TORUS BUNDLES OVER A TORUS

R. S. PALAIS AND T. E. STEWART

1. If a compact Lie group $P$ acts on a completely regular topological space $E$ then $E$ is said to be a principal $P$-bundle if whenever the relation $px = x$ holds for $p \in P$, $x \in E$ it follows that $p = e$, the identity of $P$. The orbit space $X = E/P$ is called the base space and the map $\pi: E \to X$ carrying $y$ into its orbit $P \cdot y$ is called the projection. Suppose now that $G$ is a Lie group, $H_1$ a closed subgroup of $G$ and $H$ a closed, normal subgroup of $H_1$ such that $P = H_1/H$ is compact. Then $E = G/H$ becomes a principal $P$-bundle with base space $X = G/H_1$ and projection $gH \to gH_1$ under the action $p(gH) = gp^{-1}H$. Such a principal bundle will be called canonical.

The purpose of this note is to show that if $X$ is a torus of dimension $w$ and $P$ a torus of dimension $m$ then every principal $P$-bundle over $X$ is canonical and further that the group $G$ lies in an extremely narrow class. Roughly speaking, our method is to try to lift the action of euclidean $n$-space up to the total space of the bundle and observe what obstructs this effort.

The torus of dimension $k$ will be denoted by $T^k$, the corresponding euclidean space by $\mathbb{R}^k$. We view $\mathbb{R}^k$ as acting transitively on $T^k$ with the lattice of integral points in $\mathbb{R}^k$ acting ineffectively. We denote by $p: \mathbb{R}^k \to T^k$ the usual covering map. Without loss of generality we assume the $T^m$-bundles over $T^n$ are differentiable. $\mathfrak{g}(X)$ will denote the Lie algebra of infinitely differentiable vector fields on $X$.

Exactly how narrow the class of Lie groups that $G$ lies in will be left to Theorem 2. For the present we prove:

**Theorem 1.** Suppose we have a principal bundle over $T^n$ with structural group $T^m$, total space $E$, $\pi: E \to T^n$ the projection, and $e_0 \in \pi^{-1}(0)$. Then $E$ is acted on transitively by a 2-step nilpotent Lie group $G$. Further if $\alpha: G \to E$ is defined by $\alpha(g) = g \cdot e_0$ then we have a homomorphism $\beta: G \to R^n$, and a commutative diagram

$$
\begin{array}{ccc}
G & \xrightarrow{\alpha} & E \\
\beta \downarrow & & \downarrow \pi \\
R^n & \to & T^n \\
\phi & & \\
\end{array}
$$

Received by the editors March 9, 1960.
Proof. Let $\mathfrak{g}_k$ denote the Lie algebra of the Lie group $R^k$. Since $(E, \pi)$ is a principal bundle we may consider $T^n$ acting on $E$ without fixed points. We have then an injective algebra homomorphism $\alpha_1: \mathfrak{g}_m \rightarrow \mathfrak{g}(E)$ such that if $Z \neq 0$ then $\alpha_1(Z)_p \neq 0$, $p \in E$, where $\alpha_1(Z)_p$ denotes $\alpha_1(Z)$ evaluated at $p$ [2, p. 16]. And, of course, we have the usual injective homomorphism $\phi_1: \mathfrak{g}_n \rightarrow \mathfrak{g}(T^n)$, $\phi_1(X)_x \neq 0$ for $X \neq 0$, $x \in T^n$. We suppose now we are given a connection in $(E, \pi)$ [2, p. 25]. This may be considered a linear map $\lambda: \mathfrak{g}(T^n) \rightarrow \mathfrak{g}(E)$ such that $[\lambda(X), \alpha_1(Z)] = 0$, and $\delta \pi(\lambda(X)) = X$ where $\delta \pi$ denotes the differential of the mapping $\pi$. We identify $X \in \mathfrak{g}_n$ and $\phi_1(X)$ and denote $\lambda(X)$ by $X^*$, similarly we identify $Z \in \mathfrak{g}_m$ with $\alpha_1(Z)$.

