Symmetry, criticality, and condition C

by

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Abstract

A famous innovation of Sophus Lie was his approach to the study of differential equations through their groups of symmetries. We will examine a technique called the Symmetric Criticality Principle, which is a specialization of this approach to Euler Lagrange equations.

Preface

The invitation to give a series of lectures during the Sophus Lie Memorial Week is a great honor and privilege for me. My dissertation was titled "A Global Formulation of the Lie Theory of Transformation Groups", and much of my research since then has also been concerned with ideas that can be traced back to Lie.

Perhaps the two areas of the mathematics that I have worked in the most are Differentiable Group Actions and the Calculus of Variations. About fifteen years ago I became fascinated with a mathematical principle that lies at the intersection of these two fields. I call it the Symmetric Criticality Principle (SCP for short), and in these lectures I would like to survey of some of the elegant and surprising mathematics that surrounds this principle. In the first of the three lectures we will investigate SCP itself, determine its range of validity and attempt to understand why it can be such a powerful tool for proving the existence

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1991 *Mathematics Subject Classification.* Primary 57E15, 57F35, 58E05, 58G35; Secondary 58E11, 58E20.

Key words and phrases. Condition C, Symmetric Criticality, symmetry, critial points.

of critical points (and also for locating them). In the second lecture we will apply SCP to some specific problems that arise in applications of the Calculus of Variations to geometry and physics, and see how it leads to precise descriptions of solutions to many of these problems. Finally, in the third lecture, we will look at a sophisticated recent application of SCP by Tom Parker. This application, to the Yang-Mills functional for gauge fields, helped settle a long outstanding question — namely, whether there exist solutions of the Yang-Mills equations that are unstable (and hence neither self-dual nor anti self-dual) [14], [15], [16].

Because these lectures were designed as part of a Summer School, the exposition is intentionally "uneven" — or rather, I hope, graded. I have attempted to pitch the first lecture at a level accessible to first-year graduate students, the second for advanced graduate students, and the final lecture for more mature research mathematicians.

I should point out here that I did *not* discover the Symmetric Criticality Principle (although perhaps it would be fair to say that I "uncovered" it). As you will see, it is one of those ideas that has been around for a very long time, and physicists in particular have been using it, usually implicitly, for perhaps as far back as one cares to look. My rôle has been rather to draw attention to it, by giving it a name and helping settle its true range of validity [12], and to give a rigorous general discussion of its applications in Geometry and Mathematical Physics [13].

1 Introduction

It is nearly inconceivable that a student currently doing graduate work in any area of Mathematics would not to be familiar with the name of Sophus Lie, and at least with the basis of his theories of Lie groups and Lie algebras. For these are not only in themselves major fields of modern mathematical research, but they also seem to play a basic rôle in most other areas of pure and applied Mathematics and Physics. Still, the way that Lie groups are normally taught today, a student could be excused for not realizing that Lie's primary interest in them wan *not* as abstract objects, but rather as transformation groups. In fact, for Lie, they arose in practice as the symmetry groups of important geometric, algebraic, and analytic structures, and their Lie algebras were the infinitesimal generators of one-parameter groups of such symmetries. Lie was a man of extraordinarily wide interests, and he applied his remarkable insights about symmetry to a very broad spectrum of problems. In particular, inspired no doubt by Galois' spectacular success in studying algebraic equations through their symmetry groups, Lie had the genius to suspect that a deeper understanding of the symmetries of differential equations might similarly lead to progress in classifying them and finding better methods for their solution. Of course he was correct, and this "Galois theory differential equations", initiated by Lie [1] is still an important and active field of research [10].

Now, many of the most important differential equations (both ordinary and partial) that arise in pure and applied Mathematics are "Euler-Lagrange equations", that characterize the critial points, p, of some Calculus of Variatons functional, F, i.e., these equations arise as an analytic reformulation of the fact that, dF, the differential (or "first variation") of F vanishes at p. In particular, the "field equations" that describe both the stable configurations and the dynamical evolutions of physical field theories are virtually always derived from such a Lagrangian variational principle [3]. Moreover, an important component of designing physically realistic field theoretic models consists of building into the defining Lagrangian functional appropriate symmetries that are suggested by experimental results.

Since it is clear that, for Euler-Lagrange equations, any symmetry of the defining functional F will also be a symmetry of the variational equations, this suggests that we should investigate, more generally, the following question: Give a smooth real-valued function, $F : M \to \mathbf{R}$, what is the relation between the set $\mathcal{C}(F)$ of critical points of F and the geometry of a Lie group G of diffeomorphism of M that has F as an invariant function? As we shall see below, this is precisely the sort of relation that the Symmetric Criticality Principle tells us about, and while I don't know of any work by Lie concerned with this more general question, it fits in so well with his general approach that a survey of SCP seems quite appropriate for a conference devoted to his memory.

2 The symmetric criticality principle

Before stating SCP let's look at some very simple examples. First, two that almost everyone sees in their introductory calculus course:

"Among all rectangles (triangles) with a fixed perimeter, show that the square (equilateral triangle) has the greatest area". I suspect that most people, when they first meet these problems, see immediately that the proof requires no computation — that it follows solely from symmetry considerations. A somewhat less obvious application of SCP is that any smooth function on the plane (or more generally in $\mathbf{R}^n$) that depends only on the distance from the origin must have a critical point at the origin, and similarly, any function on the Earth (i.e., the two-sphere) that is constant on lines of latitude must have critical points at the North Pole and South Pole.

These are toy examples; they capture a little of the flavor of SCP, but are too trivial to illustrate its full essence. We will present some more serious and interesting illustrations from the Calculus of Variations shortly.

