Third Assignment Answers

Here are a few exercises concerning adjoints of linear maps. We recall that if V and W are inner-product spaces and $T : V \to W$ is a linear map then there is a uniquely determined map $T^* : W \to V$, called the adjoint of T, satisfying $\langle Tv, w\rangle = \langle v, T^*w\rangle$ for all $v \in V$ and $w \in W$.

▷ **Exercise** 1. Show that $(T^*)^* = T$.

Answer By definition of adjoint linear maps, $\langle w, (T^*)^*v\rangle = \langle T^*w, v\rangle = \langle w, Tv\rangle$ for all $v \in V$ and $w \in W$. Therefore $(T^*)^* = T$.

▷ **Exercise** 2. Recall that if T_{ij} is an $m \times n$ matrix (i.e., m rows and n columns) and S_{ji} an $n \times m$ matrix, then S_{ji} is called the *transpose* of T_{ij} if $T_{ij} = S_{ji}$ for $1 \le i \le m$, $1 \le j \le n$. Show that if we choose orthonormal bases for V and W, then the matrix of T^* relative to these bases is the transpose of the matrix of T relative to the same bases.

Answer
Let $v_1, v_2, \ldots$ and $w_1, w2, \ldots$ be orthonormal bases for V and W. Then $\langle Tv_i, w_j\rangle = \langle \sum_k T_{ki} v_k, w_j\rangle = T_{ji}$ and $\langle v_i, T^*w_j\rangle = \left\langle v_i, \sum_k T^*_{kj} w_k \right\rangle = T^*_{ij}$. Since $\langle Tv_i, w_j\rangle = \langle v_i, T^*w_j\rangle$, we get that $T^*_{ij} = T_{ji}$.

▷ **Exercise** 3. Show that $\ker(T)$ and $\operatorname{im}(T^*)$ are orthogonal complements in V, and similarly, $\operatorname{im}(T)$ and $\ker(T^*)$ are each other's orthogonal complements in W. (Note that by Exercise 1, you only have to prove one of these.)

Answer Let $v \in \ker(T)$, so $Tv = 0$, and let $u \in \operatorname{im}(T^*)$, so $u = T^*w$ for some $w \in W$. Then $\langle v, u\rangle = \langle v, T^*w\rangle = \langle Tv, w\rangle = \langle 0, w\rangle = 0$, so v and u are orthogonal.

▷ **Exercise** 4. Show that a linear operator T on V is in the orthogonal group $\boldsymbol{O}(V)$ if and only if $TT^* = I$ (where I denotes the identity map of V) or equivalently, if and only if $T^* = T^{-1}$.

Answer If $T \in \boldsymbol{O}(V)$, then T preserves the inner product. Then for any $v, w \in V$, $\langle v, w\rangle = \langle Tv, Tw\rangle = \langle v, T^*(Tw)\rangle$. Therefore $T^*T = I$, or $T^* = T^{-1}$. Conversely, if $T^* = T^{-1}$, then for any $v, w \in V$, $\langle v, w\rangle = \langle v, T^*(Tw)\rangle = \langle Tv, Tw\rangle$, so T preserves the inner product, and therefore is in $\boldsymbol{O}(V)$.

If $T : V \to V$ is a linear operator on V, then T^* is also a linear operator on V, so it makes sense to compare them and in particular ask if they are equal.

Definition. A linear operator on an inner-product space V is called *self-adjoint* if $T^* = T$, i.e., if $\langle Tv_1, v_2\rangle = \langle v_1, Tv_2\rangle$ for all $v_1, v_2 \in V$.

Note that by Exercise 3 above, self-adjoint operators are characterized by the fact that their matrices with respect to an orthonormal basis are symmetric.

▷ **Exercise** 5. Show that if W is a linear subspace of the inner-product space V, then the orthogonal projection P of V on W is a self-adjoint operator on V.

Answer Let $u, v \in V$, and decompose each into a vector in W and a vector perpendicular to W: $u = u_W + u_{W^\perp}$, $v = v_W + v_{W^\perp}$. Then $\langle Pu, v\rangle = \langle u_W, v\rangle = \langle u_W, v_W + v_{W^\perp}\rangle = \langle u_W, v_W\rangle$, and $\langle u, Pv\rangle = \langle u, v_W\rangle = \langle u_W + u_{W^\perp}, v_W\rangle = \langle u_W, v_W\rangle$. Therefore $\langle Pu, v\rangle = \langle u, Pv\rangle$, so P is self-adjoined.

