SEMINAR ON TRANSFORMATION GROUPS

BY

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WITH CONTRIBUTIONS BY

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TABLE OF CONTENTS

TABLE OF CONTENTS

CONTENTS

CONTENTS

CONTENTS

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INTRODUCTION

In this book, a transformation group is a compact Lie group G acting on a topological, usually locally compact, space X. Transformation groups are discussed mainly from the point of view of algebraic topology, and the problems most often center around relations between topological properties of G, X, the fixed point set, the orbits, and the space of orbits X/G. A typical topic along these lines, and chronologically one of the first ones, is P. A. Smith's theory of prime period maps of homology spheres.

Apart from some modifications and additions, this book consists of the Notes of a Seminar held at the Institute for Advanced Study in 1958-59. Familiarity with the basic notions in the theory of transformation groups and with algebraic topology is assumed. As to the latter, the results concerning sheaves, spectral sequences and fibre bundles, for which references to the expositions of Cartan, Godement, Steenrod or the undersigned can be made, are usually taken for granted. However, certain topics for which this is not possible are given an independent discussion. As a result, five chapters (I, II, X, XI, XVI) are concerned with questions of algebraic topology and make no mention of transformation groups. We first briefly summarize those.

Chapter I is devoted to cohomology manifolds. These are spaces which have the local cohomological properties of manifolds. The definition adopted here is equivalent to that of a locally orientable generalized manifold in the sense of Wilder. Chapter II introduces a homology theory for locally compact spaces and uses it in order to derive a Poincaré duality theorem for cohomology manifolds. Chapter X is purely algebraic, and is concerned with the spectral sequences attached to a differential module endowed with two filtrations. The discussion is carried only as far as is needed to supply a convenient algebraic framework for the discussion of Fary's modification of the spectral sequence of a map (XI). It is obtained by combining Leray's filtration with a filtration stemming from a decreasing sequence of closed subspaces. To Chapter XVI we have relegated some remarks about the spectral sequence of a map, including a brief summary of its definition, a procedure for putting on it a module structure over the cohomology of the base, a discussion of families of supports, a sufficient condition under which the stalk of the Leray sheaf in a fibre bundle is the cohomology of the fibre, and an application to the cohomology of certain products.

As to Chapters I and II, the undersigned would like to point out that cohomology manifolds are emphasized for technical convenience rather than for the sake of generality. When it comes to manifolds, genuine ones are of the course the most important ones, and differentiable transformation groups already form a very interesting special case. However, if one does not wish to assume that all group actions (including those on the one point compactification of a euclidean space) are differentiable, then assumptions of a cohomological character are much more tractable with the methods used in this seminar and have a more "hereditary" character (e.g., Theorem 4.10 (b) of I, or 3.2 of V are false for manifolds).

The discussion of transformation groups starts with Chapter III, where Floyd gives a sheaf-theoretic interpretation of Smith's theory of prime period maps and applies the transfer homomorphism to the discussion of the orbit space of a finite group.

Prime period maps are also studied in Chapters IV and V, together with the action of the circle group T^1, from a different point of view. It consists of a systematic use of the twisted product $X_G = X \times_G E_G$ of X with a universal bundle for G. This space on one hand is a fibre bundle over the classifying space B_G of G, with fibre X, and on the other hand it has a projection π_1 on X/G, such that the inverse image of a point Y of X/G can be identified with the classifying space of the stability group of any point in this orbit. It allows us to tie together the cohomology groups of X, X/G, and the fixed point set F, with those of the classifying spaces of the stability groups and of G. After a few general remarks about X_G, Chapter IV proves various, mostly known, results about prime period maps on homology spheres and their analogues for the circle group. Chapter V is devoted to the local theorem for these two cases, stating mainly that F and X/G are cohomologically locally connected if X is, and that F is a cohomology manifold if X is.

The discussion of elementary abelian p-groups (by which we mean direct products of cyclic groups Z_p of order p if p is prime, of circle groups if $p = 0$) is resumed in Chapters XII, XIII, with the help of Fary's spectral sequence. In Chapter XII, it is shown that if X is totally non-homologous to zero in X_G, then G has fixed points. In

Chapter XIII a relation is obtained between the dimensions of X, F, and the fixed point sets of the subgroups of index p if p is prime, or of the closed connected subgroups of codimension 1 if $p = 0$, when X is a cohomology sphere or a cohomology manifold. In contrast with the results of Chapters IV, V, those of Chapters XII, XIII do not seem to follow by induction from the case $G = Z_p$, T^1.

Chapters V to IX are devoted to some known basic theorems: the local finiteness of the number of orbit types in a cohomology manifold, first when G is a torus (VI), and then in the general case (VII); the existence of equivariant embeddings in euclidean spaces and of a slice (VIII); various results concerning the orbits of the highest dimension in a cohomology manifold (IX).

In Chapter XIV it is shown that if a compact Lie group G acts effectively on a sphere with one class of orbits and is not transitive, then $G = T^1$, $SU(2)$ and G acts freely, and that if G acts on euclidean space with two classes of orbits, then it has a fixed point.

Chapter XV presents new results of Bredon pertaining to the action of a compact Lie group acting on a cohomology n-manifold when the fixed point set F has the greatest possible dimension allowed by the results of Chapter IX, namely $n - k - 1$ where k is the highest dimension of the orbits. The highest dimensional orbits are then integral cohomology spheres, the orbit space around a fixed point x is a cohomology manifold with boundary F and there is a local cross-section around x for the orbits.

In short, these Notes give an exposition of some basic theorems and of results obtained by cohomological methods in the theory of compact Lie groups of transformations. They make no claim at completeness, although it has been tried to give a fairly comprehensive discussion of the topics chosen, and do not cover all aspects of the theory of transformation groups. It should also be pointed out that purely cohomological methods, while forming a major part of the subject at present, have their limitations, as is shown by well known counter examples; and that in view of this, it would certainly be very desirable to make more effective use of differentiability assumptions than has been possible so far.

