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Sprays *)

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INTRODUCTION

If an affine connexien si given over a manifold it determines a "spray" of geodesics emanting from each point. This spray enables one to single out certain second order tangent vectors to be called of "pure second order", this notion dependinag on the spray; we call this a "dissection" of the second order tangent vectors. The object of this paper is to prove, conversely, that every dissection of second order tangent vectors arises in this way from the spray of geodesics of an affine connexion and that the spray is uniquely determined by the dissection. We also consider conditions under which two affine connexions have the same spray of geodesics.

1. HIGHER ORDER CONTACT ELEMENTS

We give here a brief discussion of higher order tangent vectors and differentials from the standpoint that will be used thruout this paper.

M will always denote a C^∞ manifold. If m ε M we let F_m^+ denote the "system" of all those real valued C^∞ functions whose domain is an open subset of M containing m. If f and g are in F_m^+ then so are $f+g$ and fg (their domains being the intersection of the domains of f and of g) and so is cf for c ε R(R will always denote the real numbers); but this "system" is not an algebra (it is not even a group under addition, due to troubles about domains of the functions). Nevertheless we use the usual terminology of algebra: ideal, linear subspace, quotient space, etc., in the usual way. Let F_m^c be the elements of F_m^+ that are constant on some neighborhood of m (the neighborhood varying with the function) and F_m the elements of F_m^+ which vanish at m. Then every element f ε F_m^+ can be expressed, essentially uniquely, as $f = f_o + f_c$ where

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$f_o \varepsilon F_m$ and $f_c \varepsilon F^c_m$ (the cnly non-uniqueness is in the domains of f_o and f^c and this is irrelevant for us).

We define a k-*th order tangent vector at* m to be a real linear map t of $F^+_m \longrightarrow R$ which vanishes on F^c_m and F^{k+1}_m (F^{k+1}_m = all sums of products of $k+1$ elements in F_m). If $k = 1$ one verifies easily that this is the same as saying that t is a real linear derivation of F^+_m into R, so our first order tangent vectors are tangent vectors in the usual sense. The k-th order tangent vectors form a linear space over R in the usual way, which we denote by M^k_m; we write M_m for M^1_m.

We define the space of k-th order differentials at m to be F_m/F^{k+1}_m; this is a linear space over R and we denote it by ${}^kM_m$. If $f \varepsilon F^+_m$ we define its k-th order differential at m, d^kf, to be the element of ${}^kM_m$ defined by: if $f = f_o + f_c$ as above then d^kf = the natural projection of f_o into F_m/F^{k+1}_m. We can make d^kf into a linear function on M^k_m by defining

$$(d^k f)(t) = tf \quad \text{for all} \quad t \varepsilon M^k_m. \tag{1.1}$$

This formula establishes a duality between M^k_m and ${}^kM_m$ (because every element of ${}^kM_m$ is of the form d^kf and if $d^kf = d^k g$ then there is an $h \varepsilon F^c_m$ for which $f-g+h \varepsilon F^{k+1}_m$); we shall frequently refer to this duality below

We now wish to obtain the usual coordinate expression for a second-order tangent vector; the corresponding result for k-th order tangent vectors is also true but we shall only consider the case $k=2$ (to keep the notation simple, — for it is the only case we shall need). First we need: If $x_1, \ldots, x_d$ is any coordinate system at m with all $x_i(m) = 0$ and $f \varepsilon F^+_m$ then there is a $g \varepsilon F^3_m$ such that, on some neighborhood of m,

$$f = f(m) + \Sigma \frac{\partial f}{\partial x_i}(m)\, x_i + \sum_{i \leq j} \frac{\partial^2 f}{\partial x_i \partial x_j}(m)\, x_i x_j + g \tag{1.2}$$

Proof: We use the well known theorem from advanced calculus that there exist $g_i \varepsilon F^+_m$ such that, on some neighborhood of m, $f = \Sigma g_i x_i$. It is then trivial that $g_i(m) = \dfrac{\partial f}{\partial x_i}(m)$. For the moment we write $a_i = g_i(m)$ and apply the same theorem to the $g_i - a_i$ to get the existence of $g_{ij} \varepsilon F^+_m$ such that on some neighborhood of m, $g_i - a_i = \Sigma g_{ij} x_j$. Let $a_{ij} = g_{ij}(m)$ and write

$$\alpha) \qquad f = f(m) + \Sigma a_i x_i + \Sigma a_{ij} x_i x_j + \Sigma (g_{ij} - a_{ij}) x_i x_j .$$

Differentiating this gives

$$\frac{\partial^2 f}{\partial x_i \partial x_j}(m) = a_{ij} + a_{ji}$$

Substituting this and the value of the a_i in α) gives (1.2) since the last sum in α) is clearly in F^3_m.

Using this we see that if t is any second order tangent vector at m and $x_1, \ldots, x_d$ any coordinate system at m then

$$t = \Sigma a_i \frac{\partial}{\partial x_i}(m) + \sum_{i \leq j} a_{ij} \frac{\partial^2}{\partial x_i \partial x_j}(m) \tag{1.3}$$

where the a_i and a_{ij} are unique and are given by

$$a_i = tx_i, \qquad a_{ij} = \begin{cases} t(x_i x_j) \text{ if } i \neq j \\ t(x_i^2)/2 \text{ if } i = j \end{cases}$$

(We use the notation $\frac{\partial}{\partial x_i}(m)$ for the tangent vector which assigns the number $\frac{\partial t}{\partial x_i}(m)$ to f, and similarly for second order tangent vectors).

Proof: We may assume all $x_i(m) = 0$. Applying t to (1.2) shows that (1.3) holds with the particular choice of a_i and a_{ij} indicated above. To show this is the only choice for which it holds one applies (1.3) to an x_i, an x_i^2, or an $x_i x_j$, and finds the a_i and a_{ij} are necessarily those given above.

We remark that we could equally well have taken as a canonical representation of a second order tangent vector expression

$$t = \Sigma\, b_i \frac{\partial}{\partial x_i}(m) + \Sigma\, b_{ij} \frac{\partial^2}{\partial x_i\, \partial x_j}(m)$$

with the assumption that the matrix b_{ij} be symmetric. In this case the b and b_{ij} are given by

$$b_i = tx_i, \; b_{ij} = t(x_i x_j)/2.$$

If Φ is any C^∞ mapping of M into N (where N is also a C^∞ manifold) then we have a $d^k\Phi$ carrying $M_m^k \dashrightarrow N_n^k$ (where $n = \Phi(m)$), defined, as in the first order case, so we shall not elaborate on it. We also have its dual, carrying ${}^kM_m \longleftarrow {}^kN_m$. And if Φ is a diffeomorphism of a neigborhood of m onto a neighborhood of n then this differential is an isomorphism onto, etc.