Now let's give a careful statement of SCP. As above, we let: $F : M \to \mathbf{R}$ be a smooth, real-valued function on a smooth manifold M, and denote by $\mathcal{C}(F)$ its set of critical points, i.e., the set of solutions p to the abstract variational problem $dF_p = 0$. Recall that $dF_p \in T^*M_p$ is the linear functional on TM_p defined by letting $dF_p(X)$ be the directional derivative of F in the direction X, for any tangent vector X at p. Thus the vanishing of the directional derivative in *all* directions X at p is the necessary and sufficient condition for p to belong to $\mathcal{C}(F)$.

Next suppose that a Lie group G acts smoothly on M — that is, M is a smooth G-manifold, and suppose moreover that F is an invariant function for the action of G, i.e., that F is constant on the orbits of G, or equivalently that $F \circ g = F$ for all $g \in G$. Then by the chain rule, $dF_{gp} \circ dg_p = dF_p$, and it follows that if p is a critical point of F then so is gp, i.e., $\mathcal{C}(F)$ is a G-invariant set, and hence a union of G-orbits. Those G-orbits included in $\mathcal{C}(F)$ are called *critical orbits*.

The set of *stationary points* of the action of G on M will play a central role in what follows. These are the points p of M such that $gp = p$ for all g in G, i.e., points where the isotropy group G_p is all of G, and hence where the orbit Gp consists of just the point p itself. Traditionally this set is denoted by M^G, but we shall always denote it by Σ, and we call it the set of *symmetric points* of M. This is because in the Calculus of Variations setting that we shall be dealing with, the elements of Σ represent geometrical or physical configurations that are mapped into themselves by G-those that in the physicists jargon, have "unbroken symmetry". Our object of primary interest will be the set $\mathcal{C}(F)\cap\Sigma$ of "symmetric critical points", in other words the critical orbits that are reduced to a single point.

On the other hand, we can also consider the set $\mathcal{C}(F|\Sigma)$ of "critical symmetric points". Recall that the chain-rule implies that $d(F|\Sigma)_p = dF_p|T\Sigma_p$, so that these are the points satisfying the weaker condition that the directional derivative $dF_p(X)$, while not necessarily vanishing for *all* directions at p, does at least vanish for those that are tangent to Σ. Clearly then $\mathcal{C}(F) \cap \Sigma \subseteq \mathcal{C}(F|\Sigma)$, i.e., a symmetric critical point is trivially a critical symmetric point. SCP is the statement that the converse also holds.

2.1 The Symmetric Criticality Principle

$$\mathcal{C}(F) \cap \Sigma = \mathcal{C}(F|\Sigma).$$

Let's next try to understand why this simple principle has proved to be so remarkably useful — and also why you should be a little surprised by it. First, it is useful because the set Σ of symmetric points of M is almost always a *much* "smaller" set than M itself, and this makes it far easier to find critical points of $F|\Sigma$ than to find critical points of F. For example, in Calculus of Variations problems M will always be infinite dimensional, (which is precisely why it is so hard to prove general existence theorems for Euler-Lagrange equations). But as we shall see, Σ will frequently be finite dimensional — and in many cases, Σ will even be compact, in which case the existence of critical symmetric points is obvious. There is an even more extreme case that also arises frequently, one that makes SCP useful even for finite dimensional M: namely when Σ has an isolated point. For trivially, any isolated point of Σ is a critical symmetric point, and therefore, according to SCP, a symmetric critical point. (Recall our example above of the rotations of the two-sphere about the z-axis). The reason that SCP is surprising is exactly the same as the reason it is useful. If M is infinite dimensional but Σ is finite dimensional, then there is an infinite dimensional space Θ_p transverse to $T\Sigma_p$ in TM_p. Why should the vanishing of dF_p on the finite dimensional $T\Sigma_p$ imply it on the far larger Θ_p? In the extreme case that p is an isolated point of Σ, $T\Sigma_p = 0$, while $\Theta_p = TM_p$, and this puzzle also becomes extreme.

Now some important "bad news" — *SCP is not valid in complete generality!* (which is why we called it a "principle" and not a "theorem"). Worse yet, SCP may not even be meaningful in certain circumstances. We have assumed in our discussions so far that the set Σ of symmetric points must be a smooth submanifold of M. But this may not be so, in which case we cannot even speak meaningfully of critical symmetric points. Moreover, even when Σ is a smooth manifold, there can be critical symmetric points that are not critical points. However, before giving explicit, simple examples of these failures of SCP, let me follow up with the good news:

- SCP is always valid when G is compact (and so in particular when G is finite).
- SCP is also valid when the G manifold M admits an invariant Riemannian structure — i.e., when G is a group of Riemannian isometries.

The proof of this latter fact is both easy and intuitive, and we shall give the full details after the counter-examples. Fortunately, these two cases cover most of the important situations where we would like to apply SCP.

2.2 Counter-example 1

We shall show that given any non-empty closed set C in a smooth manifold M, there exists a smooth action of $G = \mathbf{R}$ on M with $\Sigma = C$. (The same result also holds for non-compact M, but compactness simplifies the proof a little). When M is compact there is a bijective correspondence between smooth $\mathbf{R}$-actions and smooth vector fields, with the stationary set Σ of the $\mathbf{R}$-action being just the set where the generating vector field vanishes. So we only have to find a smooth vector field X on M that vanishes precisely on C. Now the existence of a smooth real-valued function f on M with $C = f^{-1}(0)$ is a standard fact, and by a basic theorem of elementary differential topology, given any p in M, there is a smooth vector field Y on M vanishing only at p. Since C is non-empty we can take $p \in C$, and then define $X = fY$. Since we can for example choose C to be a Cantor set, we see that when $G = \mathbf{R}$, Σ is not generally a smooth submanifold of M.