As to references, each chapter carries its own bibliography. The numbering of lemmas, theorems, remarks, etc. in each chapter is cumulative. For example 2.1 means Section 2.1 in the same chapter; (IV, 2.1) means Section 2.1 in Chapter IV; 3.2(1) refers to formula (1) or assertion (1) in 3.2, as the case may be.

Finally, I would like to thank G. Bredon for his help in checking the final version of these Notes.

The Institute for Advanced Study Armand Borel

VIII: SLICES AND EQUIVARIANT IMBEDDINGS

R. S. Palais

§ 1. Notation and Preliminaries

G will always denote a compact Lie group. Unless otherwise stated, the dimension of a separable metric space is the classical covering dimension.

1.1. DEFINITION. A G-space is a completely regular space X together with an action of G on X. We denote by X/G the orbit space of X and by Π_X the natural map of X on X/G.

We recall that the topology for X/G is the strongest making Π_X continuous, i.e., $S \subseteq X/G$ is open (closed) if and only if $\Pi_X^{-1}(S)$ is open (closed).

1.2. PROPOSITION. X/G is completely regular.

PROOF. Given $\tilde{x} \in X/G$ and a closed subset $\tilde{F}$ of X/G not containing $\tilde{x}$, $F = \Pi_X^{-1}(\tilde{F})$ is a closed invariant subset of X disjoint from the compact set $\tilde{x}$ and by Urysohn's lemma (applied to $\tilde{x}$ and the closure of F in the Čech compactification of X) there exists $f : X \longrightarrow I$ with $f \mid \tilde{x} \equiv 0$ and $f \mid F \equiv 1$. Let $f^*(x) = \int_G f(gx)dg$ and $\tilde{f} = f^* \circ \Pi_X^{-1}$. Then $\tilde{f} : X/G \longrightarrow I$ and $\tilde{f}(\tilde{x}) = 0$, $\tilde{f} \mid \tilde{F} \equiv 1$. q.e.d.

1.3. PROPOSITION. Π_X is open.

PROOF. If O is open in X then $\Pi_X^{-1}(\Pi_X(O)) = GO = \bigcup_{g \in G} gO$ is open, hence $\Pi_X(O)$ is open.

Since X/G can clearly have only one topology making Π_X both continuous and open:

> 1.4. COROLLARY. The topology of X/G is uniquely determined by the conditions that Π_X be open and continuous.

1.5. PROPOSITION. If X is a metrizeable G-space there is a metric ρ for X relative to which each operation of G is isometric. If we define $\tilde{\rho}(\tilde{x}, \tilde{y}) = \operatorname{Inf} \{\rho(x, y) \mid x \in \tilde{x}, y \in \tilde{y}\}$ then $\tilde{\rho}$ is a metric for X/G. If X is separable so is X/G.

PROOF. If ρ^* is any metric for X then it is easily checke that $\rho(x, y) = \int_G \rho(gx, gy)dg$ is a metric for X and $\rho(gx, gy) = \rho(x,$ by the invariance of Haar measure. It is clear that Π_X is distance de- creasing and hence continuous relative to ρ and $\tilde{\rho}$. Moreover Π_X ma the ρ ε-ball about x onto the $\tilde{\rho}$ ε-ball about $\Pi_X(x)$ and hence is open relative to ρ and $\tilde{\rho}$ so, by 1.4, $\tilde{\rho}$ induces the correct topolog If X is separable it is Lindelöf, hence $X/G = \Pi_X(X)$ is Lindelof and, being metrizeable, separable.

1.6. PROPOSITION. Π_X is a closed mapping.

PROOF. Let F be closed in X and let x be adherent to $\Pi_X^{-1}(\Pi_X(F)) = GF$. Choose nets $\{g_\alpha\}$ in G and $\{f_\alpha\}$ on F so that $g_\alpha f_\alpha \longrightarrow x$. Since G is compact we can suppose $g_\alpha \longrightarrow g$. Then $f_\alpha \longrightarrow g^{-1}x$ so $g^{-1}x \in F$ and $x \in GF$. Hence $\Pi_X^{-1}(\Pi_X(F))$ is closed so $\Pi_X(F)$ is closed.

1.7. PROPOSITION. Π_X is a proper mapping.

PROOF. It is a general fact that a map of a Hausdorff space into a Hausdorff space is proper if it is closed and the pre-image of every point is compact.

1.8. PROPOSITION. If J is an invariant subset of the G-space X then every neighborhood of J includes an invariant neighborhood of J.

PROOF. If V is an open set including J then $0 = X - \Pi_X^{-1}(\Pi_X(X - V))$ is clearly invariant and included in V. It is open by 1.6 and $J \subseteq 0$ follows easily from the invariance of J.

1.9. DEFINITION. A G-space X is:

(1) Euclidean: if X is a finite dimensional real vector sp with an orthogonal structure and the action of G on X is an orthog representation.

(2) Riemannian: if X is a Riemannian manifold and each op tion of G is an isometry.

(3) Differentiable: if X is a differentiable ($= C^\infty$) manifold and each operation of G is differentiable (it then follows by a theorem of Bochner and Montgonery [1] that the natural map of $G \times X \longrightarrow X$ is differentiable).

REMARK. Since an isometry of a Riemannian manifold is differentiable (see e.g., [2]) a Riemannian G-space is differentiable. Conversely given a differentiable G-space, X, by averaging any Riemannian tensor for X relative to Haar measure in an obvious way we get a Riemannian tensor for X which is invariant under the operations of G, so a differentiable G-space can always be considered Riemannian.

1.10. DEFINITION. Let X and Y be G-spaces. A mapping $f : X \longrightarrow Y$ is <u>equivariant</u> if $f(gx) = gf(x)$ for all $(g, x) \in G \times X$. If in addition f is one-to-one on each orbit of X we call f <u>isovariant</u>.