We now point out that ${}^2M_m = F_m/F_m^3$ contains a distinguished subspace, $H_m = F_m^2/F_m^3$ and that this H_m is naturally isomorphic to the space of symmetric bilinear functions on $M_m \times M_m$; for this reason we call H_m the space of *Hessians* at m. If $H \varepsilon H_m$ we define H(s,t) for s,t ε M_m by: $H = \Sigma\, f_i\, g_i + F_m^3$ where the f_i and g_i are in F_m; then $H(s,t) = \Sigma (sf_i)(tg_i) + \Sigma (tf_i)(sg_i)$. This is easily proved independent of the f_i, g_i used in the representation of H. Conversely, if a symmetric bilinear function B is given on $M_m \times M_m$ it gives rise to an $H \varepsilon H_m$ as follows. Let $x_1, \ldots, x_d$ be any coordinate system at m and let $h_{ij} = B\left(\frac{\partial}{\partial x_i}(m), \frac{\partial}{\partial x_j}(m)\right)$. Let $f = \Sigma\, h_{ij} x_i x_j$, so $f \varepsilon F_m^2$. Then H = the image of f in F_m^2/F_m^3 will have $H(s,t) = B(s,t)$ for all s, t. These mappings establish a real linear isomorphism between F_m^2/F_m^3 and the symmetric bilinear functions on $M_m \times M_m$.

If $f \varepsilon F_m^+$ and $df = 0$ at m (i.e. $d^1 f = 0$ at m, in the above notation) one ordinarily defines the Hessian of f at m; in these terms the definition is the following. Let $f = f_o + f_c$ as above. The statement that $df = 0$ says exactly that $f_o \varepsilon F_m^2$, so f_o has a natural projection H in $H = F_m^2/F_m^3$; this H, considered as a bilinear function on $M_m \times M_m$, is the Hessian of f. Then the Hessian matrix of f relative to a coordinate system $x_1, \ldots, d_x$ is defined by $h_{ij} =$ $= H\left(\frac{\partial}{\partial x_i}(m), \frac{\partial}{\partial x_j}(m)\right)$.

In ${}^2M_m$ we have the distinguished subspace H_m of Hessians; in M_m^2 we have the distinguished subspace of first order tangent vectors; and these subspaces are annihilators of each other under the duality between ${}^2M_m$ and M_m^2. In neither of these does the distinguished subspace have a natural complement. In terms of coordinates the trouble can be expressed by noting that a second order tangent vector whose first order components are 0 in terms of one coordinate system may have non-zero first order components in another. So "pure second order" tangent vectors are not defined in M_m^2 and "pure first order differentials" are not defined in ${}^2M_m$ (the first order differentials can be identified wirh ${}^2M_m/H_m$).

We define a *dissection* of M_m^2 to be an assignment of a linear complement that we denote by M_m^c, of M_m, in M_m^2; we call elements of M_m^c *pure* second order tangent vectors (relative to the given dissection). Thus M_m^2 is the direct sum of M_m and M_m^c and we have projections of M_m^2 into each of these. A dissection of M_m^2 gives rise to a *dissection* of ${}^2M_m$, if we define H_m^c = the annihilator of M_m^c (in the duality previously considered between M_m^2 and ${}^2M_m$). Then we have projections defined of elements of ${}^2M_m$ into H_m and H_m^c. And, conversely, a complement of H_m^c defines, thru duality, a complement M_m^c to M_m in M_m^2.

If we have a dissection given at m then we have an assignment to each $f \varepsilon F_m^+$ of a Hessian H_f at m by: If $f = f_o + f_c$ (as above) then $f_o + F_m^3$ is an element of ${}^2M_m$ and its projection into H_m (given by the dissection) will be the Hessian H_f of f. The mapping: $f \to H_f$ is: a) real linear, and b) if $df = 0$ (at m) then H_f is the usual Hessian of f. Conversely, an assignment to each $f \varepsilon F_m^+$ of a Hessian H_f with these two properties comes from a dissection of M_m^2 for from it we can define a projection of M_m^2 into H_m, by: if $\mu \varepsilon {}^2M_m$ then $\mu = f + F_m^3$ and we define the projection of μ into H_m to be the Hessian H_f of f. Because of a) and b) this is independent of the choice of f. We then define H_m^c = those elements of ${}^2M_m$ whose projection into H_m is 0. Then defines a dissection of ${}^2M_m$ which gives rise to the same Hessians as the assignment with which we started.

Up to this point we have considered dissections at a fixed $m \varepsilon M$. We define a *dissection of* M^2 to be an assignment of a dissection at each point which is C^∞ in the following sense: for each C^∞ function f and coordinate system $x_1, \ldots, x_d$ the function (defined on the intersection of the domain of the x_i and the domain of f) $H_f\left(\frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j}\right)$ is in C^∞.

If N is a d-dimensional linear space over R and $x_1, \ldots, x_d$ is any linear coordinate system then we can define a dissection as follows. If $t \varepsilon N_n^2$ then

$$t = \Sigma\, a_i \frac{\partial}{\partial x_i}(n) + \Sigma a_{ij} \frac{\partial^2}{\partial x_i \partial x_j}(n)$$

and we define $N_n^c = \left[t \varepsilon N_n^2 \mid \text{all } a_i = 0 \right]$. Since any two linear coordinate systems are related by a linear transformation (with constant coefficients) this definition of N_n^c is independent of the choice of the linear coordinate system used, and hence defines a dissection of N^2, which depends only on the linear structure of N. We remark that if H_f is the Hessian of f obtained from this dissection then

$$(*) \qquad H_f \left(\frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_i} \right) = \frac{\partial^2 f}{\partial x_i \, \partial x_j}$$

This is easily proved so we omit it.

