First Assignment

▷ **Exercise** 1. When I want to see an initial explanation of something, I have for some time now used Google to do a search and then I read a few of the top-rated items that look promising. What I have found surprising is that this seems to be quite successful even for fairly abstract mathematical subjects.

In the second lecture I plan to explain something called Felix Klein's Erlanger program. Use Google to try to get some idea what this is about, and after I talk about it in class, tell me if you think reading about it in advance this way helped you or not.

You should do the next few exercises before the third lecture. They are designed for me to get some feedback from you on whether you find it easy to learn a topic by carrying out a short series of exercises. Please try to work through the following definitions and exercises and we will discuss in class whether you feel this is a good method for you to learn something.

We start with a definition.

Definition Let X be a set. A real-valued function ρ on $X \times X$ is called a *metric* or *distance function* for X if it satisfies the following three properties:

a) Symmetry: $\rho(x, y) = \rho(y, x)$ for all x and y in X.

b) Positivity: $\rho(x, y) \geq 0$, with equality if and only if $x = y$.

c) Triangle inequality $\rho(x, z) \leq \rho(x, y) + \rho(y, z)$ for all x, y, z in X.

By a *metric space* we mean a set X together with some fixed metric for X. (We will usually denote this metric by ρ_X, but if in a certain context there is no danger of confusion, we will often use just ρ.)

Example 1. Take X to be $\mathbf{R}$, the real numbers, and define $\rho(x, y) = |x - y|$.

Example 2. More generally, take X to be $\mathbf{R}^n$, and define $\rho(x, y) = \|x - y\|$.

Example 3. Take X to be the sphere $\mathbf{S}^2$, and define $\rho(x, y)$ to be the length of the shorter of the segments of great circles joining x to y.

▷ **Exercise** 2. Suppose $\{p_n\}$ is a sequence in the metric space X and $p \in X$. Give a definition for what it means for the sequence $\{p_n\}$ to converge to p. (Hint: A seqence $\{x_n\}$ of real numbers converges to a real number x if and only if $\lim_{n\to\infty} |x_n - x| = 0$.) We usually write either $p_n \to p$ or $\lim_{n\to\infty} p_n = p$ to denote that $\{p_n\}$ converges to p. Show that if $p_n \to p$ and $p_n \to q$ then $p = q$, i.e., limits of sequences are unique. (Hint $\rho(p, q) \leq \rho(p_n, p) + \rho(p_n, q)$.)

Now suppose that X and Y are two metric spaces and that $f : X \to Y$ is a function mapping X into Y. We say that f is a *continuous* function if whenever a sequence $\{x_n\}$ in X converges to some limit x in X, it follows that the sequence $\{f(x_n)\}$ in Y converges to $f(x)$.

Definition Let X and Y be metric spaces and $f : X \to Y$ a function from X into Y. If K is a positive real number, we say that f satisfies a Lipschitz condition with constant K if $\rho_Y(f(x_1), f(x_2)) \leq K\rho_X(x_1, x_2)$ for all x_1 and x_2 in X, and we say that f is a Lipschitz function if it satisfies a Lipschitz condition with some constant K.

$\triangleright$ **Exercise** 3. Show that every Lipschitz function is continuous.

Definition A *contraction mapping* of a metric space X is a function that maps X into itself and that satisfies a Lipschitz condition with constant $K < 1$.

$\triangleright$ **Exercise** 4. Let X be a metric space and $f : X \to X$ a contraction mapping with Lipschitz constant $K < 1$. Prove the "Fundamental Inequality for Contraction Mappings":

$$\rho(x, y) \leq \frac{1}{1-K}\big(\rho(x, f(x)) + \rho(y, f(y))\big)$$

holds for all x,y in X. (Hint: This is VERY easy if you apply the triangle inequality in the right way. But where does $K < 1$ come in?)