1.11. PROPOSITION. If $f : X \longrightarrow Y$ is equivariant then $G_x \subseteq G_{f(x)}$ for all $x \in X$ and f is one-to-one on $G(x)$ if and only if equality holds. Hence if f is isovariant $G_x = G_{f(x)}$ for all $x \in X$.

PROOF. $G_x \subseteq G_{f(x)}$ is trivial. If f is one-to-one on $G(x)$ then $(f \mid G(x))^{-1}$ is equivariant and we get the reverse inclusion. Conversely if $G_x = G_{f(x)}$ and $f(g_1x) = f(g_2x)$ then $g_2^{-1}g_1f(x) = f(x)$ so $g_2^{-1}g_1x = x$ so $g_1x = g_2x$ and f is one-to-one on $G(x)$.

1.12. TIETZE-GLEASON LEMMA. If K is a compact (respectively, closed) invariant subspace of the G-space (respectively, normal G-space) X and if $f : K \longrightarrow E$ is an equivariant map of K into a Euclidean G-space, then f admits an equivariant extension $\tilde{f} : X \longrightarrow E$.

PROOF. By Tietze's extension theorem (applied, in the first case, to the Čech compactification of X) we can find a continuous extension $f^* : X \longrightarrow E$. Let $\tilde{f}(x) = \int_G g^{-1}f^*(gx)dg$. For $x \in K$ $\tilde{f}(x) = \int g^{-1}gf(x)dg = f(x)$ so $\tilde{f}$ extends f. By the invariance of Haar measure

$$\tilde{f}(\gamma x) = \int g^{-1}f^*(g\gamma x)dg = \int \gamma g^{-1}f^*(gx)dg = \gamma\tilde{f}(x)$$

so $\tilde{f}$ is equivariant.

§2. Orbit types

If H is a closed subgroup of G we denote by (H) the set of conjugate subgroups $\{gHg^{-1} \mid g \in G\}$. We call sets of the form (H) G-orbit types. If X is a G-space and $\Omega \in X/G$ an orbit in X, say $\Omega = G(x)$ then since $G_{gx} = gG_xg^{-1}$ it follows that $\{G_\omega \mid \omega \in \Omega\} = (G_x)$ is a G-orbit type which we call the orbit type of Ω and denote by $[\Omega]$ (note $[G(x)] = (G_x)$). Two orbits, in perhaps different G-spaces are call equivalent if there exists an equivariant homeomorphism of one onto the other.

2.1. PROPOSITION. Two orbits are equivalent if and only if they are of the same type.

PROOF. If Ω and Ω' are equivalent then by 1.11 they are of the same type. Conversely if Ω and Ω' are of the same type then we can find $\omega \in \Omega$ and $\omega' \in \Omega'$ with $G_\omega = G_{\omega'}$. Then $g\omega \longrightarrow g\omega'$ is clearly a well defined, continuous, one-to-one equivariant map of Ω ont Ω'. Since Ω is compact this map is a homeomorphism.

If X is a G-space then those subsets of X which are unions of all orbits of a fixed type form a partitioning of X into invariant subsets. We call this partitioning of X (or equivalently of X/G), each subset labeled by the corresponding orbit type, the orbit structure of X. We recall from Chapters VI and VII of Floyd and Bredon that if X is a cohomology manifold the orbit structure is locally finite and hence, if X is compact, finite (we shall give a very simple proof, based on the existence of slices, when X is a differentiable G-space later). The significance of this is apparent from the following theorem of Mostow [3], the proof of which is one of our main goals.

THEOREM. The following conditions are necessary and sufficient for a G-space X to admit an equivariant imbedding in some Euclidean G-space:

(1) X is metrizeable, separable, and of finite dimension.

(2) X has finite orbit structure.

The first step towards proving this is the fact, noted independently by Mostow [3] and Palais [4], that a single orbit can always be isovariantly imbedded in a Euclidean G-space.

2.2. PROPOSITION. If H is a closed subgroup of G there exists a Euclidean G-space E and a $v \in E$

such that $G_v = H$.

PROOF. Let R be right regular representation of G in the real $L^2(G)$ and let $\bar{f}$ be a continuous real valued function on the left coset space G/H which assumes the value 1 only at H. Define $f \in L^2(G)$ by $f(g) = \bar{f}(gH)$. It is clear that $H = \{g \in G \mid R_g f = f\}$. By Peter-Weyl $L^2(G) = \bigoplus_{i=1}^{\infty} E_i$ where the E_i are finite dimensional subspaces invariant under R. Let f_i be the projection of f on E_i, and $H_i = \{g \in G \mid R_g f_i = f_i\}$. Then the H_i are closed subgroups of G such that $H = \cap_i H_i$. Now the closed subgroups of a compact Lie group satisfy the descending chain condition (at each step in a properly descending chain either the dimension or number of components must drop), hence we can find $i_1, \ldots, i_n$ such that $H = \cap_{j=1}^{n} H_{i_j}$. We put $E = \bigoplus_{j=1}^{n} E_{i_j}$ and $v = f_{i_1} + \ldots + f_{i_n}$.

2.3. THEOREM. If X is a G-space and $x \in X$ there exists an equivariant map of X into a Euclidean G-space which is one-to-one on $G(x)$.

PROOF. By 2.2 we can find a Euclidean G-space E and $v \in E$ with $G_v = G_x$. Then $G(v)$ and $G(x)$ are orbits of the same type so 2.1 they are equivalent, i.e., there exists an equivariant homeomorphism of $G(x)$ onto $G(v)$. By 1.12 this can be extended to an equivariant map of X into E.

§3. Slices

3.1. DEFINITION. Let H be a closed subgroup of G. A local cross-section in G/H is a non-singular differentiable map $\chi : U \longrightarrow G$ such that U is an open neighborhood of H in G/H, $\chi(H) = e$, and $\Pi \circ \chi$ = identity where $\Pi : G \longrightarrow G/H$ is canonical.

