Answers to First Assignment

Definition Let X be a set. A real-valued function ρ on $X \times X$ is called a *metric* or *distance function* for X if it satisfies the following three properties:

a) Symmetry: $\rho(x, y) = \rho(y, x)$ for all x and y in X.

b) Positivity: $\rho(x, y) \geq 0$, with equality if and only if $x = y$.

c) Triangle inequality $\rho(x, z) \leq \rho(x, y) + \rho(y, z)$ for all x, y, z in X.

By a *metric space* we mean a set X together with some fixed metric for X. (We will usually denote this metric by ρ_X, but if in a certain context there is no danger of confusion, we will often use just ρ.)

▷ **Exercise** 1. Suppose $\{p_n\}$ is a sequence in the metric space X and $p \in X$. Give a definition for what it means for the sequence $\{p_n\}$ to converge to p. (Hint: A seqence $\{x_n\}$ of real numbers converges to a real number x if and only if $\lim_{n\to\infty} |x_n - x| = 0$.)

Answer We define $\{p_n\}$ converges to p to mean that $\lim_{n\to\infty} \rho(p_n, p)| = 0$. (Remark. many of you said, " if $\lim_{n\to\infty} \|p_n - p\| = 0$", but neither subtraction nor the norm is defined in a general metric space.)

We usually write either $p_n \to p$ or $\lim_{n\to\infty} p_n = p$ to denote that $\{p_n\}$ converges to p. Show that if $p_n \to p$ and $p_n \to q$ then $p = q$, i.e., limits of sequences are unique. (Hint $\rho(p, q) \leq \rho(p_n, p) + \rho(p_n, q)$.)

Answer By "positivity" it will suffice to show that $\rho(p, q) = 0$, and for this it suffices to show that for any $\epsilon > 0$, $\rho(p, q) < \epsilon$. Since $p_n \to p$ and $p_n \to q$, for sufficiently large n, $\rho(p_n, p) < \epsilon/2$ and $\rho(p_n, q) < \epsilon/2$, so by the triangle inequality, $\rho(p, q) \leq \rho(p_n, p) + \rho(p_n, q) < \epsilon$.

Now suppose that X and Y are two metric spaces and that $f : X \to Y$ is a function mapping X into Y. We say that f is a *continuous* function if whenever a sequence $\{x_n\}$ in X converges to some limit x in X, it follows that the sequence $\{f(x_n)\}$ in Y converges to $f(x)$.

Definition Let X and Y be metric spaces and $f : X \to Y$ a function from X into Y. If K is a positive real number, we say that f satisfies a Lipschitz condition with constant K if $\rho_Y(f(x_1), f(x_2)) \leq K\rho_X(x_1, x_2)$ for all x_1 and x_2 in X, and we say that f is a Lipschitz function if it satisfies a Lipschitz condition with some constant K.

▷ **Exercise** 2. Show that every Lipschitz function is continuous.

Answer Suppose $\{x_n\}$ in X converges to a limit x in X. Given $\epsilon > 0$, choose N so that if $n > N$ then $\rho_X(x_n, x) < \epsilon/K$. Then for $n > N$, $\rho_Y(f(x_n), f(x)) < K\rho_X(x_n, x) < \epsilon$, proving that $f(x_n) \to f(x)$, so f is continuous.

Definition A *contraction mapping* of a metric space X is a function that maps X into itself and that satisfies a Lipschitz condition with constant $K < 1$.

▷ **Exercise** 3. Let X be a metric space and $f : X \to X$ a contraction mapping with Lipschitz constant $K < 1$. Prove the "Fundamental Inequality for Contraction Mappings":

$$\rho(x, y) \leq \frac{1}{1-K}\big(\rho(x, f(x)) + \rho(y, f(y))\big)$$

holds for all x,y in X. (Hint: This is VERY easy if you apply the triangle inequality in the right way. But where does $K < 1$ come in?)

Answer By the Triangle Inequality, $\rho(x, y) \leq \rho(x, f(x)) + \rho(f(x), y)$. By a second application of the Triangle Inequality, (and symmetry) $\rho(f(x), y) \leq \rho(f(x), f(y)) + \rho(y, f(y))$, so putting these inequalities together, $\rho(x, y) \leq \rho(x, f(x)) + \rho(f(x), f(y)) + \rho(y, f(y))$. Then using $\rho(f(x), f(y)) < K\rho(x, y)$ we get $\rho(x, y) \leq \rho(x, f(x)) + K\rho(x, y) + \rho(y, f(y))$, or transposing the middle term, $(1 - K)\rho(x, y) \leq \rho(x, f(x)) + \rho(y, f(y))$. Since $K < 1$, $(1 - K)$ is positive, so we can divide this inequality by $(1 - K)$ and arrive at the desired result.