2. THE DIFFERENCE OF TWO CONNEXIONS

We now consider two affine connexions over the same M and use the following notation. B(M) is the bundle of bases over M, φ and $\overline{\varphi}$ are the 1-forms of the connexions, K_b and $\overline{K}_b$ are the horizontal subspaces assigned at $b \varepsilon B(M)$ (we use K, instead of the usual H, because we are using H for Hessians), $E_1, \ldots, E_d$ and $\overline{E}_d, \ldots \overline{E}_d$, are the canonical horizontal vector fields on B(M), $\omega_1, \ldots, \omega_d$ are the canonical 1-forms on B(M), and $E_{11}, \ldots, E_{dd}$ are the canonical vertical vector fields (the ω_1 and E_{ij} depend on no connexion so there is only one set of them). We shall use this kind of notation consistently, the K going with the E_i, the $\overline{K}$ going with the $\overline{E}_i$. We use π for the projection of $B(M) \longrightarrow M$.

Another piece of notation is this: if α is a C^∞ curve then $\alpha_*(u)$ will denote its tangent vector at $\alpha(u)$. We shall use D(s) for the tangent vector at $s \varepsilon R$ which assigns to each function its ordinary derivative. So, for f in C^∞ on part of M, $\alpha_*(u) f = D(u)(f \circ \alpha)$.

We call the 1-form $\tau = \overline{\varphi} - \varphi$ the *difference form* of these connexions. Clearly τ is horizontal (i.e. vanishes on all vertical vectors), equivariant,, is C^∞, and takes values in the Lie algebra of all $d \times d$ matrices. This τ then gives rise to a T, which assigns to each x in each M_m a linear transformation T_x of M_m into itself. $T_x y$ is defined by: if $b = (m, e_1, \ldots, e_d)$ is any point of B(M) lying over m, and if $\overline{x}$ and $\overline{y}$ are any tangent vectors at b which project to x and y, then

$$(2.1) \qquad T_x y = \Sigma_i \left(\Sigma_j \, \tau_{ij}(\overline{x}) \, \omega_j(\overline{y}) \right) e_i,$$

i. e. $T_x y = z$ is equivalent to $\omega_i(\overline{z}) = \Sigma \, \tau_{ij}(\overline{x}) \omega_i(\overline{y})$ ($\overline{z}$ being any tangent vector at b that projects to z).

Geometrically, T_x measures the difference infinitesimally between K–(parallel translation) and $\overline{K}$–(parallel translation) along curves coming into m tangent to x. i. e. if α is a C^∞ curve in M with $\alpha_*(O) = x$, if the $f_i(u)$ are the K–(parallel translates) of a base $f_1, \ldots, f_d$ along α to $\alpha(u)$, and if the $\overline{f}_i(u)$ are their $\overline{K}$–(parallel translates), then one has $\overline{f}_j(u) = \Sigma h_{ij}(u) f_i(u)$; then the matrix of T_x relative to the f_i is $(h'_{ij}(0))$.

We now say this same thing in a different way. Let α be as above, β a K–horizontal curve over α, $\overline{\beta}$ a $\overline{K}$–horizontal curve over α, with $\beta(0) = \overline{\beta}(0)$.

Then for each u there is a unique matrix h(u) for which $\bar{\beta}(u) = \beta(u)h(u)$. This h(u) is the above $(h_{ij}(u))$ and $\tau_{ij}(x)$, if $\bar{x}$ lies over x at $\beta(0)$, is $h'_{ij}(0)$; and $\tau_{ij}(\bar{x})$ is the vertical component, in the sense of the K–connexion, of $\bar{\beta}$ at $\beta(0)$.

LEMMA 2.1. In the above notation we have

a) $h^{-1}h' = -\tau(\bar{\beta}_*)$

b) $h'h^{-1} = -\tau(\beta_*)$.

Proof: We use Lemma 4.1 of (Ambrose an Singer, 1958) which, altho stated there for connexions on a bundle of frames, is clearly valid for any affine connexion. It gives, in the present notation,

$$h^{-1}h' = \varphi(\beta_*).$$

Since $\bar{\beta}$ is $\bar{K}$–horizontal, $\bar{\varphi}(\bar{\beta}_*) = 0$, hence $\tau(\bar{\beta}_*) = \bar{\varphi}(\bar{\beta}_*) - \varphi(\bar{\beta}_*) = -\varphi(\bar{\beta}_*)$, so a) holds. To prove b) we note that $K\bar{\beta}_* = dR_h\,\beta_*$, hence $\tau(\bar{\beta}_*) = \bar{\varphi}(dR_h\,\beta_*) = h^{-1}\tau(\beta_*)h$. So a) becomes $h^{-1}h' = -h^{-1}\tau(\beta_*)h$, which gives b) immediately.

THEOREM 1. Assume the above notation. Suppose for each u ε [a,b] we have a $y(u)\ \varepsilon\ M_{\alpha(u)}$ and that all these y(u) are K–parallel to y(a) along α. Let $x(u) = \alpha_*(u)$. Then all these y(u) are K-parallel to y(a) if and only if $T_{x(u)}y(u) = 0$ for all u. In particular, if α is a K-geodesic then α is a $\bar{K}$-geodesic if and only if $T_x x = 0$ for all tangent vectors x to α.

Proof: Let $y(a) = \Sigma c_i e_i$ where $\beta(0) = \bar{\beta}(0) = (m, e_1, \ldots, e_d)$, and define $E = \Sigma c_i E_i$; define the vector field F along β by $F(u) = dR_{h(u)}E(\beta(u))$. We shall write $T_x y$ for the function on [a, b] whose value at u is $T_{x(u)}y(u)$ and also consider the functions $E(\beta)$, $\omega(E(\beta))$, $\tau(\beta^*)$, etc. on [a, b]. If $a = (a_{ij})$ and $c = (c_1, \ldots, c_d)$ we define $ac = (\Sigma a_{1j}c_j, \ldots, \Sigma a_{dj}c_j)$ and denote the i–th component of ac by $(ac)_i$. We also define aE, if $E = \Sigma c_i E_i$, by $aE = (ac)_1 E_1 + \ldots + (ac)_d E_d$. Clearly we have $a(\omega(E)) = \omega(aE)$.

We first prove:

i) $T_x y = 0$ if and only if $h\,\omega(E(\beta)) = \omega(E(\beta))$ (all these considered as functions on [a,b]).