REMARK. The existence of local cross-sections follows easily from Chevalley [5] pp. 109, 110.

3.2. DEFINITION. Let X be a G-space and H a closed subgroup of G. An H-slice in X is a subset S of X such that

(1) S is invariant under H.

(2) $gS \cap S \neq \Phi \Longrightarrow g \in H$.

(3) If $\chi : U \longrightarrow G$ is a local cross section in G/H then the map $F : U \times S \longrightarrow X$ defined by $F(u, s) = \chi(u)s$ is a homeomorphism of $U \times S$ onto an open set in X.

If $x \in X$ then a slice at x is a G_x-slice in X which contains x.

3.3. PROPOSITION. If S is an H-slice in X then $\pi_X \mid S$ is an open map of S into X/G and in particular $\pi_X(S)$ is open.

PROOF. Let V be open in S. Using the notation of 3.2, $F(U \times V)$ is open in X and hence $\pi_X(F(U \times V))$ is open in X/G. Clear $\pi_X(F(U \times V)) \subseteq \pi_X(V)$ and since $F(U \times V) \supseteq F(\{H\} \times V) = \chi(H)V = eV = V$ we in fact have equality.

3.4. PROPOSITION. If S is an H-slice in X then $G_s \subseteq H$ for all $s \in S$.

PROOF. Immediate from (2) of 3.2.

3.5. COROLLARY. If there exists a slice at x then there is a neighborhood of x at each point of which the isotropy group is conjugate to a subgroup of G_x.

PROOF. Using the notation of 3.2 with $G_x = H$, $F(U \times S)$ is a neighborhood of x and $G_{\chi(u)s} = \chi(u)G_s\chi(u)^{-1}$ while $G_s \subseteq G_x$ by 3.4. We note that an H-slice S in X is an H-space. The next result says that S/H can be naturally identified with GS/G.

3.6. PROPOSITION. If S is an H-slice in X then $h : Hs \longrightarrow Gs$ is a homeomorphism of S/H onto $\pi_X(S) = GS/G$.

PROOF. If $Gs = Gs'$ with $s, s' \in S$ then $gs = s'$ for some $g \in G$ and, by (2) of 3.2, $g \in H$ so $Hs = Hs'$ and h is one-to-one. Now $\pi_S : S \longrightarrow S/H$ is open and continuous and by 3.3 so also is π_X. Since $\pi_X \mid S = h \circ \pi_S$ it follows that h is continuous and open, hence a homeomorphism.

The following powerful and elegant result is essentially due t Mostow [3] although the basic idea goes back to Gleason [6]. While eas patched up, Mostow's proof contains several errors and in fact the statement of the theorem given in [3] is easily seen to be incorrect.

3.7. THEOREM. If S' is an H-slice in the G-space Y and if f is an equivariant map of the G-space X into Y, then $S = f^{-1}(S')$ is an H-slice in X.

PROOF. Since $f(HS) = Hf(S)$ $HS' = S'$ it follows that $HS \subseteq f^{-1}(S') = S$ so S is invariant under H. Since $gS' \cap S' \supseteq gf(S) \cap f(S) = f(gS) \cap f(S) \supseteq f(gS \cap S)$ it follows that $gS \cap S \neq \Phi \Longrightarrow gS' \cap S' \neq \Phi \Longrightarrow g \in H$. Thus S satisfies (1) and (2) of 3.2 and it remains to verify (3). Let $\chi : U \longrightarrow G$ be a local cross-section in G/H and let $F' : (u, s') \longrightarrow \chi(u)s'$ be the corresponding homeomorphism of $U \times S$ onto an open set W' of Y. We will complete the proof by showing that $F : (u, s) \longrightarrow \chi(u)s$ is a homeomorphism of $U \times S$ onto $f^{-1}(W') = W$.

(1) F is continuous. Obvious.

(2) F is onto W.

PROOF. Let $w \in W$ so $f(w) = \chi(u)s'$. Then $s' = \chi(u)^{-1}f(w) = f(\chi(u)^{-1}w) = f(s)$ where $s = \chi(u)^{-1}w$ is in S because $f(s) = s' \in S'$. Then $w = \chi(u)s = F(u, s) \in F(U \times S)$, so $W \subseteq F(U \times S)$. Conversely $f(F(U \times S)) = F'(U \times f(S)) \subseteq F'(U \times S') = W'$ so $F(U \times S) \subseteq f^{-1}(W') = W$.

(3) F is one-to-one and F^{-1} is continuous.

PROOF. We prove both facts at once by showing that if $F(u_\alpha, s_\alpha) \longrightarrow F(u, s)$ then $u_\alpha \longrightarrow u$ and $s_\alpha \longrightarrow s$. In fact applying f to $f(u_\alpha, s_\alpha) \longrightarrow F(u, s)$ we see that $F'(u_\alpha, f(s_\alpha)) \longrightarrow F'(u, f(s))$ and since F' is bicontinuous $u_\alpha \longrightarrow u$. But then $\chi(u_\alpha)^{-1} \longrightarrow \chi(u)^{-1}$ which together with $\chi(u_\alpha)s_\alpha = F(u_\alpha, s_\alpha) \longrightarrow F(u, s) = \chi(u)s$ implies that $s_\alpha \longrightarrow s$. q.e.d.

REMARK. If $x \in X$ and S is a slice of x then define $f : GS \longrightarrow Gx$ by $f(gs) = gx$; $g \in G$, $s \in S$. It is easily checked that f is a well-defined equivariant retraction of GS onto Gx. What is more it can be shown that the triple (GS, G_x, f) has the structure of a fibre bundle, the fibre being S, the structural group G_x/K (where $K = \cap_{s \in S} G_s$) and the associated principal bundle being G/K.

