Fifth Assignment Due Friday, Nov. 21, 2003

▷ **Exercise** 1. Show that if $\alpha(t) = (x(t), y(t))$ is a plane parameterized curve (not necessarily parameterized by arclength) then its curvature at $\alpha(t)$ is given by the formula:

$$\frac{x'(t)y''(t) - y'(t)x''(t)}{\left(x'(t)^2 + y'(t)^2\right)^{\frac{3}{2}}}.$$

Consider the semicircle of radius r as the graph of $y = \sqrt{r^2 - x^2}$, i.e., parameterized by $x(t) = t$, $y(t) = \sqrt{r^2 - t^2}$ with $-r < t < r$. Use the above formula to show that its curvature is $\frac{1}{r}$.

Answer The angle that the tangent makes with the x-axis is $\theta(t) = \tan^{-1}\left(\frac{y'(t)}{x'(t)}\right)$, and by definition, the curvature is $\frac{d}{ds}\theta(t) = \left(\frac{dt}{ds}\right)\theta'(t)$. Recall also that $\frac{ds}{dt} = \left(x'(t)^2 + y'(t)^2\right)^{\frac{1}{2}}$ so $\frac{dt}{ds} = \left(x'(t)^2 + y'(t)^2\right)^{-\frac{1}{2}}$.

▷ **Exercise** 2. Show that if the torsion function τ of a space curve is identically zero then the curve lies in a plane.

Answer If the torsion is identically zero, then by the Frenet formulas derivative of the binormal is zero, so the binormal is a constant vector, b_0. Since the tangent $\alpha'(s)$ is orthogonal to b_0, $(\alpha \cdot b_0)' = 0$ so $(\alpha(s) - \alpha(0)) \cdot b_0) = 0$. But $(x - x_0) \cdot b_0$ is the equation of the plane that pasees through x_0 and has normal b_0.

▷ **Exercise** 3. Compute the curvature and torsion of the Helix:

$$\alpha(t) := (r\cos(t), r\sin(t), bt)$$

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Answer Clearly $s = \sqrt{r^2 + b^2}\,t$. After reparametrizing by arclength, an easy calculation gives $k = \frac{r}{r^2+b^2}$ and $\tau = \frac{b}{r^2+b^2}$.

▷ **Exercise** 4. Show that the "triple vector product" $(u \times v) \times w$ is given by the formula

$$(u \times v) \times w = (u \cdot w)v - (v \cdot w)u.$$

Answer Since both sides are linear in all three arguments, it suffices to check this when each of the arguments is part of the standard basis for $\mathbf{R}^3$, and this is straightforward. (The result can also be just checked directly using the formulas for cross and dot products.)

Definition Let V be a real vector space. A real-valued function $B : V \times V \to \mathbf{R}$ is called a *bilinear form* on V if it is linear in each variable separately (i.e., when the other variable is held fixed). The bilinear form B is called *symmetric* (respectively *skew-symmetric*) if $B(v_1, v_2) = B(v_2, v_1)$ (respectively $B(v_1, v_2) = -B(v_2, v_1)$) for all $v_1, v_2 \in V$.

▷ **Exercise** 5. Show that every bilinear form on a vector space can be decomposed uniquely into the sum of a symmetric and a skew-symmetric bilinear form.

Answer Suppose B is any bilinear form on V. Define bilinear forms B^s and B^a by $B^s(u, v) = \frac{1}{2}(B(u, v) + B(v, u))$ and $B^a(u, v) = \frac{1}{2}(B(u, v) - B(v, u))$. It is clear that B^s is symmetric, that B^a is skew-symmetric, and that B is there sum. If $B(u, v) = S(u, v) + A(u, v)$ where S is symmetric and A is skew-symmetric, then $B(v, u) = S(u, v) - A(u, v)$. Then adding and subtracting the last two equations, it follows that $S = B^s$ and $A = B^a$.

Definition A real-valued function Q on a vector space V is called a *quadratic form* if it can be written in the form $Q(v) = B(v, v)$ for some symmetric bilinear form B on V. (We say that Q is *determined by* B.)

▷ **Exercise** 6. (Polarization Again.) Show that if Q is a quadratic form on V then the bilinear form B on V such that $Q(v) = B(v, v)$ is uniquely determined by the identity $B(v_1, v_2) = \frac{1}{2}(Q(v_1 + v_2) - Q(v_1) - Q(v_2))$.

Answer The argument is just the same as for the square of the norm in an inner-product space: $Q(v_1 + v_2) = B(v_1 + v_2, v_1 + v_2) = B(v_1, v_1) + 2B(v_1, v_2) + B(v_{,2} v_2) = Q(v_1) + 2B(v_1, v_2) + Q(v_2)$.

Remark. Suppose that V is an inner-product space. Then the inner product is of course a bilinear form on V and the quadratic form it determines is just $Q(v) = \|v\|^2$. More generally, if $A : V \to V$ is any linear operator on V, then $B^A(v_1, v_2) = \langle Av_1, v_2 \rangle$ is a bilinear form on V and B^A is symmetric (respectively, skew-symmetric) if and only if A is self-adjoint (respectively, skew-adjoint).

▷ **Exercise** 7. Show that any bilinear form on a finite dimensional inner-product space is of the form B^A for a unique choice of linear operator A on V. (Hint. Recall the isomorphism of V with its dual space V^* given by the inner-product.)

Answer Suppose that B is any bilinear form on V. Given $v \in V$, define an element $v^B \in V^*$ by $v^B(u) = B(u, v)$, By the self-duality property of finite dimensional inner-product space, there is a unique element Av in V such that $\langle Av, u \rangle = v^B(u) = B(u, v)$ for all $u \in V$. SInce $B(u, v)$ is linear in v, it is evident that the map $A : V \to V$ must also be linear. (Here is another proof. The map $A \mapsto B^A$ from $L(V)$ to the space $F(V)$ of bilinear forms on V is clearly linear, so since $L(V)$ and $F(V)$ both have dimension $\dim(V)^2$, it suffices to prove that the kernel of this map is zero. But if $B^A = 0$, then for any $v \in V$, $\|Av\|^2 = \langle Av, Av \rangle = B^A(Av) = 0$, so $A = 0$.)