Third Assignment Due Friday, Sept.26, 2003

Here are a few exercises concerning adjoints of linear maps. We recall that if V and W are inner-product spaces and $T : V \to W$ is a linear map then there is a uniquely determined map $T^* : W \to V$, called the adjoint of T, satisfying $\langle Tv, w\rangle = \langle v, T^*w\rangle$ for all $v \in V$ and $w \in W$.

▷ **Exercise** 1. Show that $(T^*)^* = T$.

▷ **Exercise** 2. Recall that if T_{ij} is an $m \times n$ matrix (i.e., m rows and n columns) and S_{ji} an $n \times m$ matrix, then S_{ji} is called the *transpose* of T_{ij} if $T_{ij} = S_{ji}$ for $1 \leq i \leq m$, $1 \leq j \leq n$. Show that if we choose orthonormal bases for V and W, then the matrix of T^* relative to these bases is the transpose of the matrix of T relative to the same bases.

▷ **Exercise** 3. Show that $\ker(T)$ and $\operatorname{im}(T^*)$ are orthogonal complements in V, and similarly, $\operatorname{im}(T)$ and $\ker(T^*)$ are each other's orthogonal complements in W. (Note that by Exercise 1, you only have to prove one of these.)

▷ **Exercise** 4. Show that a linear operator T on V is in the orthogonal group $\boldsymbol{O}(V)$ if and only if $TT^* = I$ (where I denotes the identity map of V) or equivalently, if and only if $T^* = T^{-1}$.

If $T : V \to V$ is a linear operator on V, then T^* is also a linear operator on V, so it makes sense to compare them and in particular ask if they are equal.

Definition. A linear operator on an inner-product space V is called *self-adjoint* if $T^* = T$, i.e., if $\langle Tv_1, v_2\rangle = \langle v_1, Tv_2\rangle$ for all $v_1, v_2 \in V$.

Note that by Exercise 3 above, self-adjoint operators are characterized by the fact that their matrices with respect to an orthonormal basis are symmetric.

▷ **Exercise** 5. Show that if W is a linear subspace of the inner-product space V, then the orthogonal projection P of V on W is a self-adjoint operator on V.

Definition. If T is a linear operator on V, then a linear subspace $U \subseteq V$ is called a T-invariant subspace if $T(U) \subseteq U$, i.e., if $u \in U$ implies $Tu \in U$.

Remark. Note that if U is a T-invariant subspace of V, then T can be regarded as a linear operator on U by restriction, and clearly if T is self-adjoint, so is its restriction.

▷ **Exercise** 6. Show that if $T : V \to V$ is a self-adjoint operator, and $U \subseteq V$ is a T-invariant subspace of V, the $U^\perp$ is also a T-invariant subspace of V.

We next recall for convenience the definitions relating to eigenvalues and eigenvectors. We assume that T is a linear operator on V.

If λ is a real number, then we define the linear subspace $E_\lambda(T)$ of V to be the set of $v \in V$ such that $Tv = \lambda v$. In other words, if I denotes the identity map of V, then $E_\lambda(T) = \ker(T - \lambda I)$. Of course the zero vector is always in $E_\lambda(T)$. If $E_\lambda(T)$ contains a non-zero vector, then we say that λ is an *eigenvalue* of T and that $E_\lambda(T)$ is the λ-eigenspace of T. A non-zero vector in $E_\lambda(T)$ is called an *eigenvector* of T belonging to the eigenvector λ. The set of all eigenvalues of T is called the *spectrum* of T (the name comes from quantum mechanics) and it is denoted by $\mathrm{Spec}(T)$.

▷ **Exercise** 7. Show that a linear operator T on V has a diagonal matrix in a particular basis for V if and only if each element of the basis is an eienvector of T, and that then $\mathrm{Spec}(T)$ consists of the diagonal elements of the matrix.

▷ **Exercise** 8. If T is a self-adjoint linear operator on an inner-product space V and λ_1, λ_2 are distinct real numbers, show that $E_{\lambda_1}(T)$ and $E_{\lambda_2}(T)$ are orthogonal subspaces of V. In other words, eigenvectors of T that belong to different eigenvalues are orthogonal. (Hint: Let $v_i \in E_{\lambda_i}(T), i = 1, 2$. You must show that $\langle v_1, v_2 \rangle = 0$. Start with the fact that $\langle Tv_1, v_2 \rangle = \langle v_1, Tv_2 \rangle$.)