Conversely it follows from 3.7 that if O is an open invariant neighborhood of Gx and f is an equivariant retraction of O onto Gx then $f^{-1}(x)$ is a slice at x (for clearly $\{x\}$ is a G_x-slice in Gx) and we recover f from this slice by the above process. In other words there is a natural one-to-one correspondence between slices at x and equivariant retractions of open invariant neighborhoods of Gx onto Gx.

The following result is due to Koszul [6]. A shorter proof using Bochner's theorem on linearity at stationary points (which, by the way, is a consequence of the theorem) will be found in Mostow [3] page 444.

3.8. THEOREM. If X is a differentiable G-space and $x \in X$ then there exists a slice S at x. Moreover S can be so chosen that, in a suitable coordinate system about x, G_x act linearly and S is an open disc with center x in an invariant subspace.

PROOF. As noted in the remark following 1.9 we can suppose that X is a Riemannian G-space, in which case it follows that if v is a tangent vector to X for which $\mathrm{Exp}(v)$ is defined then $\mathrm{Exp}(\delta g(v))$ is defined and equals $g\mathrm{Exp}(v)$ for all $g \in G$ (here δg is the differential of g when g is considered a diffeomorphism of X). Let $\Sigma = G(x)$ and let $N(\Sigma)$ be the normal bundle of Σ in X. Since Σ is a compact submanifold of X it is well know that for ε sufficiently small Exp maps $N(\Sigma, \varepsilon) = \{v \in N(\Sigma) \mid \|v\| < \varepsilon\}$ diffeomorphically onto $B(\Sigma, \varepsilon) = \{x' \in X \mid \rho(x', \Sigma) < \varepsilon\}$. Now $\Sigma_x^{\perp}(\varepsilon) = \{v \in \Sigma_x^{\perp} \mid \|v\| < \varepsilon\}$ is an open disc in $\Sigma_x^{\perp}$ and $\Sigma_x^{\perp}$ is invariant under δg for $g \in G_x$, hence because $\mathrm{Exp}\ \delta g = g\ \mathrm{Exp}$ it follows that in any Riemannian normal coordinate system about x G_x acts linearly (in fact orthogonally) and $S = \mathrm{Exp}(\Sigma_x^{\perp}(\varepsilon))$ is an open disc centered at x in an invariant subspace. We now verify the three conditions of (3.2) that will show that S is a G_x-slice.

(1) $G_xS = S$. Already proved.

(2) $gS \cap S \neq \Phi \Longrightarrow g \in G_x$.

PROOF. Suppose $g(\mathrm{Exp}\ v) = \mathrm{Exp}\ \omega$ with $v, w \in \Sigma_x^{\perp}(\varepsilon)$. Then $\mathrm{Exp}(\delta g(v)) = \mathrm{Exp}(\omega)$ and $\delta g(v) \in N(\Sigma, \varepsilon)$. Since Exp is a diffeomorphism on $N(\Sigma, \varepsilon)$ it follows that $\delta g(v) = \omega$. But $\omega \in \Sigma_x^{\perp}$ and $\delta g(v) \in \Sigma_{g(x)}^{\perp}$ and so we must have $g(x) = x$.

(3) Let $\chi : U \longrightarrow G$ be a local cross-section in G/G_x and define $K : (u, v) \longrightarrow \delta\chi(u)v$ on $U \times \Sigma_x^{\perp}(\varepsilon)$. Then K is a diffeomorphism of $U \times \Sigma_x^{\perp}(\varepsilon)$ onto an open set W in $N(\Sigma, \varepsilon)$, its inverse in fact is $\omega \longrightarrow \delta\chi(h^{-1}\pi(\omega))^{-1}\omega$ where π is the fibre projection of $N(\Sigma) \longrightarrow \Sigma$ and h is the diffeomorphism $gG_x \longrightarrow gx$ of G/G_x onto Σ. Then $\tilde{F} = \mathrm{Exp} \circ K$ is a diffeomorphism of $U \times \Sigma_x^{\perp}(\varepsilon)$ onto an open set $V = \mathrm{Exp}(W)$ in X and since $\mathrm{Exp}(\delta\chi(u)v) = \chi(u)\ \mathrm{Exp}(v)$, $F : (u, s) \longrightarrow \chi(u)\ s$ is a diffeomorphism of $U \times S$ onto V.

3.9. THEOREM (Mostow [3]). If X is a G-space and $x \in X$ there exists a slice at x.

PROOF. By 2.3 there is an equivariant map f of X onto a Euclidean G-space E which is one-to-one on $G(x)$. By 3.8 there exists a $G_{f(x)}$-slice S' in E containing x. By 3.7 $S = f^{-1}(S')$ is a

$G_{f(x)}$-slice in X containing x. But by 1.11 $G_{f(x)} = G_x$ so S is a slice at x.

3.10. COROLLARY (Montgomery-Zippin [7]). If X is a G-space and $x \in X$ there is a neighborhood W of x such that G_w is conjugate to a subgroup of G_x for all $w \in W$.

PROOF. 3.5 and 3.9.

3.11. PROPOSITION. Let X be a G-space and $\{S_\alpha\}$ a collection of H-slices in X such that the sets $\{GS_\alpha\}$ are pairwise disjoint. Then $S = \cup_\alpha S_\alpha$ is an H slice in X.

PROOF. Clearly S is invariant under H. If $gx_\alpha = x_\beta$ with $x_\alpha \in S_\alpha$ and $x_\beta \in S_\beta$ then since $GS_\alpha \cap GS_\beta = \Phi$ we must have $\alpha = \beta$ and hence $g \in H$. Thus $gS \cap S \neq \Phi \Longrightarrow g \in H$. Now by 3.3 $GS_\alpha = \Pi_X^{-1}(\Pi_X(S_\alpha))$ is open in X hence $S_\alpha = S \cap GS_\alpha$ is open in S. If $\chi : U \longrightarrow G$ is a local cross-section in G/H then $F : (u, s) \longrightarrow \chi(u)s$ maps each $U \times S_\alpha$ homeomorphically onto an open set W_α of X. Since the $U \times S_\alpha$ form a disjoint open covering of $U \times S$ and the W_α a disjoint open covering of $W = \cup_\alpha W$, f maps $U \times S$ homeomorphically onto W.

