Lecture 4
Linear Maps And The Euclidean Group.

I assume that you have seen the basic facts concerning linear transformations and matrices in earlier courses. However we will review these facts here to establish a common notation. In all the following we assume that the vector spaces in question have finite dimension.

4.1 Linear Maps and Matrices

Let $V$ and $W$ be two vector spaces. A function $T$ mapping $V$ into $W$ is called a linear map if $T(\alpha v_1 + \beta v_2) = \alpha T(v_1) + \beta T(v_2)$ for all scalars $\alpha, \beta$ and all $v_1, v_2 \in V$. We make the space $L(V, W)$ of all linear maps of $V$ into $W$ into a vector space by defining the addition and scalar multiplication laws to be “pointwise”. i.e., if $S, T \in L(V, W)$, then for any $v \in V$ we define $(\alpha T + \beta S)(v) := \alpha T(v) + \beta S(v)$.

4.1.1 Remark. If $v_1, \ldots, v_n$ is any basis for $V$ and $\omega_1, \ldots, \omega_n$ are arbitrary elements of $W$, then there is a unique $T \in L(V, W)$ such that $T(v_i) = \omega_i$. For if $v \in V$, then $v$ has a unique expansion of the form $v = \sum_{i=1}^{n} \alpha_i v_i$, and then we can define $T$ by $T(v) := \sum_{i=1}^{n} \alpha_i \omega_i$, and it is easily seen that this $T$ is linear, and that it is the unique linear transformation with the required properties.

In particular, if $w_1, \ldots, w_m$ is a basis for $W$, then for $1 \leq i \leq n$ and $1 \leq j \leq m$ we define $E_{ij}$ to be the unique element of $L(V, W)$ that maps $v_i$ to $w_j$ and maps all the other $v_k$ to the zero element of $W$.

4.1.2 Definition. Suppose $T : V \to W$ is a linear map, and that as above we have a basis $v_1, \ldots, v_n$ for $V$ and a basis $w_1, \ldots, w_m$ for $W$. For $1 \leq j \leq n$, the element $Tv_j$ of $W$ has a unique expansion as a linear combination of the $w_i$, $T(v_j) = \sum_{i=1}^{m} T_{ij} w_i$. These $mn$ scalars $T_{ij}$ are called the matrix elements of $T$ relative to the two bases $v_i$ and $w_j$.

4.1.3 Remark. It does not make sense to speak of the matrix of a linear map until bases are specified for the domain and range. However, if $T$ is a linear map from $\mathbb{R}^n$ to $\mathbb{R}^m$, then by its matrix we always understand its matrix relative to the standard bases for $\mathbb{R}^n$ and $\mathbb{R}^m$.

4.1.4 Remark. If $V$ is a vector space then we abbreviate $L(V, V)$ to $L(V)$, and we often refer to a linear map $T : V \to V$ as a linear operator on $V$. To define the matrix of a linear operator on $V$ we only need one basis for $V$.

> 4.1—Exercise 1. Suppose that $v \in V$ has the expansion $v = \sum_{j=1}^{n} \alpha_j v_j$, and that $Tv \in W$ has the expansion $Tv = \sum_{i=1}^{m} \beta_i w_i$. Show that we can compute the components $\beta_i$ of $Tv$ from the components $\alpha_j$ of $v$ and the matrix for $T$ relative to the two bases, using the formula $\beta_i = \sum_{j=1}^{n} T_{ij} \alpha_j$. 


Caution! Distinguish carefully between the two formulas: \( T(v_j) = \sum_{j=1}^{m} T_{ij} w_i \) and \( \beta_i = \sum_{j=1}^{n} T_{ij} \alpha_j \). The first is essentially the definition of the matrix \( T_{ij} \) while the second is the formula for computing the components of \( Tv \) relative to the given basis for \( W \) from the components of \( v \) relative to the given basis for \( V \).

\[ \text{\textgreater 4.1 Exercise 2.} \quad \text{Show that } T = \sum_{i=1}^{n} \sum_{j=1}^{m} T_{ij} E_{ij}, \text{ and deduce that } E_{ij} \text{ is a basis for } L(V, W), \text{ so in particular, } L(V, W) \text{ has dimension } nm, \text{ the product of the dimensions of } V \text{ and } W. \]