Definition. If T is a linear operator on V, then a linear subspace $U \subseteq V$ is called a T-invariant subspace if $T(U) \subseteq U$, i.e., if $u \in U$ implies $Tu \in U$.

Remark. Note that if U is a T-invariant subspace of V, then T can be regarded as a linear operator on U by restriction, and clearly if T is self-adjoint, so is its restriction.

▷ **Exercise** 6. Show that if $T : V \to V$ is a self-adjoint operator, and $U \subseteq V$ is a T-invariant subspace of V, the $U^\perp$ is also a T-invariant subspace of V.

Answer Let $u \in U$. Since U is a T-invariant subspace, $Tu \in U$. Then for any $v \in U^\perp$, $0 = \langle Tu, v\rangle = \langle u, Tv\rangle$, so $Tv \in U^\perp$. Thus $U^\perp$ is a T-invariant subspace of V.

We next recall for convenience the definitions relating to eigenvalues and eigenvectors. We assume that T is a linear operator on V.

If λ is a real number, then we define the linear subspace $E_\lambda(T)$ of V to be the set of $v \in V$ such that $Tv = \lambda v$. In other words, if I denotes the identity map of V, then $E_\lambda(T) = \ker(T - \lambda I)$. Of course the zero vector is always in $E_\lambda(T)$. If $E_\lambda(T)$ contains a non-zero vector, then we say that λ is an *eigenvalue* of T and that $E_\lambda(T)$ is the λ-eigenspace of T. A non-zero vector in $E_\lambda(T)$ is called an *eigenvector* of T belonging to the eigenvector λ. The set of all eigenvalues of T is called the *spectrum* of T (the name comes from quantum mechanics) and it is denoted by $\mathrm{Spec}(T)$.

▷ **Exercise** 7. Show that a linear operator T on V has a diagonal matrix in a particular basis for V if and only if each element of the basis is an eienvector of T, and that then $\mathrm{Spec}(T)$ consists of the diagonal elements of the matrix.

Answer Suppose that T has a diagonal matrix in a particular basis $v_1, v_2, \ldots$ of V, so the diagonal entries are $T_{ii} = \lambda_i$, and the rest are 0. Then $Tv_i = \lambda_i v_i$, so the v_i are eigenvectors of T, and the λ_i are in the spectrum $\mathrm{Spec}(T)$. Moreover, the λ_i describe all the elements of $\mathrm{Spec}(T)$, since, for $w = \sum_i a_i v_i$, $Tw = \sum_i a_i \lambda_i v_i = \lambda w$ if and only if all the λ_i are equal to λ.

Conversely, suppose that the spectrum of T is $\mathrm{Spec}(T)$, and that its eigenvectors $v_1, v_2, \ldots$ form a basis of V. Then we have that $Tv_i = \lambda_i v_i$ for some number λ_i, which shows that $T_{ij} = \lambda_i \delta_i^j$, so T has a diagonal matrix.

▷ **Exercise** 8. If T is a self-adjoint linear operator on an inner-product space V and λ_1, λ_2 are distinct real numbers, show that $E_{\lambda_1}(T)$ and $E_{\lambda_2}(T)$ are orthogonal subspaces of V. In other words, eigenvectors of T that belong to different eigenvalues are orthogonal. (Hint: Let $v_i \in E_{\lambda_i}(T), i = 1, 2$. You must show that $\langle v_1, v_2 \rangle = 0$. Start with the fact that $\langle Tv_1, v_2 \rangle = \langle v_1, Tv_2 \rangle$.)

Answer Let $v_1 \in E_{\lambda_1}(T)$ and $v_2 \in E_{\lambda_2}(T)$. Then $\langle Tv_1, v_2 \rangle = \langle \lambda_1 v_1, v_2 \rangle = \lambda_1 \langle v_1, v_2 \rangle$ and $\langle v_1, Tv_2 \rangle = \langle v_1, \lambda_2 v_2 \rangle = \lambda_2 \langle v_1, v_2 \rangle$. Since T is self-adjoint, $\langle Tv_1, v_2 \rangle = \langle v_1, Tv_2 \rangle$, so $\lambda_1 \langle v_1, v_2 \rangle = \lambda_2 \langle v_1, v_2 \rangle$, Since λ_1 is not equal to λ_2, we have that $\langle v_1, v_2 \rangle = 0$, so the eigenvectors of T that belong to different eigenvalues are orthogonal.