The following metatheorem will have several important applications in the sequel.

3.12. METATHEOREM. Let S be a statement valued function defined for all compact Lie groups. If the truth of $S(H)$ for every proper closed subgroup of an arbitrary compact Lie group G entails the truth of $S(G)$, then $S(G)$ is true for every compact Lie group G.

PROOF. Assume $S(G)$ is false for some compact Lie group G and let n be the least integer which is the dimension of a compact Lie group G for which $S(G)$ is false. Among all compact Lie groups of dimension n for which $S(G)$ is false choose one with fewest connected components and note that $S(H)$ is true for all proper closed subgroups of G.

As a first application we prove a special case of the result proved earlier in the book by Floyd and Bredon.

3.13. THEOREM. A differentiable G-space has locally finite orbit structure.

PROOF. By 3.12 we can assume that the theorem holds for all proper closed subgroups H of G. If $\dim X = 0$ the theorem is obvious and we can therefore also assume that the theorem holds for differentiable G-spaces of dimension less than $\dim X$. Let $x \in X$. If x is not a fixed point let S be a slice at x, which by 3.8 can be chosen to be a differentiable G_x-space. Since G_x is a proper closed subgroup of G we can find a neighborhood V of x in S in which only a finite number of G_x-orbit types occur. Then for $v \in V$ G_v is conjugate in G_x to one of a finite number of subgroups of G. Then by 3.3 GV is a neighborhood of x and clearly only a finite number of orbit types occur in GV.

If x is a fixed point then by Bochner's theorem (or by 3.8 which generalizes Bochner's theorem) we can find a coordinate system about x in which the action of G is linear and in fact orthogonal. Then if S is a sphere with center x in the coordinate system S is a differentiable G-space of dimension less than $\dim X$ so S has locally finite orbit structure and, being compact, actually has finite orbit structure. Since the isotropy group of a point is clearly constant on open rays only a finite number of orbit types occur within the coordinate system. Thus for any x in X we can find a neighborhood in which only a finite number of orbit types occur.

3.14. PROPOSITION. If X is a G-space with (locally) finite orbit structure and if H is a closed subgroup of G then X has (locally) finite orbit structure when considered as an H-space.

PROOF. It clearly suffices to show that if Σ is an orbit of type (K) then Σ has finite orbit structure as an H-space. But since Σ is equivalent to G/K we see that Σ is a compact differentiable H-space and the proposition follows from 3.13.

3.15. COROLLARY. If X is a G-space with (locally) finite orbit structure then any H-slice has (locally) finite orbit structure as an H-space.

We now make a second application of 3.12.

3.16. THEOREM. If X is a separable metric G-space of dimension n then $\dim X/G \leq n$.

PROOF. We recall that by 1.5 X/G is separable metric. Let F be the set of stationary points of X. Then F is a closed subspace of X/G of dimension $\leq n$. Let $U = X/G - F$. Since U is open and hence an F_σ it follows from the sum theorem that it suffices to prove that X/G has dimension $\leq n$ at points of U. Let $\tilde{x} \in U$ and let S be a slice at a point x of $\tilde{x}$. We can suppose by 3.12 that the theorem holds for the proper closed subgroup G_x of G and hence that $\dim S/G_x \leq \dim S \leq n$. By 3.6 $\dim \Pi_x(S) = \dim GS/G = \dim S/G_x \leq n$ and since by 3.7 $\Pi_x(S)$ is a neighborhood of $\tilde{x}$, U has dimension $\leq n$ at $\tilde{x}$.

REMARK. The formal properties of dim used in the above proof include only topological invariance, the sum and monotonicity theorems, and the fact that $\dim X \leqq n$ provided each point x of X has a neighborhood with $\dim U \leqq n$. Thus if we have a G-space X with such a dimension function defined for X and X/G, then $\dim X/G \leqq \dim X$. In particular if L is a principal ideal domain and $\dim_L$ is the cohomological dimension of H. Cohen, defined for locally compact spaces (cf. reference in I), then if X is a locally compact G-space, $\dim_L X/G \leqq \dim_L X$.

The following is apparently due to Milnor, at least in its present form. It replaces the much more complicated Theorem 4.1 of Mostow [3].

3.6. THEOREM. Let X be a paracompact space with covering dimension n and let $\{U_\alpha\}$ be an open covering of X. Then there is an open covering of X refining $\{U_\alpha\}$, $\{G_{i\beta}\}_{\beta \in B_i}$ $i = 0, \ldots, n$ such that $G_{i\beta} \cap G_{i\beta'} = \Phi$ if $\beta \neq \beta'$.

PROOF. By making an initial refinement of $\{U_\alpha\}$ we can suppose that the order of $\{U_\alpha\}$ is at most n, i.e., no $x \in X$ is contained in more than $n + 1 U_\alpha$. Let $\{\varphi_\alpha\}$ be a locally finite partition of unity with support $\varphi_\alpha \subseteq U_\alpha$. Given $i = 0, \ldots, n$ let B_i = the set of unordered i+1-tuples from the indexing set of the $\{U_\alpha\}$. Given $\beta = (\alpha_0, \ldots, \alpha_i) \in B_i$ we set $G_{i\beta} = \{x \in X \mid \varphi_{\alpha_j}(x) > 0$ and $\alpha \notin \beta \Longrightarrow \varphi_\alpha(x) < \varphi_{\alpha_j}(x)\ j = 0, \ldots, i\}$. Since in a neighborhood of any point of X only a finite number of φ_α are not identically zero each $G_{i\beta}$ is open. Clearly $G_{i\beta} \cap G_{i\beta'} = \Phi$ if $\beta \neq \beta'$ and $G_{i\beta} \subseteq \cap_{\alpha \in \beta}$ support $\varphi_\alpha \subseteq \cap_{\alpha\in\beta} U_\alpha$ so $\{G_{i\beta}\}$ refines $\{U_\alpha\}$. It remains to show that $\{G_{i\beta}\}$ covers X. Given $x \in X$ let $(\alpha_0, \ldots, \alpha_m)$ be the indices such that $\varphi_\alpha(x) > 0$. Since $x \in \cap_{i=0}$ support $\varphi_\alpha \subseteq \cap_{i=0} U_\alpha$, $m \leq n$. We can suppose $\varphi_\alpha(x) = \ldots = \varphi_{\alpha_i}(x) > \varphi_{\alpha_{i+1}}(x) \geq \ldots \geq \varphi_{\alpha_m}$ in which case clearly $x \in G_{i(\alpha_0 \ldots \alpha_i)}$.

