Fourth Assignment Due Tuesday, Oct. 6, 2003

▷ **Exercise** 1. Let X be a metric space. Recall that a subset of X is called *open* if whenever it contains a point x it also contains all points "sufficiently close" to x, and that a subset F of X is called closed if whenever a sequence in F converges to a point p of X it follows that $p \in F$. Show that a set is open if and only if its complement is closed.

▷ **Exercise** 2. Let X and Y be metric spaces, $f : X \to Y$ continuous, and A an open (resp. closed) subset of Y. Show that $f^{-1}(A)$ is open (resp. closed) in X. Deduce from this that the unit sphere, $S(V)$, in an inner-product space V is a closed and hence compact subset of V.

▷ **Exercise** 3. Recall that a subset K of a metric space X is called *compact* if every sequence in K has a subsequence that converges to some point of K (the Bolzano-Weierstrass property), Show that if $f : X \to \mathbf{R}$ is continuous real-valued function on X then f must be bounded on any compact subset K of X and in fact there is a point p of K where f assumes its maximum value. (Hint # 1: If it were not bounded above on K, then there would be a sequence $k_n \in K$ such that $f(k_n) > n$. Hint #2: Choose a sequence $k_n \in K$ so that $f(k_n)$ converges to the least upper bound of the values of f on K.)

▷ **Exercise** 4. Prove the so-called Chain Rule.

Chain Rule. *Let U, V, W be inner-product spaces, Ω an open set of U and O and open set of V. Suppose that $G : \Omega \to V$ is differentiable at $\omega \in \Omega$ and that $F : O \to W$ is differentiable at $p = G(\omega) \in O$. Then $F \circ G$ is differentiable at ω and $D(F \circ G)_\omega = DF_p \circ DG_\omega$.*

▷ **Exercise** 5. Let $f : \mathbf{R}^n \to \mathbf{R}^m$ be a function from $\mathbf{R}^n$ to $\mathbf{R}^m$. We use the usual conventions, so that if $x = (x_1, \ldots, x_n) \in \mathbf{R}^n$ then its image point $f(x)$ has the m components $(f_1(x_1, \ldots, x_n), \ldots, f_m(x_1, \ldots, x_n))$. Show that if f is differentiable at a point p of $\mathbf{R}^n$ then the matrix of the differential, Df_p, with respect to the standard bases of $\mathbf{R}^n$ and $\mathbf{R}^m$, is just the so-called "Jacobian matrix" at p, namely the matrix of partial derivatives $\frac{\partial f_i}{\partial x_j}(p)$.

▷ **Exercise** 6. Let $\sigma : (a, b) \to V$ and $\gamma : (a, b) \to V$ be differentiable curves in an inner-product space V. Show that $\frac{d}{dt} \langle \sigma(t), \gamma(t) \rangle = \langle \sigma'(t), \gamma(t) \rangle + \langle \sigma(t), \gamma'(t) \rangle$, and in particular $\frac{d}{dt} \|\sigma(t)\|^2 = 2 \langle \sigma(t), \sigma'(t) \rangle$. Deduce that if $\sigma : (a, b) \to V$ has its image in $S(V)$, i.e., if $\|\sigma(t)\|$ is identically one, then $\sigma(t)$ and $\sigma'(t)$ are orthogonal for ll t.