Proof: Since $d\pi F = y$ and $d\pi\beta_* = \alpha_*$ the definition of T makes $T_x y = 0$ if and only if $\tau(\beta_*)\,\omega(F) = 0$. By the previous lemma, $h^{-1}h' = -\tau(\beta_*)$ and we know $\omega(F) = \omega(dR_h E(\beta)) = h^{-1}\omega(E(\beta))$, hence $T_x\,y = 0$ if and only if $h^{-1}h'h^{-1}\omega(E(\beta)) = 0$. But $h^{-1}h'h^{-1} = -(h^{-1})'$ and $\omega(E(\beta))$ is constant, so $h^{-1}h'h^{-1}\omega(E(\beta)) = 0$ is equivalent to $(h^{-1}\omega(E(\beta)))' = 0$, which is equivalent to $h^{-1}\omega(E(\beta)) =$ constant. Because $h(0) =$ the identity matrix this is equivalent to $h^{-1}\omega(E(\beta)) = \omega(E(\beta))$, which is trivially equivalent to $h^{-1}\omega(E(\beta)) = \omega(E(\beta))$. Hence i) is proved.

To finish the proof of the theorem we now show

ii) All y(u) are K–parallel along α if and only if $h\omega(E(\beta)) = \omega(E(\beta))$.

Proof: The y(u) are K–parallel to y(a) if and only if $d\pi E(\beta) = d\pi F$; since $d\pi E = d\pi E$ and $d\pi F = d\pi E(\beta)$ this shows the y(u) are K-parallel if and only if $d\pi E(\beta) = d\pi E(\beta)$. And $d\pi E(\beta) = d\pi E(\beta h) = d\pi(hdR_h E(\beta)) = d\pi(dR_\beta hE(\beta)) = = d\pi(hE(\beta))$, so the y(u) are all K-parallel if and only if $d\pi E(\beta) = d\pi(hE(\beta))$ and this is equivalent to $E(\beta) = hE(\beta)$, which is in turn equivalent to $\omega(E(\beta)) = \omega(hE(\beta))$, or to $h\omega(E(\beta)) = \omega(E(\beta))$. This proves ii), and together i) and ii) imply the theorem.

If $T_x x = 0$ for all x in all M_m we say the two connexions have the same geodesics. This means, of course, the same parametrized geodesics.

The expression $T_x y$, as a function of x and y, is a bilinear function from $M_m \times M_m$ to M_m. Hence it has a symmetric part S and an anti-symmetric part A, where $S_x y = (T_x y + T_y x)/2$ and $A_x y = (T_x y - T_y x)/2$, with $T = S + A$. Since S is determined by its values on the diagonal our theorem above gives:

Corollary. Two affine connexions over M have the same geodesics if and only if the symmetric part of their difference transformation is zero, i.e. if and only if their difference transformation is anti-symmetric.

We now point out that if W and $\overline{W}$ are the torsion bilinear functions of φ and $\overline{\varphi}$ then the anti-symmetric par A of their difference transformation is given by

$$A = \overline{W} - W. \tag{2.2}$$

Proof: The structural equations of the connexions give, in an obvious notation, $d\omega = -\varphi\omega + \Omega$, $d\omega = -\overline{\varphi}\omega + \overline{\Omega}$, where Ω and $\overline{\Omega}$ are the torsion forms. These torsion forms (on B(M)) being equivariant and horizontal give rise in the usual way to bilinear functions, W and $\overline{W}$, from $M_m \times M_m$ to M_m (for each $m \varepsilon M$). Hence $\overline{\varphi}\omega - \varphi\omega = \overline{\Omega} - \Omega$, i.e. $x\, \tau\omega = \overline{\Omega} - \Omega$. Expressing $\tau\omega$, Ω, $\overline{\Omega}$ in terms of A,W,$\overline{W}$ (and noting that our product of forms contains an anti-symmetrisation in it) we get (2.2).

3. SPRAYS AND CONNEXIONS

An affine connexion over M gives rise, for each x in each M_m, to a unique (parametrize) geodesic α_x whose tangent vector at m is x. Wenow want to consider, without an affine connexion being given, an assignment which gives, for each x in each M_m, a curve α_x whose tangent vector at m is x. We call such an assignment a *spray* if it satisfies certain natural conditions ("natural" in the sense that they hold when the α_x are the geodesics of an affine connexion). The object of this section is to show that every spray on M comes from an affine connexion over M, that the torsion of the affine connexion can be assigned arbitrarily, and that the spray plus the torsion uniquely determine the connexion.

In general, even if the α_x are the geodesics of an affine connexion, the $\alpha_x(t)$ can not be defined for all real t (α_x may run into a "hole" in M). Hence we must assume that our $\alpha_x(t)$ are defined only for some t. For uniqueness statements we need, however, to know they are defined for as big a range of t as possible. This explains the elaborateness about the domains in the following definition.

Def. A *spray* on M is an assignment which gives, for each x in each M_m, a C^∞ curve α_x in M, such that the following hold:

1) The domain of each α_i is an open interval of real numbers containing 0,

2) $(\alpha_x)_*(0) = x$,

3) $\alpha_{(\alpha_x)_*(s)}(t) = \alpha_x(s+t)$,

4) $\alpha_{sx}(t) = \alpha_x(st)$

5) Let 0 be the subset of $R \times T(M)$ consisting of all (t, x) ($t \varepsilon R$, $x \varepsilon T(M)$) for which $\alpha_x(t)$ is defined, and F the mappinf of 0 into T(M): $F(t,x) = (\alpha_x)_*(t)$. Then $F \varepsilon C^\infty$.

6) ψ is maximal relative to 1) — 5).

The above F is the usual flow on T(M), i.e. F is a 1-parameter family of transformations of T(M) into T(M), defined by

$$F_t(x) = (\alpha_x)_*(t).$$

F clearly has the following properties.

1′) For each x, $F_t(x)$ is defined for an open interval of real numbers containing 0,

2′) $F_0(x) = x$,

3′) $F_t(F_s(x)) = F_{s+t}(x)$,

4′) $F_t(sx) = sF_{st}(x)$

5′) $F \varepsilon C$ (from 0 into T(M))

6′) F is maximal relative to 1′) — 5′)

7′) $F_t(x) = (\alpha_x)_*(t)$.

Conversely, if a 1-parameter group of transformations of T(M) into T(M) is given satisfying 1′) — 6′) and ax(t) is defined as the projection of $F_t(x)$ into M, then these α_x form a spray satisfying 1′ — 7′).

