEOREM. *If* $\phi: M \to N$ *is an embedding of n-manifolds then* $F(\phi)_x: F(M)_x \to F(N)_x$ *depends only on* $j_\infty(\phi)_x$. In particular the kernel of $\tilde{F}: G^n \to \mathrm{Diff}(F(\mathbf{R}^n)_0)$ *includes* $\mathcal{J}^n$.

Proof. Let ϕ_1 and ϕ_2 be origin preserving diffeomorphisms of $\mathbf{R}^n$ such that $D^\alpha\phi_1(0)=D^\alpha\phi_2(0)$ for all α. It will suffice to show that $F(\phi_1)_0=F(\phi_2)_0$, and to this end we will construct a germ of a diffeomorphism ψ of $\mathbf{R}^n$ (at the origin) such that 0 is in the closure of each of the two open sets $\mathcal{O}_1$ and $\mathcal{O}_2$, where $\mathcal{O}_i$ is the interior of the set of x in $\mathbf{R}^n$ such that $\phi_i(x)=\psi(x)$. By 0.2 $F(\phi_i)_x=F(\psi)_x$ for $x \in \mathcal{O}_i$, so by continuity of $F(\phi)$ it will follow that $F(\phi_1)_0=F(\psi)_0=F(\phi_2)_0$. To construct ψ let S_0 and S_1 be disjoint closed subjects of S^{n-1} with non-empty interiors and let $\phi: \mathbf{R}^n \to \mathbf{R}$ be as in Lemma 1.1. Since $D^\alpha(\phi_1-\phi_2)(0)=0$ for all α, by (4) of Lemma 1.1 $x \to \phi_2(x)+\phi(x)(\phi_1(x)-\phi_2(x))$ is a smooth map $\mathbf{R}^n \to \mathbf{R}^n$ whose differential at the origin agrees with that of ϕ_2, so in particular it is non-singular. It follows that there is a diffeomorphism ψ of $\mathbf{R}^n$ whose germ at 0 agrees with that of this map. Now for $x/|x|$ in S_i, $\phi(x)=i$ $(i=0,1)$ so $\phi_2(x)+\phi(x)(\phi_1(x)-\phi_2(x))=\phi_2(x)$ for $x/|x|$ in S_0, and is $\phi_1(x)$ for $x/|x|$ in S_1. It follows that ψ has the required property. ■

It is worth noting that the proof of the above Theorem used only properties (1) and (2) ("naturality") of the Definition 0.1 of natural bundle and *not* property (3) ("continuity").

§2. THE GROUPS G_k^n AND G_∞^n

In this section we will develop some structure theorems for the groups G_k^n and G_∞^n, which are needed to prove the main theorem.

Let $P^{(i)}(R^n, R^n)$ denote the space of all homogeneous polynomial maps of degree i from R^n to R^n. Then G_k^n can be identified with the open subset $GL(n)+\bigoplus_{i=2}^{k} P^{(i)}(R^n,R^n)$ of the vector space $\bigoplus_{i=1}^{k} P^{(i)}(R^n,R^n)$. It follows that the tangent space of G_k^n at the identity can be naturally identified with the vector space of polynomial maps of R^n to itself of degree less than or equal to k which vanish at the origin, or equivalently with the k-jets of vector fields on R^n at the origin which vanish at the origin.

2.1. THEOREM ([5]). *The Lie algebra* $\mathcal{G}_k^n$ *of* G_k^n *is* $\{j_k(X)_0 | X \in C^\infty(TR^n)$ *and* $X(0)=0\}$, *with the bracket* $[j_k(X)_0, j_k(Y)_0]=j_k([Y,X])_0$. *Moreover, if* $\exp(tX)$ *denotes the flow in* $\mathbf{R}^n$ *generated by* X, *then* $\exp(j_k(X)_0)=j_k(\exp(X))_0$.