2.3 Counter-example 2

In this example we again take $G = \mathbf{R}$ and let $M = \mathbf{R}^2$. We define a smooth G-action (in fact a linear representation) of G on M by $t(q, p) = (q + pt, p)$. Notice that this is a well-known "physical" action — it is the Hamiltonian flow corresponding to a one-dimensional free particle of unit mass. Clearly Σ coincides with the q-axis, and the other orbits are just the lines $p =$ constant, parallel to the q-axis. It follows that a smooth function F on M is invariant if and only if it is of the form $F(q, p) = h(p)$ for some smooth function h on $\mathbf{R}$. But such a function is constant on Σ, so every point of Σ is a critical symmetric point. On the other hand, if $h'(0) \neq 0$, then there are no symmetric critical points.

Validity of SCP for Riemannian G-manifolds

In this section we assume that M is a *Riemannian* G-manifold, and, as usual, $F : M \to \mathbf{R}$ is a G-invariant function. To say that the G-manifold M is Riemannian means not only that M has a Riemannian structure, but also that every element of G acts as an isometry of M. We denote the Riemannian inner product by $\langle , \rangle$. For each p in M, we have the so-called *isotropy representation* $g \mapsto dg_p$ of the isotropy group G_p on TM_p, and we note that it is clearly orthogonal. For points p of Σ this of course given an orthogonal representation of G on TM_p.

Let exp denote the exponential map of TM into M. That is, if v is a tangent vector to M at p, then $\gamma(t) = \exp(tv)$ is the geodesic, parametrized proportionally to arc-length, with $\gamma(0) = p$ and $\dot\gamma(0) = v$. Since isometries map geodesics to geodesics, if $g \in G$ then $\lambda = g \circ \gamma$ is the geodesic with $\lambda(0) = g(p)$ and $\lambda'(o) = dg(v)$, and it follows that $\exp \circ dg = g \circ \exp$. If $\exp_p$ denotes the restriction of exp to TM_p, then from the definition of exp it follows that $d\exp_p$ is the identity map of TM_p so, by the inverse function theorem, $\exp_p$ maps a neighborhood of zero in the Hilbert space TM_p diffeomorphically onto a neighborhood of p in M. This is a canonical chart or coordinate system for M near p, so-called "Riemannian normal coordinates". The relation $\exp \circ dg = g \circ \exp$ above now says that:

Proposition *If $p \in \Sigma$, then in a neighborhood of p the action of G on M is linear in Riemannian normal coordinates at p, and in fact is just* *the isotropy representation of G.*

And as an immediate corollary

Corollary *Σ is a smooth submanifold of M, in fact a totally geodesic submanifold. Moreover, for each p in Σ, $T\Sigma_p = \{v \in TM_p | dg(v) = v$ for all $g \in G\}$, i.e., the tangent space to Σ consists of all the elements of TM_p fixed by the isotropy representation at p.*

Note that Σ need not be connected, and moreover different components of Σ can even have different dimensions. Thus one cannot speak of the dimension of Σ — only its dimension at some point.

Next, recall that the gradient of a smooth real-valued function $u : M \to \mathbf{R}$ is the vector field ∇u dual to the 1-form du with respect to the Riemannian structure, i.e., ∇u is characterized by $du(X) = (X, \nabla u)$ (so of course $C(u)$ consists of the zeros of ∇u). Because dg_p is an isometry of TM_p onto TM_{gp} for every $g \in G$, the identity $dF_{gp} \circ dg_p = dF_p$, noted earlier, translates immediately into $dg(\nabla F_p) = \nabla F_{gp}$ — in other words, the vector field ∇F is G-equivariant. In particular, if p is a symmetric point then it follows that $(\nabla F)_p$ is fixed by the isotropy representation at p, and by the above Corollary we see that:

Proposition *At each point of Σ, ∇F is tangent to Σ.*

Let X be any submanifold of M (with the induced Riemannian structure), and given $p \in X$, let P denote the orthogonal projection of TM_p onto the subspace TX_p. Clearly $P(\nabla u)_p$ satisfies the characteristic defining property of $\nabla(u|X)_p$, so they are in fact equal. In particular, the critical points of $u|X$, i.e., the zeros of $\nabla(u|X)$, are exactly the points where ∇u is orthogonal to X. Specializing to $u = F$ and $X = \Sigma$, we see that the critical symmetric points are exactly the points p where ∇F is orthogonal to Σ. But, by the preceding Proposition, ∇F is *always* tangent to Σ, so at a critical symmetric point, $\nabla F = 0$, i.e., a critical symmetric point is a symmetric critical point, and we have proved:

Theorem *The Symmetric Criticality Principle is a valid theorem for Riemannian G-manifolds.*

Since we always assume our manifolds are paracompact, if M is modeled on a Hilbert space (in particular, if it is finite dimensional), then we can choose a Riemannian structure for M. Then, if G is any *compact* group acting on M, by the well-known process of "averaging over the group" we can get an invariant Riemannian structure for M, and it follows as a corollary of the above Theorem that SCP is valid for all Hilbert G-manifolds, provided G is compact. Of course, for an infinite dimensional M not modeled on Hilbert space, this argument breaks down. Nevertheless, a somewhat more elaborate argument ([12, Theorem 5.4]) proves:

Theorem *If G is a compact Lie group then the Symmetric Criticality Principle is valid for all G-manifolds.*

Before leaving this abstract discussion of SCP I should point out that it does *not* answer all the important questions concerning variational problems with symmetry. Here are two examples.

First, just because an equation exhibits some symmetry, it does not follow that we only care about solutions that exhibit this same symmetry. In fact, as often as not, we also need to know information about the non-symmetric solutions of symmetric variational problems and regarding these SCP is silent.