Let $\{X_i\}_{i=1}^n$ be a basis of $\mathfrak{g}_n$, $X_i = (\delta_{ij})_{j=1}^n$, and $\{Z_k\}$ a basis of $\mathfrak{g}_m$. For $X, Y \in \mathfrak{g}_n$ we have $\delta \pi [X^*, Y^*]_z = [\delta \pi (X^*), \delta \pi (Y^*)]_z = [X, Y]_z = 0$, hence $[X^*, Y^*]_p = \Phi_p(X, Y) = \sum a^k(X, Y, p)(Z_k)_p$. It is easily seen that $\Phi_p(X, Y)$ is actually a function of $\pi(p)$ since $[X^*, Z]_p = 0$, $Z \in \mathfrak{g}_m$, i.e., $\Phi_p(X, Y)$ is an exterior two form on $T^n$. In particular we set $\Phi_p(X_i, X_j) = \sum_{k=1}^n a^k_i(x)(Z_k)$. Now $[[X^*, X^*], X^*]_p = \sum_{k=1}^n (X_r(a^k(r(x)))Z_k$. Applying the Jacobi identity $[[X^*, X^*], X^*]_p + [[X^*, X^*], X^*]_p + [[X^*, X^*], X^*]_p = 0$ we obtain for $k = 1, \ldots, m$

\[(1.2) \quad X_r(a^k_i(x)) + X_i(a^k_j(x)) + X_j(a^k_i(x)) = 0.\]

If $\{\omega_i\}_{i=1}^n$ is the set of exterior one forms dual to $X_i$, i.e. $(\omega_i(X_j))_p = \delta_{ij}$ then equation 1.2 states that $\Phi_p = (\sum_{i<j} a^k_i \omega_i \wedge \omega_j)_z$ is a closed form, i.e., $d \Phi^k = 0$. Now as is well known every closed form of the torus is cohomologous to an invariant form [1, p. 95]. Hence for each $k$ there exists a two form $\Psi^k$ on $T^n$ which does not vary with $x \in T^n$, and a one form $\theta^k$ on $T^n$ such that

\[(1.3) \quad \Phi^k - \Psi^k = d \theta^k.\]

In terms of our chosen basis we have a set of constants $c^k_{ij}$ and a set of functions $b^k_i(x)$ such that for $i, j, k$

\[(1.4) \quad a^k_{ij}(x) - c^k_{ij} = X_r(b^k_i(x)) - X_i(b^k_j(x)).\]

For each $i$, set $X^*_{i} = X^*_{i} + \sum_{k=1}^n b^k_i(x)Z_k$. It follows easily from (1.4) that

\[(1.5) \quad [X^*_{i}, X^*_{j}]_p = \sum_{k=1}^n c^k_{ij}(Z_k)_p.\]

Extending the function $X_i \rightarrow X^*_{i}$ linearly over $\mathfrak{g}_n$ we obtain a linear map $\mathfrak{g}_n \rightarrow \mathfrak{g}(E)$ satisfying
Define the Lie algebra $\mathfrak{g}$ with underlying vector space $\mathfrak{g}_n + \mathfrak{g}_m$ and bracket operation defined by

\[
[X, Z] = 0, \quad X \in \mathfrak{g}, \quad Z \in \mathfrak{g}_m,
\]

\[
[X, Y] = \sum_k \Psi^k(X, Y)Z_k, \quad X, Y \in \mathfrak{g}_n.
\]

According to (1.6) and (1.7) we have an isomorphism $\chi: \mathfrak{g} \to \mathfrak{g}(E)$ such that $(\chi(X))_p \neq 0$ for $X \neq 0$, $p \in E$. If $G$ is the simply connected Lie group associated to $\mathfrak{g}$ it follows from the compactness of $E$ that we can integrate the isomorphism $\chi$ to obtain an action of $G$ on $E$. Since each orbit will be open and $E$ is connected this action is transitive. $G$ is nilpotent since $\mathfrak{g}$ is and $\beta: G \to R^n$ is the homomorphism associated to the Lie algebra homomorphism $X + Z \to X$, $X \in \mathfrak{g}_n$, $Z \in \mathfrak{g}_m$. The diagram (1.1) follows easily, and hence the theorem.