Proof. $\{j_k(X)_0 | X \in C^\infty(TR^n)$ and $X(0)=0\}$ is the tangent space of G_k^n at the identity. $t \to j_k(\exp(tX)_0$ is a group homomorphism from R to G_k^n with $(\mathrm{d}/\mathrm{d}t)|_{t=0} j_k(\exp(tX))_0 = j_k[(\mathrm{d}/\mathrm{d}t)|_{t=0}\exp(tX)]_0 = j_k(X)_0$ as the tangent vector at the identity, hence, $\exp(tj_k(X)_0)=j_k(\exp(tX))_0$. And

$$\begin{aligned}
[j_k(X)_0, j_k(Y)_0] &= \frac{1}{2}\frac{\mathrm{d}^2}{\mathrm{d}t^2}\bigg|_{t=0} j_k(\exp(-tX))_0 \cdot j_k(\exp(-tY))_0 \cdot j_k(\exp(tX))_0 \cdot j_k(\exp(tY))_0 \\
&= \frac{1}{2}\frac{\mathrm{d}^2}{\mathrm{d}t^2}\bigg|_{t=0} j_k(\exp(-tX)\circ\exp(-tY)\circ\exp(tX)\circ\exp(tY))_0 \\
&= \frac{1}{2} j_k\left(\frac{\mathrm{d}^2}{\mathrm{d}t^2}\bigg|_{t=0} \exp(-tX)\circ\exp(-tY)\circ\exp(tX)\circ\exp(tY)\right)_0 \\
&= j_k([Y,X])_0. \quad ■
\end{aligned}$$

2.2. COROLLARY ([15]). *Let X, Y be two vector fields on R^n such that $X(0)=Y(0)=0$ and $j_k(X)_0=j_k(Y)_0$. Then $j_k(\exp tX)_0=j_k(\exp tY)_0$.*

Let $j_{k,l}\colon G_k{}^n\to G_l{}^n$ be the projection sending $j_k(\phi)$ to $j_l(\phi)$ for $k>l$. Then $\{G_k{}^n, j_{k,l}\}$ is an inverse system of Lie groups. By the well-known extension lemma of E. Borel, given any system $\{a_\alpha\in\mathbf{R}^n|\alpha\in\mathbf{Z}^n,\ \alpha_i\geq 0\}$ there exists a smooth map $f\colon\mathbf{R}^n\to\mathbf{R}^n$ such that $D^\alpha f(0)=a_\alpha$.

It follows that $G_\infty{}^n$ is the inverse limit of $\{G_k{}^n, j_{k,l}\}$ with $j_k\colon G_\infty{}^n\to G_k{}^n$ sending $j_\infty(\phi)_0$ to $j_k(\phi)_0$ as projections. $B^n_{l+1,k}$ will denote the normal subgroup $\ker j_{k,l}$ of $G_k{}^n$, and N^n_{k+1} the normal subgroup $\ker j_k$ of $G_\infty{}^n$. Since the differential of $j_{k,l}$ also sends $j_k(X)_0$ to $j_l(X)_0$, we denote it by $j_{k,l}\colon \mathscr{G}_k{}^n\to\mathscr{G}_l{}^n$. So $\{\mathscr{G}_k{}^n, j_{k,l}\}$ also forms an inverse system of Lie algebras and the inverse limit is the infinite dimensional Lie algebra $\mathscr{G}_\infty{}^n=\{j_\infty(X)_0|X\in C^\infty(TR^n)$ and $X(0)=0\}$ with bracket $[j_\infty(X)_0, j_\infty(Y)_0]=j_\infty([Y,X])_0$ and $j_k\colon\mathscr{G}_\infty{}^n\to\mathscr{G}_k{}^n$ as projections. Let $\mathscr{N}^n_{k+1}$ denote the ideal $\ker j_k$ of $\mathscr{G}_\infty{}^n$; then the following proposition tells us that $\mathscr{N}^n_{k+1}$ is "the Lie algebra of N^n_{k+1}" and that the exponential map is bijective.

2.3. PROPOSITION. *Given $j_\infty(\phi)_0\in N^n_{k+1}$ there exists a vector field $X\in C^\infty(TR^n)$ such that $j_\infty(X)_0\in\mathscr{N}^n_{k+1}$ and $j_\infty(\exp(X))_0=j_\infty(\phi)_0$. Moreover $j_\infty(X)_0$ is uniquely determined.*