And second, we can say little about the second order properties of F at a symmetrical critical point starting from knowledge of teh second order properties of $F|\Sigma$. Recall that at a critical point p of F, there is a well-defined symmetric bilinear form, $\mathrm{Hess}(F)_p$ on TM_p, the Hessian of F at p, that characterizes the second order behavior of F near p. If X and Y are vector fields on M then $\mathrm{Hess}(F)_p(X, Y) = X_p(YF)$. (In local coordinates at p, its matrix is the matrix of second partial derivatives of F at p). If the Hessian is positive definite then p is a local minimum, while if it is negative definite then p is a local maximum, and in general the index and coindex of the Hessian are called the index and co-index of the critical point p. At a symmetric critical point, the Hessian is invariant under the isotropy representation, so for example, if the isotropy representation is irreducible, then (as is pointed out in [17]) it follows from Schur's lemma that p must be either a minimum or a maximum. But in infinite dimensions a representation of a compact group is never irreducible, so this is of no help. Of course, the index and coindex of F at p are greater than or equal to the index and coindex of $F|\Sigma$ at p. But this is trivial — a real second order SCP would allow us to infer properties of the Hessian of F transverse to Σ from the Hessian of $F|\Sigma$, and this is unfortunately not possible.

3 The calculus of variations

The most interesting and striking uses of SCP occur as applications to problems in the Calculus of Variations. Before setting up the machinery to explain this in full generality, let's examine a couple of simple but important examples.

First, let's consider the problem of finding closed geodesics on a surface of revolution S, formed say by rotating the graph of a positive function $y = f(x)$ about the x-axis in $\mathbf{R}^3$. Let $G = S^1$ denote the compact group of complex numbers of modulus 1. If we identify $e^{i\theta} \in G$ with the rotation R_θ through an angle θ about the x-axis, then G acts on S, and the orbits are the "circles of latitude", γ_a, of radius $f(a)$ (and hence length $2\pi f(a)$) formed by intersecting S with the planes $x = a$ (a parallel to the y,z-plane). I think it is clear intuitively that those orbits γ_a corresponding to critical points a of f must be geodesics of S. (Imagine stretching a piece of thread tightly around S). We will use SCP to prove this. Let M denote the manifold of "C^1 loops in S", i.e. C^1 maps σ of S^1 into S. The length function F on M is of course defined by $F(\sigma) = \int_0^{2\pi} \|\sigma'(e^{it})\| dt$. (Why M really is a smooth Banach manifold and $F : M \to \mathbf{R}$ a smooth function will be explained later). There is a natural action of G on M: namely $g = e^{i\theta}$ acts on σ by $(g\sigma)(e^{it}) = R_\theta(\sigma(e^{i(t-\theta)}))$ — so g just translates the base point of a loop by $-\theta$ and then rotates its image on S through the angle θ. Since R_θ is an isometry, it is clear that F is a G-invariant function on M, so we are in a position to apply SCP. But which elements of M are symmetric? Note that if $g\sigma = \sigma$ for all $g = e^{i\theta}$ in G, then taking $t = 0$ in the above formula for the action of g on σ gives $\sigma(e^{i\theta}) = R_\theta\sigma(1)$, i.e., each symmetric loop σ is an orbit of G on S. Conversely, the orbits are clearly symmetric, and so Σ coincides with the one-dimensional manifold of orbits γ_x. Finally, since $F(\gamma_x) = 2\pi f(x)$, the critical symmetric points are the γ_x for which x is a critical point of f, and then by SCP these are the symmetric geodesics.

Next, a classic example of the technique of making a "symmetry *ansatz*". This is an implicit appeal to SCP that physicists love dearly and use frequently to discover particular solutions of their variational problems.

Let A denote an n dimensional "annulus", $A = \{x \in \mathbf{R}^n | r_1 \leq \|x\| \leq r_2\}$, and let g be a smooth real-valued function on its boundary ∂A. Let M denote the Banach space of C^2 real-valued functions U on A that agree with g on the boundary. The Dirichlet Functional,

$$F(U) = \int_A \|\nabla U(x)\|^2 dx,$$

is easily seen to be a smooth quadratic function on M, and according to Dirichlet's Principle, a function U on A is harmonic (and has g as its boundary values) if and only if it is a critical point of F.

Now suppose we want to find all the radial harmonic functions on R^n. A function U is radial if it has the form $U(x) = \tilde{U}(\|x\|)$ for some $\tilde{U}$ in $C^2([r_1, r_2])$, and for such a function clearly $\nabla U(x) = \tilde{U}'(\|x\|)x/\|x\|$. Thus, resticting to radial functions and using spherical polar coordinates to evaluate the integral, the Dirichlet functional takes the form:

$$F(U) = \Omega \int_{r_1}^{r_2} \tilde{U}'(r)^2 r^{n-1} dr,$$

where Ω denotes the volume of S^{n-1}. The Euler-Lagrange equation for the right hand side of the latter is just $(2\tilde{U}'(r)r^{n-1})' = 0$, or $r\tilde{U}'' + (n-1)\tilde{U}' = 0$, and this is easily solved explicitly: it has the general solution $\tilde{U}(r) = a + br^{2-n}$ (except, $\tilde{U}(r) = a + b \log r$ when $n = 2$), and of course these are indeed the radial harmonic functions.

But wait a minute! By taking the Euler-Lagrange equations of this one-variable functional, we are implicitly restricting not only the function U to satisfy the radial *ansatz* (which is legitimate), but we are as well only looking at first variations that also satisfy this same *ansatz*, and there is nothing in Dirichlet's Principle to justify the latter. One rigorous way to proceed is to write the Laplacian in polar spherical coordinates, and substitute the symmetry *ansatz* into the harmonic equation $\nabla U = 0$. (This is basically the method of group-invariant solutions, explained in Chapter 3 of [10].)