3.17. DEFINITION. Let X be a G-space and H a closed subgroup of G. A subset of X/G is called H-liftable if it is the image under π_X of an H-slice in X.

3.18. LEMMA. An open subset of an H-liftable set is H-liftable.

PROOF. If S is an H-slice and V is open in $\pi_X(S)$ then i is immediate from the definition that $S \cap \pi_X^{-1}(V)$ is an H-slice.

The following is one of the crucial steps in getting equivaria imbeddings in Euclidean space.

3.19. THEOREM. Let X be a separable metric G-space of dimension h, H a closed subgroup of G and $X_{(H)}$ the union of all orbits in X of type (H). Then there exist $n + 1$ H-slices in X $S_o, \ldots, S_n$ such that $X_{(H)} \subseteq GS_o \cup \ldots \cup GS_n$.

PROOF. Let $\tilde{X}_{(H)} = \pi_X(X_{(H)})$. We must show that $\tilde{X}_{(H)}$ is included in the union of $n + 1$ H-liftable sets. Let $\{U_\alpha\}$ be the collection of all H-liftable subsets of X/G and let $\tilde{Y} = \cup_\alpha U_\alpha$. By 3.9 $\tilde{X}_{(H)} \subseteq Y$. Since Y is metrizeable (and hence paracompact) by 1.5 and of dimension $\leq n$ by 3.15 and since the U_α are open by 3.3 it follows fr 3.16 that we can find an open covering of Y refining $\{U_\alpha\}$, $\{G_{i\beta}\}_{\beta \in B}$ $i = 0, \ldots, n$ such that $G_{i\beta} \cap G_{i\beta'} = \Phi$ if $\beta \neq \beta'$. By 3.18 each $G_{i\beta}$ is H-liftable and it follows from 3.11 that $\tilde{S}_i = \cup_{\beta \in B_i} G_{i\beta}$ is also H-liftable. Clearly $Y = \tilde{S}_o \cup \ldots \cup \tilde{S}_n$ and since $\tilde{X}_{(H)} \subseteq Y$ we are thr

§4. Equivariant imbeddings in Euclidean space

4.1. LEMMA. If a G-space admits an equivariant imbedding in a Euclidean G-space then it admits an equivariant imbedding in the unit sphere of a Euclidean G-space.

PROOF. Let f_1 be an equivariant imbedding of X in a Euclidean G-space V, W a one-dimensional Euclidean G-space on which G acts trivially, w a non-zero vector in W and define $f_2 : X \longrightarrow V$ by $f_2(x) = (f_1(x), w)$. Then $f(x) = f_2(x)/\|f_2(x)\|$ defines an equivari imbedding of X in the unit sphere of $V \oplus W$.

4.2. THEOREM (Mostow [3]). Let X be a metrizeable

G-space and F a closed invariant subspace of X. If both F and $X - F$ admit equivariant imbeddings in Euclidean G-spaces so does X.

PROOF. Let f_1 be an equivariant imbedding of F in a Euclidean G-space V, extended by 1.12 to an equivariant map of X into V. Let ρ be an invariant metric for X (see 1.5) and define $h(x) = \text{Inf}\,\{(\rho(x, z) + \|f_1(x) - f_1(z)\| \mid z \in F\}$. Then h is continuous on X, positive on $X - F$, invariant under the action of G, and $x_n \longrightarrow x \in F \Longrightarrow h(x_n) \longrightarrow 0$.

Now let f_2^* be an equivariant imbedding of $X - F$ in a Euclidean G-space W which, by 4.1, we can suppose satisfies $\|f_2^*(x)\| \equiv 1$. Define $f_2(x) = h(x)f_2^*(x)$. That f_2 is continuous, one-to-one and equivariant is clear. If $f_2(x_n) \longrightarrow f_2(x)$ then $h(x_n) = \|f_2(x_n)\| \longrightarrow \|f_2(x)\| = h(x)$ so $f_2^*(x_n) \longrightarrow f_2^*(x)$ and hence $x_n \longrightarrow x$, so f_2 is an imbedding. Moreover we can extend f_2 to a continuous equivariant map of X into W by defining $f_2(x) = 0$ for $x \in F$. We now define $f : X \longrightarrow V \oplus W$ by $f(x) = (f_1(x), f_2(x))$. It is clear that f is continuous, equivariant and a homeomorphism on each of F and $X - F$. Since $f_2(x) = 0$ $x \in F$ and $f_2(x) \neq 0$ $x \in X - F$ it follows that f is one-to-one. Now suppose $f(x_n) \longrightarrow f(x)$. The proof will be complete if we can show that $x_n \longrightarrow x$. If $x \in X - F$ then $f_2(x) \neq 0$ hence $f_2(x_n) \neq 0$ for large n, hence $x_n \in X - F$ for large n and, since f is a homeomorphism on $X - F$, $x_n \longrightarrow x$. Now suppose $x \in F$. Then by definition of h we can choose $z_n \in F$ so that $\rho(x_n, z_n) \leq 2h(x_n)$ and $\|f_1(x_n) - f_1(z_n)\| \leq 2h(x_n)$. Since $h(x_n) = \|f_2(x_n)\| \longrightarrow \|f_2(x)\| = 0$ it follows that $\lim f_1(z_n) = \lim f_1(x_n) = f_1(x)$, and since f_1 is a homeomorphism on F $z_n \longrightarrow x$. But then since $\rho(x_n, z_n) = 2h(x_n) \longrightarrow 0$, $x_n \longrightarrow x$. q.e.d.