Proof. Since $B^n_{k+1,m}$ is a simply connected nilpotent subgroup of $G_k{}^n$ ([5]) for $m>k$, the exponential map is a diffeomorphism of its Lie algebra $\mathscr{B}^n_{k+1,m}$ onto $B^n_{k+1,m}$. Let $j_m(X^m)_0\in\mathscr{B}^n_{k+1,m}$ be the unique element such that $\exp(j_m(X^m)_0)=j_m(\phi)_0$. Then $\exp(j_l(X^m)_0)=\exp(j_{m,l}(j_m(X^m)_0))=j_{m,l}(\exp(j_m(X^m)_0))=j_{m,l}(j_m(\phi)_0)=j_l(\phi)_0$, so $j_l(X^m)_0=j_l(X^l)_0$ for $l<m$. Hence by the E. Borel extension lemma again there is a vector field X such that $j_m(X)_0=j_m(X^m)_0$ for all $m>k$ and certainly $j_k(X)_0=j_k(X^m)_0=0$. By Corollary 2.2 $j_m(\exp(X))_0=j_m(\exp(X^m))_0$. But $j_m(\exp(X^m))_0=\exp(j_m(X^m)_0)$ by Theorem 2.1, and we have $\exp(j_m(X^m)_0)=j_m(\phi)_0$, so $j_m(\exp(X))_0=j_m(\phi)_0$ for all $m>k$, i.e., $j_\infty(\exp(X))_0=j_\infty(\phi)_0$. ∎

2.4. *Definition.* For a diffeomorphism ϕ of $\mathbf{R}$ we define a diffeomorphism $\tilde\phi$ of $\mathbf{R}^n$ by sending $(x_1,\ldots,x_n)$ to $(\phi(x_1),x_2,\ldots,x_n)$. This induces a canonical embedding $i\colon G_\infty{}^1\to G_\infty{}^n$.

2.5. PROPOSITION. *The normal subgroup N of $G_\infty{}^n$ generated by $i(N_k{}^1)$ is $N_k{}^n$.*

Proof. By proposition 2.3 it will suffice to show that the ideal $\mathscr{N}$ generated by $\mathscr{N}_k{}^1$ in $\mathscr{G}_\infty{}^n$ is $\mathscr{N}_k{}^n$ and since the latter is an ideal which includes $\mathscr{N}_k{}^1$ it will be enough to show that $\mathscr{N}_k{}^n\subseteq\mathscr{N}$, i.e., that elements of $\mathscr{G}_\infty{}^n$ of the form $\sum_{|\alpha|\geq k} a_\alpha x^\alpha(\partial/\partial x_i)$ are in $\mathscr{N}$. If $q(x)(\partial/\partial x_1)$ is in $\mathscr{N}$ then so is $[q(x)(\partial/\partial x_1), x_i(\partial/\partial x_1)]=(\partial q/\partial x_1)x_i(\partial/\partial x_1)$. Using this remark recursively, starting from $q(x)=x_1{}^l$ with $l\geq k$ we see that $p(x)(\partial/\partial x_1)\in\mathscr{N}$ for any homogeneous polynomial $p(x)$ of degree $\geq l$. Hence more generally $\sum_{|\alpha|=k}^{2kn} a_\alpha x^\alpha(\partial/\partial x_1)$ is in $\mathscr{N}$. Since $\mathscr{N}$ is an ideal containing $x_1{}^k(\partial/\partial x_1)$ and $x_i{}^k(\partial/\partial x_1)$ it also contains the elements

$$\sum_{\alpha_1\geq 2k} a_\alpha x^\alpha\frac{\partial}{\partial x_1}=-\left[x_1{}^k\frac{\partial}{\partial x_1},\sum_{\alpha_1\geq 2k}\frac{a_\alpha}{\alpha_1-2k+1}x_1{}^{\alpha_1-k+1}x_2{}^{\alpha_2}\cdots x_n{}^{\alpha_n}\frac{\partial}{\partial x_1}\right]$$

and

$$\sum_{\alpha_i\geq k} a_\alpha x^\alpha\frac{\partial}{\partial x_1}=-\left[x_i{}^k\frac{\partial}{\partial x_1},\sum_{\alpha_n\geq k}\frac{a_\alpha}{\alpha_1+1}x_1{}^{\alpha_1+1}x_2{}^{\alpha_2}\cdots x_i{}^{\alpha_i-k+1}x_n{}^{\alpha_n}\frac{\partial}{\partial x_1}\right]$$

where in the latter $i\geq 2$. It follows in particular that $\mathscr{N}$ contains any element of the form $\sum_{|\alpha|\geq 2nk} a_\alpha x^\alpha(\partial/\partial x_1)$ and so by a preceding remark it also contains all elements of the form

$\sum_{|\alpha|\geq k} a_\alpha x^\alpha(\partial/\partial x_1)$.