Of course this gives the correct answer, but by a much more complicated derivation. The contrast is even more dramatic for the analogous problem of finding the static, radial solutions of the Einstein field equations of General Relativity (the so-called Schwarzschild line element), where pages of complex calculations involving a system of ten partial differential equation to solve a system of two ordinary differential equations. Having worked through the original calculation in Eddington's "Mathematical Theory of Relativity", I was very impressed with this enormous simplification when I saw it in Herman Weyl's "Space, Time, and matter". But I remember looking in vain for some explanation from Weyl as to how one overcomes the logical hiatus mentioned above.

Of course the answer is the Symmetric Criricality Principle. For the case of radial harmonic functions, the compact group $G = SO(n)$ acts (linearly) on M by $(gU)(x) = U(g^{-1}x)$, the Dirichlet Functional, F, is invariant, and the set Σ of symmetric points consists of the radial functions. So, what is specified by the Euler-Lagrange equation $(2\tilde{U}'(r)r^{n-1})' = 0$, is the set of critical symmetry points — and by SCP these are also the symmetric critical points, i.e., the radial harmonic functions. An entirely similar argument justifies Weyl's derivation of the Schwarzschild solution.

Manifolds of Sections of Fiber Bundles

The above two simple examples give an indication of how SCP can be applied to realistic problems of the Calculus of Variations. In order to explain the general theory of such applications, we must first review some of the important concepts and machinery of the subject, and in particular explain just what are the manifolds M, symmetry groups G, functions F, and symmetric points Σ in a general Calculus of Variations setting.

Let X be a smooth manifold and $\pi : E \to X$ a smooth fiber bundle over X. We call E a smooth G fiber bundle if both the total space, E and the base space X are smooth G-manifolds, and the bundle projection π is G-equivariant. This implies that each operation of a g in G on E is a fiber bundle morphism over the corresponding operation of g on X. While not essential, it will simplify the discussion to assume that the base manifold X is compact, and we shall do so, similarly, we will assume that the group G is compact, so we can safely apply SCP to any G manifold.

One important special case of a G fiber bundle is the product G-

manifold $E = X \times Y$, where X and Y are smooth G-manifolds, the action of G is the product action, $g(x, y) = (gx, gy)$, and π is the natural projection onto X. Another important special case is a G vector bundle over X. This means that E is a smooth vector bundle over X and each operation of G on E is a vector bundle morphism.

We denote by $C(E)$ the space of continuous sections of E. In case $E = X \times Y$, we make the standard identification of $C(E)$ with $C(X, Y)$. As usual, we topologize fiber bundle, the group G acts naturally on $C(E)$ — if $s \in C(E)$ and $g \in G$, then $gs \in C(E)$ is defined by $(gs)(x) = g(s(g^{-1}x))$. When E is a G vector bundle over X, then clearly $C(E)$ es a Banach space and the action of G on $C(E)$ is a strongly continuous linear representation. It is perhaps not obvious, but $C(E)$ is a Banach G-manifold, and in fact it can be considered as the basic example of the type of G-manifold M that arises in the Calculus of Variations. Here is how one puts a natural smooth structure on $C(E)$.

A vector bundle ξ over X is called a vector bundle neighborhood (VBN) for a fiber bundle E over X if the total space of ξ is an open sub-bundle of the total space of E and if the inclusion map $\xi \to E$ is a fiber bundle morphism. Clearly the Banach space $C(\xi)$ is then an open subset of $C(E)$. It is not hard to show (by a proof analogous to that of the tubular neighborhood theorem) that if $s \in C(E)$ then there is a VBN ξ for E with $s \in C(\xi)$, and we call such a ξ a VBN for s in E, so as ξ varies aver all VBN's for E, the corresponding Banach spaces $C(\xi)$ give an open covering of $C(E)$ by charts, and since it can be shown that these charts are smoothly related, we have a natural smooth atlas defining a smooth manifold structure for $C(E)$. Of course many details have been omitted from this discussion (and also from the following paragraphs): the interested reader will find these details in [11]. We note further that this assignment of a Banach manifold $C(E)$ to each fiber bundle E over X is even functorial — given a smooth fiber bundle morphism $\varphi : E_1 \to E_2$ (over the identity map of x), the map $s \to \varphi \circ s$ is a smooth map $C(\varphi) : C(E_2)$. Such maps $C(\varphi)$ are called *differential operators of order zero* from (sections of) E_2. Usually we will write simply $\varphi(s)$ instead of $C(\varphi)(s)$, so $\varphi(s)(x) = \varphi(s(x))$.

As remarded above, when E is a G fiber bundle then G acts naturally on the Banach manifold $C(E)$, and it is easy to check that this action is smooth. It will be essential for our later applications of SCP to have a good characterization of the set Σ of symmetric points of $C(E)$, and fortunately this is easy. In fact, it is immediate from the definition of the action of G on $C(E)$ that $gs = s$ for all g if and only if $s(gx) = g(s(x))$ for all g and x. Thus:

Proposition *If E is any G fiber bundle, then the set Σ of symmetric elements og the Banach G manifold $C(E)$ consits of the sections s that are equivariant as maps of the base into the total space.*