For notational purposes we restrict ourselves now to the case $m = 1$. The following statements are easily generalized to the case $m > 1$.

Let $Z$ be a unit vector of $\mathfrak{g}_1$. The Lie algebra $\mathfrak{g}$ defined in the last theorem is given by the space $\mathfrak{g}_n + \mathfrak{g}_1$ with

\[
[X, Z] = 0, \quad X \in \mathfrak{g}, \quad Z \in \mathfrak{g}_1,
\]

\[
[X, Y] = \Psi(X, Y)Z, \quad X, Y \in \mathfrak{g}_n,
\]

$\Psi$ bilinear and skew-symmetric.

We describe now the group $G$. As a space we have clearly $G = R^n \times R^1$. As one can readily verify (or else by the Baker-Hausdorff formula) the multiplication in $G$ is given by

\[
(x, y) \cdot (x', y') = \left(x + x', y + y' + \frac{1}{2} \Psi(x, x') \right).
\]

We have here identified $\mathfrak{g}_k$ and $R^k$. We have then:

**Theorem 2.** If both $x$ and $x'$ belong to the lattice of integers in $R^n$, then $\Psi(x, x')$ is integral.

**Proof.** We show first that if $x$ is in the lattice of integers in $R^n$ then there exists $y \in R^1$ such that $(x, y)$ is in the isotropy group of $e_0 \in \pi^{-1}(0)$.

Indeed it follows from the commutativity of the diagram (1.1) that $(x, 0) \cdot e_0 = \pi^{-1}(0)$ and hence there is $y \in R^1$ such that $e_0 = (0, y)(x, 0) = (x, y)$.

We have then identified $\mathfrak{g}_k$ and $R^k$. We have then:

**Theorem 2.** If both $x$ and $x'$ belong to the lattice of integers in $R^n$, then $\Psi(x, x')$ is integral.

**Proof.** We show first that if $x$ is in the lattice of integers in $R^n$ then there exists $y \in R^1$ such that $(x, y)$ is in the isotropy group of $e_0 \in \pi^{-1}(0)$.

Indeed it follows from the commutativity of the diagram (1.1) that $(x, 0) \cdot e_0 = \pi^{-1}(0)$ and hence there is $y \in R^1$ such that $e_0 = (0, y)(x, 0) = (x, y)$.
Now let \(x\) and \(x'\) be in the lattice of integers in \(\mathbb{R}^n\) and \(y, y'\) elements of \(\mathbb{R}^1\) such that \((x, y)\) and \((x', y')\) are in the isotropy group \(G_{e_0}\) of \(e_0\). Then the commutator of these two elements is in \(G_{e_0}\). Since \((x, y)^{-1} = (-x, -y)\) we have
\[
(x, y)(x', y') \cdot (x, y)^{-1} \cdot (x', y')^{-1} = (0, c(x, x')) \in G_{e_0}.
\]

Since \((0, y) \in G_{e_0}\) if and only if \(y\) is integral the theorem follows.

It should be noted that Theorem 1 is not true if the torus is only a fibre and not actually the structural group. For example, the two dimensional Klein bottle is not acted on transitively by a nilpotent group, but is the total space of a bundle over the circle with circle as fibre.

2. The authors would like to express their appreciation to the referee for pointing out the following theorem which strengthens Theorem 2.

**Theorem 3.** A space \(X\) is a compact 2-step nilmanifold if and only if it is the total space of a principal \(T^m\)-bundle over \(T^n\).

**Proof (Following the referee).** The sufficiency follows from Theorem 1. Now suppose \(X\) is acted on transitively by a connected, simply connected 2-step nilpotent group, \(N\), with stability group \(\Gamma\) which is discrete [2, p. 12]. Set \(P = [N, N]\), the commutator group of \(N\), and \(G = P \cdot \Gamma\). Then \(P\) is central since \(N\) is a 2-step nilpotent group and \(\Gamma\) is invariant in \(G\), and \(G\) is invariant in \(N\). Since \(N/G\) and \(G/\Gamma\) are then compact, connected, abelian Lie groups, they are tori. Then the assertion follows from the bundle \(X = N/\Gamma \to N/G\).

**References**