Then

$$\left[\sum_{|\alpha|\geq k} a_\alpha x^\alpha\frac{\partial}{\partial x_1}, x_1\frac{\partial}{\partial x_i}\right]=\sum_{|\alpha|\geq k} a_\alpha\frac{\partial(x^\alpha)}{\partial x_i}x_1\frac{\partial}{\partial x_1}-\sum_{|\alpha|\geq k} a_\alpha x^\alpha\frac{\partial}{\partial x_i}$$

is in $\mathscr{N}$ and hence so is $\sum_{|\alpha|\geq k} a_\alpha x^\alpha(\partial/\partial x_i)$. ∎

§3. POINTWISE FINITE ORDER

In this section we shall show that the order of natural bundles is at least "pointwise finite", i.e., we shall prove that:

3.1. THEOREM. *Let $F: \mathcal{M}_n \to \mathcal{F}$ be a natural bundle. Then for each p in $F(\mathbf{R}^n)_0$ there is an integer $k(p)$ such that $(G_\infty{}^n)_p$, the isotropy subgroup of the action of $G_\infty{}^n$ on $F(\mathbf{R}^n)_0$, includes $N^n_{k(p)}$.*

First we prove "infinitesimal pointwise finite order" and for this we need the following:

3.2. LEMMA. *Let $\mathcal{A}$ be a subalgebra of $\mathcal{G}_\infty{}^n$ of finite codimension and suppose $\mathcal{A} = \varprojlim j_l(\mathcal{A})$. Then there is an integer k such that $\mathcal{N}_k{}^n \subseteq \mathcal{A}$.*

Proof. We have a decreasing sequence of subalgebras of $\mathcal{G}_\infty{}^n$:

$$\mathcal{G}_\infty{}^n = \mathcal{A} + \mathcal{N}_1{}^n \supseteq \mathcal{A} + \mathcal{N}_2{}^n \supseteq \cdots \supseteq \mathcal{A} + \mathcal{N}_r{}^n \supseteq \cdots \supseteq \mathcal{A}.$$

Since $\mathcal{A}$ has finite codimension there is an integer k for which $\mathcal{A} + \mathcal{N}_k{}^n = \mathcal{A} + \mathcal{N}_l{}^n$ for all $l \geq k$, and we claim $\mathcal{N}_k{}^n \subseteq \mathcal{A}$. For suppose $j_\infty(f)_0 \in \mathcal{N}_k{}^n \subseteq \mathcal{A} + \mathcal{N}_l{}^n$ for $l \geq k$. Then there exists $j_\infty(f_l)_0 \in \mathcal{A}$ such that $j_l(f_l)_0 = j_l(f)_0$, i.e., $j_\infty(f)_\infty \in \varprojlim j_l(\mathcal{A}) = \mathcal{A}$. ∎

Now let X be a smooth vector field on $\mathbf{R}^n$ vanishing at the origin. By the continuity assumption 0.1(3) the smooth local flow $\exp tX$ on $\mathbf{R}^n$ induces a smooth flow $F(\exp tX)|F(\mathbf{R}^n)_0$ on $F(\mathbf{R}^n)_0$ and therefore defines a generating smooth vector field on $F(\mathbf{R}^n)_0$, which by 1.2 and 2.2 depends only on $j_\infty(X)_0$. We denote this vector field by $d\tilde{F}(j_\infty(X)_0)$. It also follows from the continuity assumption that the map $d\tilde{F}: \mathcal{G}_\infty{}^n \to C^\infty(T(F(\mathbf{R}^n)_0))$ is a Lie algebra homomorphism. Then for $p \in F(\mathbf{R}^n)_0$ the map $j_\infty(X)_0 \to d\tilde{F}(j_\infty(X)_0)(p)$ is a linear map from $\mathcal{G}_\infty{}^n$ to the finite dimensional vector space $T(F(\mathbf{R}^n)_0)$ and its kernel $(\mathcal{G}_\infty{}^n)_p$ is a subalgebra of $\mathcal{G}_\infty{}^n$ with finite codimension, (consisting of the $j_\infty(X)_0 \in \mathcal{G}_\infty{}^n$ such that the flow $F(\exp tX)|F(\mathbf{R}^n)_0$ fixes p, i.e., the isotropy subalgebra of $\mathcal{G}_\infty{}^n$ at p). Moreover, by 3.4 below $(\mathcal{G}_\infty{}^n)_p = \varprojlim j_k((\mathcal{G}_\infty{}^n)_p)$ and hence by Lemma 3.2 there is an integer $k(p)$ such that $(\mathcal{G}_\infty{}^n)_p \supseteq \mathcal{N}^n_{k(p)}$. This proves infinitesimal point-wise finite order which coupled with Proposition 2.3 implies Theorem 3.1 on pointwise finite order.