### 4.2 Isomorphisms and Automorphisms

If \( V \) and \( W \) are vector spaces, then a linear map \( T : V \to W \) is called an isomorphism of \( V \) with \( W \) if it is bijective (i.e., one-to-one and onto), and we say that \( V \) and \( W \) are isomorphic if there exists an isomorphism of \( V \) with \( W \). An isomorphism of \( V \) with itself is called an automorphism of \( V \), and we denote the set of all automorphisms of \( V \) by \( \text{GL}(V) \). (\( \text{GL}(V) \) is usually referred to as the general linear group of \( V \)—check that it is a group.)

\[ \text{\textgreater 4.2 Exercise 1.} \quad \text{If } T : V \to W \text{ is a linear map and } v_1, \ldots, v_n \text{ is a basis for } V \text{ then show that } T \text{ is an isomorphism if and only if } Tv_1, \ldots, Tv_n \text{ is a basis for } W. \text{ Deduce that two finite-dimensional vector spaces are isomorphic if and only if they have the same dimension.} \]

There are two important linear subspaces associated to a linear map \( T : V \to W \). The first, called the kernel of \( T \) and denoted by \( \ker(T) \), is the subspace of \( V \) consisting of all \( v \in V \) such that \( T(v) = 0 \), and the second, called the image of \( T \), and denoted by \( \text{im}(T) \), is the subspace of \( W \) consisting of all \( w \in W \) of the form \( Tv \) for some \( v \in V \).

Notice that if \( v_1 \) and \( v_2 \) are in \( V \), then \( T(v_1) = T(v_2) \) if and only if \( T(v_1 - v_2) = 0 \), i.e., if and only if \( v_1 \) and \( v_2 \) differ by an element of \( \ker(T) \). Thus \( T \) is one-to-one if and only if \( \ker(T) \) contains only the zero vector.

**Proposition.** A necessary and sufficient condition for \( T : V \to W \) to be an isomorphism of \( V \) with \( \text{im}(T) \) is for \( \ker(T) \) to be the zero subspace of \( V \).

**Theorem.** If \( V \) and \( W \) are finite dimensional vector spaces and \( T : V \to W \) is a linear map, then \( \dim(\ker(T)) + \dim(\text{im}(T)) = \dim(V) \).

**Proof.** Choose a basis \( v_1, \ldots, v_k \) for \( \ker(T) \) and extend it to a basis \( v_1, \ldots, v_n \) for all of \( V \). It will suffice to show that \( T(v_{k+1}), \ldots, T(v_n) \) is a basis for \( \text{im}(T) \). We leave this as an (easy) exercise.

**Corollary.** If \( V \) and \( W \) have the same dimension then a linear map \( T : V \to W \) is an isomorphism of \( V \) with \( W \) if it is either one-to-one or onto.

Recall that if \( V \) is an inner product space and \( v_1, v_2 \in V \), then we define the distance between \( v_1 \) and \( v_2 \) as \( \rho(v_1, v_2) := \|v_1 - v_2\| \). This makes any inner-product space into a metric space. A mapping \( f : V \to W \) between inner-product spaces is called an isometry if it is distance preserving, i.e., if for all \( v_1, v_2 \in V \), \( \|f(v_1) - f(v_2)\| = \|v_1 - v_2\| \).
4.2.1 Definition. If \( V \) is an inner product space then we define the Euclidean group of \( V \), denoted by \( \text{Euc}(V) \), to be the set of all isometries \( f : V \rightarrow V \). We define the orthogonal group of \( V \), denoted by \( O(V) \) to be the set of \( f \in \text{Euc}(V) \) such that \( f(0) = 0 \).

4.2.2 Remark. We will justify calling \( \text{Euc}(V) \) a group shortly. It is clear that \( \text{Euc}(V) \) is closed under composition, and that elements of \( \text{Euc}(V) \) are one-to-one, but at this point it is not clear that an element \( f \) of \( \text{Euc}(V) \) maps onto all of \( V \), so \( f \) might not have an inverse in \( \text{Euc}(V) \). A similar remark holds for \( O(V) \).