There is still another formulation of the notion of a spray, which is well known, in terms of the infinitesimal generator V of the flow F. If the flow F is given then one defines the vector field V, on T(M), by V(x) = the tangent vector to $(\alpha_x)_*$ at $(\alpha_x)(0)$. This vector field has the property: If π is the projection of T(M) into M, then $d\pi V(x) = x$. Conversely, any vector field with this property gives rise to a unique spray, via its integral curves.

We say a spray is the geodesic spray of an affine connexion if anf only if for each x in each M_m the α_x given by the spray is a geodesic of the connexion.

Prop. For each spray on M there exists a unique affine connexion of torsion zero whose geodesic spray is the given spray.

Proof: We shall use the notation α_x and F as above for the mappings given with the spray. For each m ε M we define a mapping ψ_m of $M_m \to M$ by:

$$\psi_m(x) = \bar{\pi} F_1(m, x)$$

where $\bar{\pi}$ is the natural projection of $T(M) \to M$. For each x ε M_m let p_x be the ray: $p_x(t) = tx$ (defined for all real t). We first prove

α) $\psi_m \circ p_x = \alpha_x$

Proof of α): We have, from 3′), taking t = 1,

$$F_1(m, \cdot) \circ p_x(s) = sF_s(m, x),$$

hence

$$\bar{\pi} F_1(m, \cdot) \circ p_x(s) = \bar{\pi} F_s(m, x).$$

The definition of ψ_m and 5′) then give α).

There is a natural isomorphism of M_m onto $(M_m)_0$, namely, $x \to p_x^*(0)$, and we now see that $d\psi_m$ is the inverse to this for α) with 2) gives

$$d\psi_m p_{x*}(0) = \alpha_x^*(0) = x.$$

Hence, by the inverse function theorem, ψ_m maps some neighborhood of 0 in M_m diffeomorphically onto some neighborhood of m.

We define the desired connexion as follows. Let $b = (m, e_1, \ldots, e_d)$ ε ε B(M) and we shall define K_b. Let $x_1, \ldots, x_d$ be the dual base (of M_m) to $e_1, \ldots, e_d$. Define the mapping ψ_b of a neighborhood of 0 in M_m to B(M) by

$$\psi_b(n) = \left(\psi_m(n), d\psi_m \frac{\partial}{\partial x_1}(n), \ldots, d\psi_m \frac{\partial}{\partial x_d}(n)\right).$$

Clearly $\psi_b(0) = b$. We now define

$$K_b = d\psi_b(M_m)_0.$$

It is easily checked that the K thus defined is a connexion on B(M) so it remains to show that: 1) the spray generated by K is the given one, 2) K has torsion zero, 3) uniqueness of K.

Proof of 1): If a spray is given on M then its associated flow is generated by a certain vector field V on T(M), where V is defined by: if $\tilde{\alpha}_x(t) = F_t(m, x)$ then $V(m, x) = \tilde{\alpha}_x^*(0)$. By 3′) we then have

β) $\alpha_x = \bar{\pi} \circ \tilde{\alpha}_x$

where $\bar{\pi}$ is the natural projection of $T(M) \to M$. Clearly V determines F.

If the spray comes from a connexion K with canonical horizontal fields E_i then

γ) $V(m, x) = d\bar{\pi}_1 E_1(b)$, if $\bar{\pi}_1(b) = (m, x)$,

where $\bar{\pi}_1$ is the projection of $B(M) \to T(M)$ carrying each base into its first element. Since α_x is constant when x is the zero element of M_m this determines V.

Hence to prove 1) it is sufficient, if V is the vector field of our given spray and E_1 is the vector field of the connexion we have defined, to prove that γ) holds for this V and this E_1.

So let $b=(m, e_1, \ldots, e_d)$ be fixed and we write $x=e_1$. To prove γ) it will be sufficient to find a curve δ in B(M) with $\delta(0) = b$, $\delta_*(0) = E_1(b)$, and $\bar{\pi} \circ \delta = \tilde{\alpha}_x$. We define δ by

$$\delta = \psi_b \circ p_x.$$

Letting $x_1, \ldots, x_d$ be the dual base (of M_m) to $e_1, \ldots, e_d$, we have

$$\delta(t) = \left(\psi_m(p_x(t)), d\psi_m \frac{\partial}{\partial x_1}(p_x(t)), \ldots, d\psi_m \frac{\partial}{\partial x_d}(p_x(t))\right).$$

Since δ clearly projects under π to $\psi_m \circ p_x = \alpha_x$ (using α) and since $\delta_*(0)$ is horizontal by definition of K, we have

i) $E_1(b) = \delta_*(0)$.

Clearly

$$\bar{\pi}_1 \circ \delta(t) = \left(\psi_m(p_x(t)), d\psi_m \frac{\partial}{\partial x_1}(p_x(t))\right).$$

Because $\frac{\partial}{\partial x_1}(p_x(t)) = \frac{\partial}{\partial x_1}(p_{e_1}(t)) = p_x{}^*(t)$ and α) shows that

$$d\psi_m \circ p_{x^*}(t) = \alpha_{x^*(t)}$$

we have

ii) $\bar{\pi}_1 \circ \delta(t) = (\alpha_x(t), \alpha_{x^*}(t)) = F_t(m, x) = \tilde{\alpha}_x(t)$.

As remarked above, i) and ii) prove γ) and hence 1).

Proof of 2): One defines the torsion forms of the connexion to be the horizontal 2-forms $\Omega_1, \ldots, \Omega_d$ by

$$\Omega_i(s, t) = d\,\omega_i(Ks, Kt)$$

(for all s, t ε $B(M)_b$) so we want to show $d\omega_i(s,t) = 0$ if s and t are horizontal in $B(M)_b$. Again letting $b = (m, e_1, \ldots, e_d)$ and $x_1, \ldots, x_d$ be the dual base of $e_1, \ldots, e_d$ it is celar that $\omega_i \circ d\psi_b = dx_i$, hence $d\omega_i \circ d\psi_b = 0$. If s,t are horizontal at b then, since $K_b = d\psi_0(M_m)_0$, $d\omega_i(s,t) = d\omega_i \circ d\psi_b(s,t) = 0$. This proves 2).