3.3. PROPOSITION [5, Lemma 1.5]. *Let ϕ and $\{\phi_m\}$ be origin preserving diffeomorphisms of $\mathbf{R}^n$ and suppose $j_m(\phi_m)_0 = j_m(\phi)_0$ for all m. Then there exists a sequence $\{\psi_m\}$ of origin preserving diffeomorphisms of $\mathbf{R}^n$ such that ϕ_m and ψ_m have the same germ at the origin and $\psi_m \to \phi$ in the C^∞ topology.*

3.4. COROLLARY. *Let $\mathcal{A} = (\mathcal{G}_\infty{}^n)_p$. Then $\mathcal{A} = \varprojlim j_l(\mathcal{A})$.*

Proof. Let $j_\infty(X_m)_0 \in (\mathcal{G}_\infty{}^n)_p$ and $j_m(X_m)_0 = j_m(X)_0$. We want to show that $j_\infty(X)_0 \in (\mathcal{G}_\infty{}^n)_p$, i.e., that $F(\exp tX)(p) = p$ for all t. Now by 2.2 $j_m(\exp tX_m)_0 = j_m(\exp tX)_0$, and so by 3.3 for each t there exists $\psi_m{}'$ with $j_\infty(\psi_m{}') = j_\infty(\exp tX_m)$ and $\psi_m{}' \to \exp tX$ in the C^∞ topology. By the continuity assumption $F(\psi_m{}') \to F(\exp tX)$ in the compact open topology. But by 1.2 $F(\psi_m{}')_0 = F(\exp tX_m)_0$, and $F(\exp tX_m)_0(p) = p$ by assumption. If follows that $F(\exp tX)(p) = p$. ∎

§4. PROOF OF THE MAIN THEOREM

We have shown in the last section that every point of the $G_\infty{}^n$-space $F(R^n)_0$ has finite order, which may vary from point to point. However, there is an upper bound, namely 2^{f+1}, where $f = \dim(F(R^n)_0)$. We now prove this.

Recall we identify $\mathcal{G}_k{}^1$ with the polynomials in one variable of degree less than $k+1$ without constant term. $(p(x) \sim j_k(p(x)(\partial/\partial x))_0)$.

4.1. PROPOSITION. *Let H_k be a Lie subalgebra of $\mathcal{G}_k{}^1$, $x^k \notin H_k$, and $\dim(\mathcal{G}_k{}^1/H_k) = m$. Then $k < 2^{m+1}$.*

Proof. By induction on m. For $m = 1$, we have $H_k \oplus Rx^k = \mathcal{G}_k{}^1$. So there exist real numbers a, b, c such that $x + ax^k$, $x^2 + bx^k$, $x^3 + cx^k \in H_k$. Suppose $k \geq 4$. Since $[x + ax^k, x^2 + bx^k] + (x^2 + bx^k) = -b(k-2)x^k \in H_k$ and $x^k \notin H_k$, $b = 0$. Similarly, $[x + ax^k, x^3 + cx^k] + 2(x^3 + cx^k) = -c(k-3)\, x^k \in H_k$ and $x^k \notin H_k$ imply that $c = 0$. Hence $x^2, x^3 \in H_k$ which imply that $x^i \in H_k$ for all $i \geq 2$, contradicting the fact that $x^k \notin H_k$. So $k < 4 = 2^{l+1}$. Now suppose it is true for integers less than or equal to m. Let H_k be a Lie subalgebra of $\mathscr{G}_k{}^1$ such that $x^k \notin H_k$ and $\dim(\mathscr{G}_k{}^1/H_k) = m+1$, $m \geq 1$. Consider $\bar{H}_k = H_k + Rx^k = j_{k,k-1}^{-1}(j_{k,k-1}(H_k))$, a Lie subalgebra of $\mathscr{G}_k{}^1$; then $\dim(\mathscr{G}_k{}^1/\bar{H}_k) = m$. We claim that if l is the smallest positive integer such that $B^1_{l+1,k} \subset \bar{H}_k$ then $k-1 \geq l \geq m$ and $(k-1)/2 \leq l$. The first part of the claim is clear, for $Rx^k = B^1_{k,k} \subset \bar{H}_k$ and $\dim(\mathscr{G}_k{}^1/B^1_{l+1,k}) \geq \dim(\mathscr{G}_k{}^1/\bar{H}_k)$. For the second part of the claim, suppose on the contrary that $(k-1)/2 > l$, i.e. $k - l > l + 1$. Since $B^1_{l+1,k} \subset \bar{H}_k$ and $\bar{H}_k = H_k + Rx^k$, there exist real numbers a, b such that $x^{l+1} + ax^k$, $x^{k-l} + bx^k \in H_k$. Hence $[x^{l+1} + ax^k, x^{k-l} + bx^k] = -(k-1-2l)\, x^k \in H_k$. But $k - 1 - 2l > 0$, so $x^k \in H_k$ a contradiction. Let $\bar{H}_l = j_{k,l}(\bar{H}_k)$, then $x^l \notin \bar{H}_l$. For if $x^l \in \bar{H}_l$ then there is an element h in $B^1_{l+1,k} \subset \bar{H}_k$ such that $x^l + h \in \bar{H}_k$, which implies $x^l \in \bar{H}_k$. So $B^1_{l,k} \subset \bar{H}_k$, contradicting the choice of l. It is clear that $p = \dim(\mathscr{G}^1/\bar{H}_l) \leq \dim(\mathscr{G}_k{}^1/\bar{H}_k) = m$. Therefore we can use the induction hypothesis $l < 2^{p+1}$. But $p \leq m$ and $(k-1)/2 \leq l$, so $k \leq 2(2^{m+1}-1)+1 = 2^{m+2}-1$. ■