4.3. COROLLARY. Let X be a separable, metric, finite dimensional G-space and F the set of stationary points of X. If $X - F$ admits an equivariant imbedding in a Euclidean G-space then so does X.

PROOF. F is closed in X and being a finite dimensional separable metric space admits an imbedding in a Euclidean space V which is automatically equivariant if we let G act trivially on V.

4.4. COROLLARY. Let X be a metrizeable, separable G-space and $U_1, \ldots, U_n$ a covering of

X by open invariant subsets. If each U_i admits an equivariant imbedding in a Euclidean G-space then so does X.

PROOF. If $n = 1$ the theorem is trivial so we can assume inductively that $U_1 \cup \dots \cup U_{n-1}$ admits an equivariant imbedding in a Euclidean G-space. Then $F = X - U_n \subseteq U_1 \cup \dots \cup U_{n-1}$ admits an equivariant imbedding in a Euclidean G-space, and since $X - F = U_n$ does als so does X by 4.2.

4.5. THEOREM. Let X be a G-space and S an H-slice in X. If S admits an H-equivariant imbedding in a Euclidean H-space then GS admits a G-equivariant imbedding in a Euclidean G-space.

PROOF. Let f_1 be an H-equivariant imbedding of S in a Euclidean H-space V. It is immediate from the Frobenius reciprocity theorem that there is a Euclidean G-space W which considered as an H-space contains V as an invariant linear subspace (see [8], italicize remark bottom of page 83). Let U be a Euclidean G-space with a vector $u \in U$ such that $G_u = H$ (see 2.2). Define $f : GS \longrightarrow W \oplus U$ by $f(gs) = (gf_1(s), gu)$. We note first that f is well defined, for if $g_1s_1 = g_2s_2$ then $g_2^{-1}g_1s_1 = s_2$ and by (2) of 3.2 $g_2^{-1}g_1 \in H$. Since f is H-equivariant $g_2^{-1}g_1f_1(s_1) = f_1(s_2)$ so $g_1f(s_1) = g_2f(s_2)$ and since $G_u = H$ $g_1u = g_2u$. It is clear that f is continuous and equivariant so it remains to show that f is one-to-one and that f^{-1} is continuous. We can kill both birds with one stone by showing that $f(g_ns_n) \longrightarrow f(gs$ implies $g_ns_n \longrightarrow gs$. Now $f(g_ns_n) \longrightarrow f(gs)$ gives $g_nu \longrightarrow gu$ and $G_u = H$ it follows that $g_n = \gamma_nh_n$ where $h_n \in H$ and $\gamma_n \longrightarrow g$. But $f(g_ns_n) \longrightarrow f(gs)$ also gives $g_nf_1(s_n) \longrightarrow gf_1(s)$ or $\gamma_nf_1(h_ns_n) = \gamma_nh_nf_1(s_n) \longrightarrow gf_1(s)$ and since $\gamma_n \longrightarrow g$ we get $f_1(h_ns_n) \longrightarrow f_1(s$. Since f_1 is a homeomorphism on S we see that $h_ns_n \longrightarrow s$ and hence $g_ns_n = \gamma_nh_ns_n \longrightarrow gs$. q.e.d.

4.6. THEOREM [Mostow]. If G is a compact Lie group and X is a separable, metrizeable G-space of finite dimension and with finite orbit structure then S admits an equivariant imbedding in a Euclidean G-space.

PROOF. By 3.12 we can assume that the theorem holds if in the statement we replace G by any of its proper closed subgroups, and by we can suppose that X has no stationary points. Let $(H_1), \dots, (H_k)$

be the orbit types occurring in X and note that each H_i is a proper closed subgroup of G. By 3.19 we can find subsets S^i_j $i = 1, \ldots, k$ $j = 0, \ldots, \dim X$, of X such that S^i_j is an H_i-slice in X and $X = \bigcup_{i,j} GS^i_j$. Now by 3.15 each S^i_j has finite orbit structure as an H_i-space and of course each S^i_j is separable, metric, and of finite dimension. Thus each S^i_j admits an H_i-equivariant imbedding in a Euclidean H_i-space. By 5.4 each GS^i_j admits a G-equivariant imbedding in a Euclidean G-space. Since each GS^i_j is open in X (3.3) and $X = \bigcup GS^i_j$ it follows by 4.4 that X admits an equivariant imbedding in a Euclidean G-space. q.e.d.

4.7. COROLLARY. If in 4.6 we assume in addition that X is locally compact, then the equivariant imbedding of the conclusion can be chosen so as to be a proper map.

PROOF. By 4.1 and 4.6 we can find an equivariant imbedding f of X into the unit sphere in some Euclidean G-space V. Let Φ be a positive real valued invariant function on X which vanishes at infinity (e.g., $\Phi(x) = \Sigma_n 2^{-n}\psi_n(\Pi_X(x))$ where $\{\psi_n\}$ is a locally finite partition of unity for X/G with support ψ_n compact) and define $f^*(x) = (1/\Phi(x))$ $f(x)$. It is clear that f^* is an equivariant imbedding of X in V (cf. proof that f^*_2 is an imbedding in 4.2). Let K be any compact subset of V and put $M = \sup\{\|k\| \mid k \in K\}$. Then $f^{*-1}(K) \subset \{x \in X \mid \Phi(x) \geq M\}$ which is a compact subset of X, hence f^* is proper.

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