There is an important class of G fiber bundles E for which the structue of the symmetric points of $C(E)$ can be made even more explicit. Namely, E is called a *homogeneous* G fiber bundle if the action of G on the base, X, is transitive. If we choose p_0 in X and let $H = G_{p_0}$ denote the isotropy group at p_0 then, by the equivariance of the projection π, the fiber E_{p_0} over p_0 is an H manifold, so its set of H-stationary points (which we will denote by $E_{p_0}^H$) is a smooth submanifold of E_{p_0}. If $s : X \to E$ is an equivariant section, then since $hp_0 = p_0$ for $h \in H$, it follows by equivariance that $h(s(p_0)) = s(hp_0) = s(p_0)$, so $s(p_0) \in E_{p_0}^H$. Conversely, given any $e \in E_{p_0}^H$, it is easy to see that $s(gp_0) = ge$ gives a well-defined smooth and equivariant section of E, and that this is the unique equivariant section whose value at p_0 is e. This gives a bijection between the equivariant sections and $E_{p_0}^H$, and it is not hard to check that it is smooth in both directions. This proves:

Proposition *If E is a homogeneous G fiber bundle over X and H is the isotropy group at some point p_0 of X, then the map $s \mapsto s(p_0)$ is a diffeomorphism between the set Σ of symmetric (i.e. equivariant) elements of $C(E)$ and $E_{p_0}^H$.*

Corollary *If E is a homogeneous G fiber bundle, then the set Σ of symmetric elements of $C(E)$ is a finite dimensional manifold, and is even compact if the fibers of E are compact.*

If G is not transitive on X, but has an orbit, Ω, of codimension one, then the G manifold X is said to have cohomogeneity one. In this case we can also often identify the equivariant sections of E with something simpler, as follows. Assume X is a complete Riemannian G manifold. A geodesic C normal to Ω at one point p_0 will meet all other orbits, and meet them orthogonally. If in addition C is regularly embedded in X we call it a *section* of the G manifold X. (For example, any line through the origin is a section for the action of $SO(n)$ of R^n. Similarly, a circle of longitude is a section for rotations of the two-sphere about the z-axis.) The isolated group H at p_0 will also be the isotropy group at other points of C (except at isolated points where it may be larger). Form the bundle E^H over C whose fiber at p is the fixed point set of H on the fiber E_p. Then it is not hard to see that the equivariant sections of E can be identified with sections of this bundle E^H over C.

All the above constuctions work equally well (*mutatis mutandis*) if we replace the space $C(E) = C_0(E)$ of continuous sections and the compact-open topology by the spaces $C^k(E)$ of C^k sections and the C^k topology. That is, we again get by the same process a natural smooth Banach manifold structure on $C^k(E)$, and the assignment is again functorial. Moreover, when E is a G fiber bundle then $C^k(E)$ is again a Banach G manifold, and the set Σ of symmetric elements of $C^k(E)$ are the equivariant C^k sections. And if further E is a homogeneous G bundle then we have the same identification of Σ with $E^H_{p_0}$. (No, there is no contradiction here: the point is that in the homogeneous case an equivariant section is automatically C^∞).

Remark

For the applications in this second lecture we will be able to get by with these manifolds $M = C^k(E)$. But it is important to realize that these classical spaces are no longer considered the manifolds of choice for more sophisticated applications of non-linear analysis. If $\dim(X) = n$ we can also define manifolds $L^p_k(E)$, modeled on the Sobolev Banach spaces of L^p_k sections of vector bundles over X, provided $k > n/p$. By definition $L^p_k(\xi)$ consists of sections of ξ with "distributional derivatives" of order $\leq k$ in $L^{pm}(\xi)$ are smoothly related. In particular, the $L^2_k(E)(k > n/2)$ are Hilbert manifolds and will be denoted by $H^k(E)$. While perhaps less intuitive than the manifolds $C^k(E)$, these Sobolev manifolds turn out to be much better suited for problems involving non-linear PDE, and in particular for "hard" Calculus of Variations problems, and these are the manifolds that we must work with in the final lecture.

We next discuss the nature of the functions F studied by the Calculus of Variations.

We first recall the concept of a (non-liear) k-th order differential operator of order k from (sections of) E_1 to (sections of) E_2 is a certain kind of map $D : C^k(E_1) \to C(E_2)$. We have alreadey defined this above for $k = 0$. For general k, D has the form $D = C(\varphi) : C(J^k(E_1)) \to C(E_2)$ is a differential operator of order zero. Thus, a k-th order differential operator is, by definition, just the composition of two maps. The first is a very special "universal k-th order differential operator", (the k-jet extension map from E_1 to $J^k(E_1)$, the k-jet bundle of E_1) and the second is a zero order operator from $J^k(E_1)$ to E_2.

Remark

We will assume a basic familiarity with jet bundles and the k-jet extension map (details can be found in [11]). But here is a brief description. The k-jet of a smooth section s of E at a point x of X, is just a coordinate free way of describing the k-th order Taylor polynomial of s at x. In fact, given two sections s_1, s_2 defined near x, by definition $j_k(s_1)_x = j_k(s_2)_x$ if and only if $s_1(x) = s_2(x) = e$ and with respect to some of choice of local coordinates for X near x and for E near e the k-th order Taylor polynomials of s_1 and s_2 at x agree. The set $J^k(E)$ of all such equivalence classes of local sections at all points x has a natural bundle structure over X, with the projection of course mapping $j_k(s)_x$ to x. So what this all means is that, exactly as the name suggests, after introducing local coordinates and local trivializations, a k-th order differential operator has the form $Du = F(u, Du, D^2u, \ldots, D^k u)$, (where $D^j u$ is a symbol denoting the set of all partial derivatives of u of order j). It follows from the definition of the k-jet bundle $J^k(E)$ that the k-jet extension map $j_k : C^k(E) \to C(J^k(E))$ is a smooth map of Banach manifolds, and it is elementary that differential operators of order zero are smooth. It follows that k-th order differential operators $D : C^k(E_1) \to C(E_2)$ are always smooth maps of Banach manifolds.