[Added in revision. In [1] Epstein and Thurston obtain the much better bound $k \leq 2m+1$ (which they show is sharp for $n = 1$) as follows: from the relation $[x^i, x^j] = (i-j)x^{i+j-1}$ and the assumption $x^k \notin H_k$ we see that H_k cannot contain both terms of the form (x^l + higher order) and (x^{k+1-l} + higher order) unless $2l = k+1$, and from this we conclude the codimension of H_k in $\mathscr{G}_k{}^1$ cannot be less than $[k/2]$. This of course implies that in our main Theorem, 0.3, we can also replace the bound 2^{f+1} by $2f+1$.]

Now we are ready to prove the Main Theorem for natural bundles over 1-manifolds.

4.2. THEOREM. *Every natural bundle $F: \mathcal{M}_1 \to \mathscr{F}$ is of finite order less than $r = 2^{f+1}$, where* $f = \dim(F(R^n)_0)$.

Proof. By Proposition 2.3, it suffices to prove that $\mathcal{N}_r{}^1$ is included in $(\mathscr{G}_\infty{}^1)_p$ for all $p \in F(R)_0$. By Theorem 3.1 there is a least integer $k(p)$ with $(\mathscr{G}_\infty{}^1)_p$ including $\mathcal{N}^1_{k(p)+1}$, so the linear map $d\tilde{F}_p: \mathscr{G}_\infty{}^1 \to T(F(R))_p$ defined by $d\tilde{F}_p(j_\infty(X)_0) = d\tilde{F}(j_\infty(X)_0)(p)$ factors through $\mathscr{G}^1_{k(p)}$ with kernel $H_p = j_{k(p)}((\mathscr{G}_\infty{}^1)_p)$. Then $x^{k(p)} \notin H_p$ and $m = \dim(\mathscr{G}^1_{k(p)}/H_p) \leq \dim T(F(R)_0)_p = f$. By 4.1 $k(p) < 2^{m+1}$, so *a fortiori* $k(p) < 2^{f+1} = r$, i.e. $\mathcal{N}_r{}^1 \subset (\mathscr{G}_\infty{}^1)_p$. ■

4.3. *Proof of the Main Theorem.* Recall the canonical embedding $i: G_\infty{}^1 \to G_\infty{}^n$ defined in 2.4. $\tilde{F} \circ i: G_\infty{}^1 \to \mathrm{Diff}(F(R^n))$ corresponds to a natural bundle $F_1: M_1 \to \mathscr{F}(F_1(M) = F(M \times R^{n-1})|M \times 0$ and $F_1(\varphi) = F(\varphi \times \mathrm{id}_{R^{n-1}})|M \times 0)$. Then by 4.2 F_1 has an order $l < 2^{f+1}$, i.e. $i(N^1_{l+1}) \subset \ker \tilde{F}$. But $\ker \tilde{F}$ is a normal subgroup of $G_\infty{}^n$, so by proposition 2.5 $N^n_{l+1} \subset \ker \tilde{F}$. Therefore $\tilde{F}$ factors through $G_l{}^n$, i.e. F has order less than 2^{f+1}. ■

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