Second Assignment Answers

Some exercises on linear transformations and matrices.

▷ **Exercise** 1. Let $v_1, \ldots, v_n$ be a basis for a vector space V and let S and T be two linear operators on V. If the matrices of S and T relative to this basis are respectively S_{ij} and T_{ij}, then show that the matrix elements of the composed linear operator ST are given by $(ST)_{ij} = \sum_{k=1}^n S_{ik}T_{kj}$, and that the matrix elements of the sum operator $S+T$ are given by $(S+T)_{ij} = S_{ij} + T_{ij}$.

Answer By definition of S_{ij} and T_{ij} we have: $Tv_j = \sum_{i=i}^n T_{ij}v_i$ and similarly $Sv_j = \sum_{i=i}^n S_{ij}v_i$. Since (by definition of the addition of linear operators) $(S+T)(v) = S(v) + T(v)$, the formula for $(S+T)_{ij}$ is immediate. On the other hand the "product" ST is defined to be the composition of S and T, so $(ST)(v_j) = S(T(v_j)) = S(\sum_{i=i}^n T_{ij}v_i) = \sum_{i=i}^n T_{ij}S(v_i) = \sum_{i=i}^n T_{ij}\sum_{k=1}^n S_{ki}v_k = \sum (ST)_{kj}v_k$.

In what follows, $\mathcal{P}^n$ denotes the space of polynomials functions $a_0 + a_1x + a_2x^2 + \ldots + a_nx^n$ of degree $\leq n$. Clearly $\mathcal{P}^n$ is a vector space of dimension $n+1$ and $1, x, x^2, \ldots, x^n$ is a basis for $\mathcal{P}^n$ (called the standard basis).

▷ **Exercise** 2. Differentiation defines an operator D on $\mathcal{P}^n$, and of course D^k denotes the k-th derivative operator.

a) What is the matrix of D in the standard basis?

Answer Let's denote the standard basis by $v_0 = 1, v_1 = x, \ldots, v_n = x^n$. Then since $Dv_k = kv_{k-1}$, the matrix D_{ij} of D is given by $D_{ij} = j$ if $i = j-1$ and $D_{ij} = 0$ otherwise. (Note that this says that the first or "row-index" must be one less than the second or "column index" for a matrix element to be non-zero.) So the non-zero entries are $D_{01} = 1, D_{12} = 2, \ldots, D_{n-1\,n} = n$, i.e., the matrix has $1, 2, \ldots, n$ just above the diagonal and zero elsewhere.

b) What is the kernel of D^k?

Answer $\mathcal{P}^{k-1}$

c) What is the image of D^k?

Answer $\mathcal{P}^{n-k}$

▷ **Exercise** 3. Define an inner-product on $\mathcal{P}^n$ by $\langle P_1, P_2 \rangle = \int_{-1}^1 P_1(x)P_2(x)\,dx$, and note that the standard basis is **not** orthonormal (or even orthogonal). Let us define orthonormal polynomials $L_k(x)$ by applying the Gram-Schmidt Algorithm to the standard basis. (The L_k are usually called the normalized Legendre Polynomials.) Compute L_0, L_1, and L_2.

Answer $L_0 = \frac{\sqrt{2}}{2}$, $L_1 = \frac{\sqrt{6}}{2}x$, $L_2 = \frac{\sqrt{10}}{2}(3x^2 - 1)$.