Proof of 3): Let $\bar{\varphi}$ be any other connexion with the same geodesics as φ and torsion 0 and let τ be the difference form. Then τ gives rise to a difference transformation T as in § 2. We write T as a sum of a symmetric and anti-symmetric part: $T=S+A$. Because φ and $\bar{\varphi}$ have the same geodesics, $S=0$, by the Corollary to Theorem 1. Because they have the same torsion, $A=0$ by (2.2). Hence $T=0$, thus $\tau=0$, thus $\varphi = \bar{\varphi}$.

If W is any mapping which assigns to each pair of vectors, x and y, in each M_m, an element W(x,y) in M_m, and such that W is bilinear, W is anti-symmetric, and W ε C^∞ we call W a *torsion tensor* on M.

Theorem 2. If a spray and torsion tensor are given on M then there exists a unique affine connexion over M whose spray of geodesics is the given spray and whose torsion is the given torsion tensor.

Proof: Let φ be the 1-form of the affine connexion given above with torsion zero and whose geodesic spray is the given spray. Let $\overline{W}$ be the given torsion tensor, and let $T = \overline{W}/2$, so $W(x, y) = T_x y - T_y x$. Let τ be the equivariant 1-form on B(M) corresponding to T and we define the desired connexion $\bar{\varphi}$ by $\bar{\varphi} = \varphi + \tau$. Then T is the difference transformation of φ and $\bar{\varphi}$. Because $T_x x = 0$ we have, by Theorem 1, that $\bar{\varphi}$ has the same geodesics as φ, and by (2.2) we see that $\overline{W}$ is the torsion of $\bar{\varphi}$. Uniqueness follows as in the previous proposition for if $\tilde{\varphi}$ is another such then having the same geodesics and torsion again implies the symmetric and anti-symmetric part of its difference tiansformation with $\bar{\varphi}$ is zero.

4. DISSECTIONS AND SPRAYS

We show here that there is a natural 1:1 correspondence between sprays on M and dissections of M^2. If a spray is given it provides a mapping (denoted by ψ_m in §3) of $M_m \rightarrow M$ and we can carry over the natural dissection of $(M_m)^2$ (i.e. the one obtained from its linear structure) via ψ_m to get a dissection of M^2 at m, for each m, thus obtaining a dissection of M^2. However we prefer to describe this dissection starting from the α_x of the spray, and to give the dissection in terms of the Hessian H_f which it assigns to f. If a spray is given, consisting of a family of curves $\{\alpha_x\}$ (using our previous notation) we then define the Hessian H_f of f by

$$H_f(x, x) = D^2(0)(f \circ \alpha_x). \tag{4.1}$$

Here D(s) denotes differentiation on the real line at s. This determines H_f as a bilinear function on each $M_m \times M_m$ for which $f \in F_m^+$, making

$$H_f(x, y) = (1/4)\left[D^2(0)(f \circ \alpha_{x+y}) - D^2(0)(f \circ \alpha_{x-y}) - D^2(0)(f \circ \alpha_x) - \right.$$
$$\left. - D^2(0)(f \circ \alpha_y)\right].$$

One proves the bilinearity of H_f most easily by reference to the mapping ψ_m above, and in terms of that definition of the dissection, but we omit the details here. The assignment: $f \rightarrow H_f$, has the properties mentioned in §1 which ensure that it arises from a dissection of M^2. We call this the dissection *induced*, by the spray $\{\alpha_x\}$.

In this section we wish to show that every dissection arises in this way from a spray and that this correspondence between sprays and dissections is 1:1. We shall do this by showing that a dissection gives rise to a connexion whose geodesic spray induces the given dissection.

At this point we wish to consider the condition on a connexion that its geodesic spray induce a given dissection. From now on our dissection will be described by an assignment: $f \rightarrow H_f$, which associated to each C^∞ function f (whose domain is an open subset of M) an H_f which is a bilinear function on each $M_m \times M_m$ for which m is in the domain of f, and which has the properties: a) the assignment is C^∞ in the sense of §1, b) it is linear, c) at any critical point of any f, H_f, shall be the usual Hessian of f. As shown in §1, this is equivalent to a dissection of M^2.

The condition that a spray $\{\alpha_x\}$ induce a given dissection of M^2 is (4.1). Now suppose a connexion is given on B(M) with canonical horizontal vector fields E_i and let $\{\alpha_x\}$ be the geodesic spray of this connexion. If $x \neq 0$ and $b = (m, e_1, \ldots, e_d)$ with $x = e_j$, and if $\bar{\alpha}_x$ is the integral curve of E_j with $\bar{\alpha}_x(0) = b$, then $\pi \circ \bar{\alpha}_i = \alpha_x$. Hence for $f \varepsilon F_m^+$,

$$(E_j^2(b))(f \circ \pi) = D^2(0)(f \circ \pi \circ \bar{\alpha}_x) = D^2(0)(f \circ \alpha_x).$$

Combining this with (4.1) we see that *a given dissection:* $f \longrightarrow H_f$ *is induced by the geodesic spray of the connexion* K *if and only if for each* $b = (me_1, \ldots, e_d)$, *and each* $f \in F_m^+$

(4.2) $$(E_j^2(b))(f \circ \pi) = H_f(e_j, e_j).$$

It is trivially sufficient to know this at a single b over each $m \varepsilon M$ and to know it only for $j=1$.

Next we recall a few notions associated with a coordinate system of M. Let $x_1, \ldots, x_d$ be any coordinate system of M, with domain 0. This gives rise to a local cross section χ of B(M), over 0, defined by

$$\chi(m) = \left(m, \frac{\partial}{\partial x_1}(m), \ldots, \frac{\partial}{\partial x_d}(m)\right).$$

From the x_i we obtain a coordinate system $y_1, \ldots, y_d, y_{11}, \ldots, y_{dd}$ of B(M) with domain $\pi^{-1}(0) = B(0)$, defined by

$$y_i = x_i \circ \pi$$

$$y_{ij}(b) = dx_i(e_j) \text{ if } b = (m, e_1, \ldots, e_d).$$

If H *is any connexion on* B(O) *with canonical horizontal vector fields* E_i *then*

(4.3) $$E_j(x_i \circ \pi) = y_{ij}.$$

Proof: $(E_j(b))(x_i \circ \pi) = (d\pi E_j(b))\, x_i = e_j\, x_i = dx_i(e_j) = y_{ij}(b)$.