$\Omega(X)$ will denote the line bundle of differential forms on X of degree $= \dim X$ (and of odd type) — so sections ω of $\Omega(X)$ define signed measures on X. We will denote by μ some fixed section of $\Omega(X)$ that defines a positive smooth measure on X, e.g., the Riemannian measure associated to some fixed Riemannian structure for X. Then every other section ω of $\Omega(X)$ can be written uniquely in the form $\omega = h\mu$, where h is a continuous real valued function on X. We note that "integration over X" is a natural continuous linear functional on the Banach space $C(\Omega(X))$. We denote it by $\int_X : C(\Omega(X)) \to \mathbf{R}$. We recall that there is a natural "pull-back" acton $\varphi^*\omega$ of diffeomorphisms φ of X on sections ω of $C(\Omega(X))$, and the "change of variables formula" says that $\int_X \varphi^*\omega = \int_{\varphi(X)} \omega = \int_X \omega$.

Definition A *k-th order Lagrangian* for a bundle E over X is a k-th order differential operator $\mathcal{L} : C^k(E) \to C(\Omega(X))$, i.e., $\mathcal{L}(s) = \dot{\mathcal{L}}(j_k(s))$, with $\dot{\mathcal{L}} : J^k(E) \to \Omega(X)$ as a bundle morphism. If μ is some canonical choice of smooth measure on X, then $\mathcal{L}(s) = \dot{\mathcal{L}}_\mu(j_k s)\mu$, and $\dot{\mathcal{L}}_\mu : J^k(E) \to \mathbf{R}$ is called the Lagrangian function. Associated to a k-th order Lagrangian $\mathcal{L}$ we define a real-valued function $F_{\mathcal{L}} : C^K(E) \to \mathbf{R}$ by $F_{\mathcal{L}}(s) = \int_X \mathcal{L}(s)$.

We note that $F_{\mathcal{L}}$, as a composition of a smooth map and a continuous linear funcional, is smooth real-valued function. It is such functions $F_{\mathcal{L}} : C^k(E) \to \mathbf{R}$, and in particular the structure of their critical points, that is subject matter of the Calculus of Variations. We will not here repeat the well-known derivation of the Euler Lagrange equations corresponding to the condition $dF_{\mathcal{L}_s} = 0$, but only remind the reader that for a k-th order Lagrangian, these equations are of order $2k$. Since Mother Nature (and Geometers) seem to have a preference for second order PDE, the great preponderance of Lagrangians that arise in practice are in fact of first order (and the few remaining are of second order).

Of course, the next thing to investigate is the conditions under which a Lagrangian for a G fiber bundle E defines a G-invariant function $F_{\mathcal{L}} : C^k(E) \to \mathbf{R}$, and fortunately this is straightforward. The action of G on E induces a natural action of G on the k-jet bundle $J^k(E)$, defined by $g(j_k(s)_x) = j_k(gs)_{gx}$, and this induces an action of G on $C^k(J^k(E))$, making the k-jet extension map $j_k : C^k(E) \to C(J^k(E))$ equivariant, i.e., $gj_k(s) = j_k(gs)$. Since $\mathcal{L}(gs) = \dot{\mathcal{L}}(j_k(gs)) = \dot{\mathcal{L}}(gj_k(gs))$, if $\dot{\mathcal{L}} : J^k(E) \to \Omega(X)$ is G-equivariant, i.e., if $\dot{\mathcal{L}}(gj_k(s)) = g^*\dot{\mathcal{L}}(j_k(s))$, then so is $\mathcal{L}$, i.e., $\mathcal{L}(gs) = g^*\mathcal{L}(s)$, and so:

$$F_{\mathcal{L}}(gs) = \int \mathcal{L}(gs) = \int_X g^*\mathcal{L}(s) = \int_X \mathcal{L}(s) = F_{\mathcal{L}}(s),$$

by the change of variable formula. This proves:

Proposition *If $\mathcal{L}(s) = \dot{\mathcal{L}}(j_k(s))$ is a k-th order Lagrangian for a G fiber bundle E over X, then the corresponding functional $F_{\mathcal{L}} : C^k(E) \to \mathbf{R}$ is G-invariant provided the bundle morphism $\dot{\mathcal{L}} : J^k(E) \to \Omega(X)$ is G-equivariant.*

In particular, suppose now that the base manifold X is a Riemannian G manifold, so we have a natural choice for a section μ for $\Omega(X)$, the Riemannian measure. As above we define $\dot{\mathcal{L}}_\mu : J^k(E) \to \mathbf{R}$ by $\dot{\mathcal{L}}(j_k(s)) = \dot{\mathcal{L}}_\mu(j_k(s))\mu$. Then since $g^*\mu = \mu$, it follows that if the function $\dot{\mathcal{L}}_\mu$ is G-invariant, then the morphism $\dot{\mathcal{L}}$ will be G-equivariant, so $F_{\mathcal{L}} : C^k(E) \to \mathbf{R}$ will be G-invariant.

We can now recapitulate the above as a general prescription for applying SCP to problems in the Calculus of Variations.

1) Let G be a compact Lie group, X a compact G manifold, and E a smooth G fiber bundle over X. Then $M = C^k(E)$ is a Banach G manifold, and each k-th order Lagrangian $\mathcal{L}(s) = \dot{\mathcal{L}}(j_k(s)) =$ for E defines a Calculus of Variations functional $F : M \to \mathbf{R}$, $F(s) = \int_X \mathcal{L}(s)$.

2) F is G-invariant provided $\dot{\mathcal{L}} : J^k(E) \to \Omega(X)$ is G-equivariant.

3) The manifold Σ of symmetrical points of M is the space of G-equivariant sections $s : X \to E$, so the critical symmetric points are those equivariant sectons for which the first variation of F vanishes for all equivariant variations of its argument.

