# Second Assignment Due Friday, Sept.12, 2003

Some exercises on linear transformations and matrices.

▷ **Exercise** 1. Let $v_1, \ldots, v_n$ be a basis for a vector space $V$ and let $S$ and $T$ be two linear operators on $V$. If the matrices of $S$ and $T$ relative to this basis are respectively $S_{ij}$ and $T_{ij}$, then show that the matrix elements of the composed linear operator $ST$ are given by $(ST)_{ij} = \sum_{k=1}^{n} S_{ik}T_{kj}$, and that the matrix elements of the sum operator $S + T$ are given by $(S + T)_{ij} = S_{ij} + T_{ij}$.

In what follows, $\mathcal{P}^n$ denotes the space of polynomials functions $a_0 + a_1 x + a_2 x^2 + \ldots + a_n x^n$ of degree $\leq n$. Clearly $\mathcal{P}^n$ is a vector space of dimension $n + 1$ and $1, x, x^2, \ldots, x^n$ is a basis for $\mathcal{P}^n$ (called the standard basis).

▷ **Exercise** 2. Differentiation defines an operator $D$ on $\mathcal{P}^n$, and of course $D^k$ denotes the $k$-th derivative operator.

a) What is the matrix of $D$ in the standard basis?

b) What is the kernel of $D^k$?

c) What is the image of $D^k$?

▷ **Exercise** 3. Define an inner-product on $\mathcal{P}^n$ by $\langle P_1, P_2 \rangle = \int_{-1}^{1} P_1(x)P_2(x)\, dx$, and note that the standard basis is **not** orthonormal (or even orthogonal). Let us define orthonormal polynomials $L_k(x)$ by applying the Gram-Schmidt Algorithm to the standard basis. (The $L_k$ are usually called the normalized Legendre Polynomials.) Compute $L_0, L_1$, and $L_2$.