The coordinate system $y_1, \ldots, y_d, y_{11}, \ldots, y_{dd}$ defines a product representation of $\pi^{-1}(0)$ as $0 \times GL(d,R)$ and thus defines on $\pi^{-1}(0)$ an affine connexion of curvature and torsion zero. We call this the affine connexion *given by* the coordinate system $x_1, \ldots, x_d$. If its canonical vector fields are $\bar{E}_i$ and $\bar{\varphi}$ is its 1-form then at all $m \varepsilon 0$,

(4.4) $$\begin{cases} \bar{\varphi}_{ij} = dy_{ij} \\ \bar{E}_i(\chi(m)) = d\chi\left(\frac{\partial}{\partial x_i}(m)\right) \end{cases}$$

Let $\underline{K}$ be any connexion on B(0) with canonical horizontal vector fields E_i and $\overline{K}$ be given by the coordinates $x_1, \ldots, x_d$; let φ and $\overline{\varphi}$ be the 1-forms of these connexions and τ their difference form: $\tau = \overline{\varphi} - \varphi$. Then at any $b = (m, e_1, \ldots, e_d) \varepsilon \pi^{-1}(0)$,

$$\tau_{ij}(E_j(b)) = H_{x_i}(e_j, e_j) = (E_j^2(b))(x_i \circ \pi), \tag{4.5}$$

where this Hessian is that induced by the geodesic spray of the connexion K.

Proof: $\tau_{ij}(E_j) = \overline{\varphi}_{ij}(E_j) - \varphi_{ij}(E_j) = \overline{\varphi}_{ij}(E_j) = dy_{ij}(E_j) = E_j\, y_{ij} = E_j^2(x_i \circ \pi) = H_{x_i}(e_j, e_j)$ (the last two equalities holding by (4.3) and (4.2).

In case K is also given by a coordinate system, say $z_1, \ldots, z_d$, with domain 0 and ψ is the corresponding cross section of B(M) over 0 then we have, at all $b \varepsilon \psi(0) \cap \chi(0)$, letting $m = \pi b$,

$$\tau_{ij}(E_k(b)) = \frac{\partial^2 x_i}{\partial z_j\, \partial z_k}(m). \tag{4.6}$$

Proof: In this case we have, at all $n \varepsilon 0$,

$$y_{ij}(\psi(n) = dx_i\left(\frac{\partial}{\partial z_j}(n)\right) = \frac{\partial x_i}{\partial z_j}(n),$$

i.e.

$$y_{ij} \circ \psi = \frac{\partial K_i}{\partial z_j}$$

Hence, as in the above proof,

$$\tau_{ij}(E_k(b)) = dy_{ij}(E_k(b)) = E_k(b)\, y_{ij} = d\psi\left(\frac{\partial}{\partial z_k}(m)\right) y_{ij} =$$

$$= \frac{\partial}{\partial z_k}(m)\,(y_{ij} \circ \psi) = \frac{\partial}{\partial z_k}(m)\left(\frac{\partial x_i}{\partial z_j}\right) = \frac{\partial^2 x_i}{\partial z_j\, \partial z_k}(m).$$

Theorem 3. If a dissection of M^2 is given then there is a unique spray $\{\alpha_x\}$ on M which induces it.

Proof: We first prove the existence of a spray inducing a given dissection. So we assume a dissection: $f \longrightarrow H_f$ given; and we prove the existence of a connexion K for which (4.2) holds.

We call a coordinate system $x_1, \ldots, x_d$ of M *special* at m if and only if all $x_i(m) = 0$ and all $H_{x_i} = 0$ at m. We first prove that for each m there exists a special coordinate system at m. To show this choose any coordinate system $y_1, \ldots, y_d$ at m with all $y_i(m) = 0$; thus the $y_i \varepsilon F_m$. Let $\mu_1, \ldots, \mu_d$ be their images under the natural projection of $F_m \longrightarrow F_m/F_m^3 = {}^2M_m$. Our dissection of M^2 now enables us to write $\mu_i = \mu_i^1 + \mu_i^2$ where $\mu_i^2 \varepsilon H_m$ and $\mu_i^1 \varepsilon H_m^c$. Let $f_1, \ldots, f_d$ be functions in F_m^2 such that $\mu_i^2 = f_i + F_m^3$. We define the desired $x_1, \ldots, x_d$ by $x_1 = y_i - f_i$. We see that these x_i are a coordinate system on some neighborhood of m because $dx_i = dy_i$ at m, so the dx_i are linearly independent at m. And the projection, given by the dissection, of $x_i + F_m^3$

into F^2_m/F^3_m is zero because the y_i and f_i have the same component in F^2_m/F^3_m. Hence the x_i are a special coordinate system at m. Furthermore, if $e_1, \ldots, e_d$ is any base of M_m then there exists a special coordinate system at m, say $x_1, \ldots, x_d$, for which $\frac{\partial}{\partial x_i}(m) = e_i$ (for all i); this is obtained by performing a linear transformation on the coordinate system already obtained.

Now we define the desired connexion. Let $b = (m, e_1, \ldots, e_d)$, choose any special coordinate system $x_1, \ldots, x_d$ at m with $\frac{\partial}{\partial x_i}(m) = e_i$ (for all i), let χ be the local cross section of B(M) obtained from this coordinate system and we wish to define

$$K_b = d\chi M_m.$$

To legitimate this definition however we must show it independent of the choice of the special coordinate system. So let $z_1, \ldots, z_d$ be another such, $\left(\text{including of course that } \frac{\partial}{\partial z_i}(m) = e_i\right)$ and ψ the local cross section of B(M) which it gives. For the moment let us write $\overline{K}_b = d\psi M_m$, and we need to show that $K_b = \overline{K}_b$.

We have really defined K_b and $\overline{K}_b$ to be the horizontal spaces at b of the connexions defined by the coordinate systems $x_1, \ldots, x_d$ and $z_1, \ldots, z_d$ so if we can show the difference form of these two connexions is zero at b we shall have that $K_b = \overline{K}_b$. By (4.6) it is sufficient to show that $\frac{\partial^2 x_i}{\partial z_j \partial z_k}(m) = 0$. We now prove this.