4) In case X is a homogeneous G manifold (say $X = G/H$ with p_0 the identity coset) then Σ is finite dimensional, and can be identified with the closed submanifold $E^H_{p_0}$ of the fiber E_{p_0} via the map $s \mapsto s(p_0)$. In particular, for homogeneous G bundles with compact fibres, Σ is always compact, so provided H has at least one fixed point on E_{p_0}, (i.e., if $E^H_{p_0} = \Sigma$ is non-empty) every invariant F has symmetric critical points.

5) In case X has cohomogeneity one and C is a "section", i.e., a regularly embedded geodesic of X meeting all the orbits orthogonally, we can identify the equivariant sections of E with sections of the

bundle E^H over C, where H is the isotropy group along C. Moreover $F|\Sigma$ becomes a variation problem for this reduced bundle, so finding the critical symmetric points is an ODE problem.

Now that we have developed this general prescription, we can easily explain how SCP is used for finding solutions to some important variational problems.

Critical Maps

We shall first consider "critical maps". That is we take as our bundle E a product bundle $X \times Y$, so sections of E are just maps of X into Y, and we will consider only first order Lagrangians, $\mathcal{L} : C^1(X, Y) \to C(\Omega(X))$.

The first case we shall consider is perhaps the most classical of variational functionals, the "volume". Here we assume that Y is a compact Riemannian manifold and that $\dim(X) < \dim(Y)$, and we take as our manifold M of admissible maps not the entire space $C^1(X, Y)$, but only the open submanifold of immersions of X, into Y. Each immersion $s \in M$, defines and induced Riemannian structure on X, and we define $\mathcal{L}(s)$ to be the volume element on X of this induced structure, so that $F_{\mathcal{L}}(s) = \int_X \mathcal{L}(s)$ is the volume of the image of s. The critical points of $F_{\mathcal{L}}(s)$ are called *minimal immersions*, and they have a long history and rich theory. In particular, when $\dim(X) = 1$ we have the special case of geodesics, and when $n = 2$ of minimal surfaces.

When SCP is considered in this context, it leads to a natural and far reaching generalization (due to W.-Y. Hsiang [7]) of the example we consider earlier, of geodesic orbits on a surface of revolution. We assume that Y is a Riemannian G manifold, and look for *orbits* of critical volume in Y. Let H be any closed subgroup of G that occurs as an isotropy group of some point y_0 of Y. Let Y^H denote the set of points on Y fixed under H, (i.e., points y such that $H \subseteq G_y$), and let $\tilde{Y}^H$ denote the subset of Y^H where the isotropy group is exactly H. As we have seen, Y^H is a totally geodesic submanifold of Y, and by assumption $\tilde{Y}^H$ is a non-empty subset of Y^H, it is a standard fact of transformation group theory [4] that $\tilde{Y}^H$ is a dense open submanifold of the compact manifold Y^H. Let X be the homogeneous G manifold G/H and x_0 the identity coset. As above, M denotes the manifold of C^1 immersions of X into Y. As we have noted, since X and Y are G manifolds, so is M under the action $(gs)(x) = g(s(g^{-1}x))$. We saw earlier that the set of symmetric (i.e., equivariant) maps of X into N is diffeomorphic to $\tilde{N}^H$ under the correspondence $s \mapsto s(x_0)$. It is easily seen that for s to be an immersion, we must have $s(x_0) \in \tilde{N}^H$, i.e., the set Σ of symmetric points of M is just $\tilde{N}^H$. It is also easily seen that as y in $\tilde{N}^H$ converges to a point not in $\tilde{N}^H$ the volume of the orbit G_y converges to zero. Since $\bar{N}^H$ is compact, it follows that the volume of G_y must have a maximum at some point $y \in \tilde{N}^H$, so there exists a critical symmetric point, and hence a symmetric critical point. This proves:

Hsiang's Theorem *Let G be a compact Lie group and N a compact Riemannian G manifold. If a compact subgroup H of G occurs as an isotropy group in N, then among all orbits of N having H as an isotropy group, there is one of maximum volume, and it is minimally embedded in N.*

There is an interesting corollary to this theorem. It was proved independently by the author and G. D. Mostow that, given any compact Lie group G and closed subgroup H, there exists an orthogonal representation of G on some R^n having H as an isotropy group. It follows that:

Corollary *Every compact homogeneous manifold G/H admits a G-equivariant minimal embedding in some S^n.*

Next we assume that X and N are both compact Riemannian manifolds. For each C^1 map $s : X \to N$ we define a smooth function $e(s)$ on X, called the *energy density* of s, by $e(s)(x) = \mathrm{Tr}(ds_x^* ds_x)$ — the Hilbert-Schmidt norm of the linear map $ds_x : TX_x \to TN_{s(x)}$. Letting μ denote the Riemannian volume element of X, we define a first order Lagrangian ε on $C^1(X, N)$ by $\varepsilon(s) = e(s)\mu$, and the functional $E(s) = F_\varepsilon(s) = \int_X e(s)\mu$ is called the *total energy* of the map s. Critical points of E are called *harmonic maps* from X into N, and they have been actively studied in the past decades, especially by J. Eells and his co-workers [5], [6]. When $n = \dim(X) = 1$, then harmonic maps are just geodesics parametrized proportionally to arc length, and for $n = 1$ they play an important role in the study of minimal surfaces.

It is immediate from the definition of E that if φ is an isometry of

Y and ψ an isometry of Y then $E(\psi \circ s \circ \varphi) = E(s)$, so in particular, if X and Y are G spaces, it follows that E is a G-invariant function on $M = C^1(X, Y)$.

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