Fifth Assignment Due Friday, Nov. 21, 2003

▷ **Exercise** 1. Show that if $\alpha(t) = (x(t), y(t))$ is a plane parameterized curve (not necessarily parameterized by arclength) then its curvature at $\alpha(t)$ is given by the formula:

$$\frac{x'(t)y''(t) - y'(t)x''(t)}{\left(x'(t)^2 + y'(t)^2\right)^{\frac{3}{2}}}.$$

Consider the semicircle of radius r as the graph of $y = \sqrt{r^2 - x^2}$, i.e., parameterized by $x(t) = t$, $y(t) = \sqrt{r^2 - t^2}$ with $-r < t < r$. Use the above formula to show that its curvature is $\frac{1}{r}$.

▷ **Exercise** 2. Show that if the torsion function τ of a space curve is identically zero then the curve lies in a plane.

▷ **Exercise** 3. Compute the curvature and torsion of the Helix:

$$\alpha(t) := (r\cos(t), r\sin(t), bt)$$

▷ **Exercise** 4. Show that the "triple vector product" $(u \times v) \times w$ is given by the formula

$$(u \times v) \times w = (u \cdot w)v - (v \cdot w)u.$$

Definition Let V be a real vector space. A real-valued function $B : V \times V \to \mathbf{R}$ is called a *bilinear form* on V if it is linear in each variable separately (i.e., when the other variable is held fixed). The bilinear form B is called *symmetric* (respectively *skew-symmetric*) if $B(v_1, v_2) = B(v_2, v_1)$ (respectively $B(v_1, v_2) = -B(v_2, v_1)$) for all $v_1, v_2 \in V$.

▷ **Exercise** 5. Show that every bilinear form on a vector space can be decomposed uniquely into the sum of a symmetric and a skew-symmetric bilinear form.

Definition A real-valued function Q on a vector space V is called a *quadratic form* if it can be written in the form $Q(v) = B(v, v)$ for some symmetric bilinear form B on V. (We say that Q is *determined by* B.)

▷ **Exercise** 6. (Polarization Again.) Show that if Q is a quadratic form on V then the bilinear form B on V such that $Q(v) = B(v, v)$ is uniquely determined by the identity $B(v_1, v_2) = \frac{1}{2}(Q(v_1 + v_2) - Q(v_1) - Q(v_2))$.

Remark. Suppose that V is an inner-product space. Then the inner product is of course a bilinear form on V and the quadratic form it determines is just $Q(v) = \|v\|^2$. More generally, if $A : V \to V$ is any linear operator on V, then $B^A(v_1, v_2) = \langle Av_1, v_2 \rangle$ is a bilinear form on V and B^A is symmetric (respectively, skew-symmetric) if and only if A is self-adjoint (respectively, skew-adjoint).

▷ **Exercise** 7. Show that any bilinear form on a finite dimensional inner-product space is of the form B^A for a unique choice of linear operator A on V. (Hint. Recall the isomorphism of V with its dual space V^* given by the inner-product.)