By (1.2), applied with $f = x_i$ and with $z_1, \ldots, z_d$ in place of the $x_1, \ldots, x_d$ of (1.2) we have, on some neighborhood of m,

$$x_i = \Sigma_j \frac{\partial x_i}{\partial z_j}(m) z_j + \Sigma_{j \leq k} \frac{\partial^2 x_i}{\partial z_j \partial z_k}(m) z_j z_k + g_j$$

where $g_j \varepsilon F^2_m$. Since the x_i and z_j are both special at m we have, applying the mapping, $f \longrightarrow H_f$,

$$0 = \Sigma_{j \leq k} \frac{\partial^2 x_i}{\partial z_j \partial z_k}(m) H_{z_j z_k} + H_{g_i}.$$

The functions $z_j z_k$ and g_i have critical points at m so their Hessians are the usual Hessians. Since $g_i \varepsilon F^3_m$ its Hessian is zero, thus

$$0 = \Sigma_{j \leq k} \frac{\partial^2 x_i}{\partial z_j \partial z_k}(m) H_{z_j z_k}$$

Because the usual Hessian of $z_j z_k$ satisfies

$$H_{z_j z_k}\left(\frac{\partial}{\partial x_p}(m), \frac{\partial}{\partial x_q}(m)\right) = \delta_{jp}\delta_{kq}$$

the previous equality shows

$$\frac{\partial^2 x_i}{\partial z_p \, \partial z_q}(m) = 0 \text{ for each p, q.}$$

Thus $K_b = \overline{K}_b$ and our connexion is well defined.

Next we must show that (4.2) holds, where the Hessian H_f is that given by the dissection and the E_j are those of the connexion just defined. So consider any fixed $b = (m, e_1, \ldots, e_d)$ and choose a coordinate system $x_1, \ldots, x_d$ which is special at m, and with $\frac{\partial}{\partial x_i}(m) = e_i$ (for all i). We first show (using that the x_i are special at m) that (4.2) holds when $f = x_i$; then we show (this part not depending upon the x_i being special) that this implies (4.2) for general f.

Proof of (4.2) when $f = x_i$: Because the x_i are special we have $H_{x_i}(e_j, e_j) = 0$ for all i,j. Let τ be the difference form of K and the connexion given by the $x_1, \ldots, x_d$, so $\tau_{ij}\big(E_k(b)\big) = 0$ for all i,j,k, — by the definition of K. Then (4.5) shows $E_j^2(b)(x_i \circ \pi) = 0$, hence (4.2) holds for $f = x_i$ (We can not use the equality with $H_{x_i}(e_j, e_j)$ given in (4.5) for that H_{x_i} is the one given by the connexion whereas we need it for the one given in advance).

Proof of (4,2) for general $f \in F_m^+$: (We are really proving here, for any coordinate system, that the Hessian of the coordinate functions determines the Hessian of all functions). We have, at all points in some neighborhood of m,

$$\alpha) \quad f = f(m) + \Sigma\, a_p x_p + \Sigma\, g_p x_p$$

where the $g_p \in F_m$ and the a_p are real numbers. Then

$$\beta) \quad H_f(e_j, e_j) = 2\, \Sigma\, (e_j g_p)\, (e_j x_p).$$

On the other hand, applying E_j^2 to $f \circ \pi$,

$$E_j^2(f \circ \pi) = \Sigma\, a_p E_j^2(x_p \circ \pi) + E_j^2(\Sigma\, g_p x_p \circ \pi) =$$

$$= \Sigma\, a_p E_j^2(x_p \circ \pi) + \Sigma\, \big(E_j^2(g_p \circ \pi)\big)(x_p \circ \pi) + \Sigma\, (g_p \circ \pi)\, \big(E_j^2(x_p \circ \pi)\big) +$$

$$+ 2\Sigma\, \big(E_j(g_p \circ \pi)\big)\, \big(E_j(x_p \circ \pi)\big).$$

Evaluating this at b, using that $x_p(m) = g_p(m) = 0$, and $E_j^2(b)(x_p \circ \pi) = 0$ (this last because we already have (4.2) when $f = x_p$), we have

$$E_j^2(f \circ \pi) = 2\Sigma\, \big(E_j(b)\,(g_p \circ \pi)\big)\, \big(E_j(b)\,(x_p \circ \pi)\big) = 2\Sigma\, (e_j g_p)\, (e_j x_p).$$

With β) this proves (4.2), so we have proved the existence of a connexion whose geodesic spray induces the given dissection of M^2.

Proof of uniqueness of the spray: Let $\{\alpha_x^1\}$ and $\{\alpha_x^2\}$ be two sprays which induce the same dissection of M^2 and we now prove they are the same. By what we have proved above we know $\{\alpha_x^1\}$ is the geodesic spray of some affine connexion K^1 and $\{\alpha_x^2\}$ is the geodesic spray of some affine connexion K^2. Let φ^1 and φ^2 be the 1-forms of these connexions and $\tau = \varphi^2 - \varphi^1$. We shall prove, under the assumption that these sprays induce the same dissection, that, for the difference transformation T of τ, we have $T_x x = 0$ for all x in all M_m. By §2 this will imply that K^1 and K^2 have the same geodesics, i.e. that our two sprays are the same.

Let m be any point of M, $b = (m, e_1, \ldots, e_d)$ any point of B(M) over m, and choose a coordinate system $x_1, \ldots, x_d$ at m such that $\frac{\partial}{\partial x_i}(m) = e_i$ (for all i). Let $\bar{\varphi}$ be the connexion obtained from this coordinate system and let $\tau^1 = \bar{\varphi} - \varphi^1$, $\tau^2 = \bar{\varphi} - \varphi^2$. Hence $\tau^2 = \tau^2 - \tau^1$. Because the geodesic sprays of φ^1 and φ^2 induce the same dissection of M^2 we see by (4.5), applied to τ^1 and τ^2, that

$$\tau^2_{ij}(E_j(b)) = \tau^1_{ij}(E_j(b)).$$

Hence $\tau_{ij}(E_j(b)) = 0$. This implies using the relation (2.1) between τ and T, and letting $E_j(b) = \bar{x} = \bar{y}$, and writing $x = e_1$,

$$T_x x = \Sigma_p (\Sigma_q \tau_{pq}(E_j(b)\, \omega_q (E_j(b))\, e_p$$
$$= \Sigma_p (\tau_{pj}(E_j(b)))\, e_j = 0.$$

Hence $T_x x = 0$. Since b is arbitrary this proves the desired uniqueness.

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