Proposition. If \( f \in O(V) \) then \( f \) preserves inner-products, i.e., if \( v_1, v_2 \in V \) then \( \langle f v_1, f v_2 \rangle = \langle v_1, v_2 \rangle \).

PROOF. Clearly \( f \) preserves norms, since \( \| f(v) \| = \| f(v) - f(0) \| = \| v - 0 \| = \| v \| \), and we also know that, \( \| f(v_1) - f(v_2) \| = \| v_1 - v_2 \| \). Then \( \langle f v_1, f v_2 \rangle = \langle v_1, v_2 \rangle \) now follows easily from the polarization identity in the form: \( \langle v, w \rangle = \frac{1}{2} (\| v + w \|^2 - \| v - w \|^2) \).

Theorem. \( O(V) \subseteq GL(V) \), i.e., elements of \( O(V) \) are invertible linear transformations.

PROOF. Let \( e_1, \ldots, e_n \) be an orthonormal basis for \( V \) and let \( e_i = f(e_i) \). By the preceding proposition \( \langle e_i, e_j \rangle = \langle e_i, e_j \rangle = \delta_{ij} \), so that the \( e_i \) also form an orthonormal basis for \( V \). Now suppose that \( v_1, v_2 \in V \) and let \( \alpha_i, \beta_i \) and \( \gamma_i \) be respectively the components of \( v_1, v_2 \), and \( v_1 + v_2 \) relative to the orthonormal basis \( e_i \), and similarly let \( \alpha_i', \beta_i' \) and \( \gamma_i' \) be the components of \( f(v_1), f(v_2) \), and \( f(v_1 + v_2) \) relative to the orthonormal basis \( e_i \). To prove that \( f(v_1 + v_2) = f(v_1) + f(v_2) \) it will suffice to show that \( \gamma_i' = \alpha_i' + \beta_i' \). Now we know that \( \gamma_i = \alpha_i + \beta_i \), so we will suffice to show that \( \alpha_i = \alpha_i', \beta_i = \beta_i', \gamma_i = \gamma_i' \). But since \( \alpha_i = \langle v_i, e_i \rangle \) while \( \alpha_i' = \langle f(v_1), e_i \rangle = \langle f(v_1), f(e_i) \rangle, \alpha_i' = \alpha_i \) follows from the fact that \( f \) preserves inner-products, and the other equalities follow likewise. A similar argument shows that \( f(\alpha v) = \alpha f(v) \). Finally, since \( f \) is linear and one-to-one, it follows that \( f \) is invertible.

4.2.3 Remark. It is now clear that we can equivalently define \( O(V) \) to be the set of linear maps \( T : V \rightarrow V \) that preserves inner-products.

Every \( a \in V \) gives rise to a map \( \tau_a : V \rightarrow V \) called translation by \( a \), defined by, \( \tau_a(v) = v + a \). The set \( \mathcal{T}(V) \) of all \( \tau_a, a \in V \) is clearly a group since \( \tau_{a+b} = \tau_a \circ \tau_b \) and \( \tau_0 \) is the identity. Moreover since \( (v_1 + a) - (v_2 + a) = v_1 - v_2 \), it follows that \( \tau_a \) is an isometry, i.e. \( \mathcal{T}(V) \subseteq \text{Euc}(V) \)

Theorem. Every element \( f \) of \( \text{Euc}(V) \) can be written uniquely as an orthogonal transformation \( O \) followed by a translation \( \tau_a \).

PROOF. Define \( a := f(0) \). Then clearly the composition \( \tau_{-a} \circ f \) leaves the origin fixed, so it is an element \( O \) of \( O(V) \), and it follows that \( f = \tau_a \circ O \). (We leave uniqueness as an exercise.)

Corollary. Every element \( f \) of \( \text{Euc}(V) \) is a one-to-one map of \( V \) onto itself and its inverse is also in \( V \), so \( \text{Euc}(V) \) is indeed a group of transformations of \( V \).

PROOF. In fact we see that \( f^{-1} = O^{-1} \circ \tau_{-a} \).