Final Exam Due Friday, December 5, 2003

▷ **Problem 1.** Recall that if $f : \mathcal{O} \to \mathbf{R}$ is a real-valued function we get a parametric surface $\mathcal{F} : \mathcal{O} \to \mathbf{R}^3$, called the graph of f, by $\mathcal{F}(t_1, t_2) := (t_1, t_2, f(t_1, t_2))$.

Show that the First Fundamental Form coefficients are:

$$g_{11} = 1 + f_{t_1}^2, \quad g_{12} = f_{t_1} f_{t_2}, \quad g_{22} = 1 + f_{t_2}^2,$$

and that the Second Fundamental Form coefficients are:

$$\ell_{11} = \frac{f_{t_1 t_1}}{\sqrt{1 + f_{t_1}^2 + f_{t_2}^2}}, \quad \ell_{12} = \frac{f_{t_1 t_2}}{\sqrt{1 + f_{t_1}^2 + f_{t_2}^2}}, \quad \ell_{22} = \frac{f_{t_2 t_2}}{\sqrt{1 + f_{t_1}^2 + f_{t_2}^2}}.$$

▷ **Problem 2.** Let $t \mapsto \alpha(t) = (x(s), z(s))$ be a curve parametrized by arclength lying in the x, z-plane and not meeting the z-axis—i.e., $x(s) > 0$ for all s in the domain (a, b) of α, and let $\mathcal{O} = (0, 2\pi) \times (a, b)$. The surface of revolution defined by α is the parametrized surface $\mathcal{F} : \mathcal{O} \to \mathbf{R}^3$, defined by $\mathcal{F}(t_1, t_2) := (x(t_2)\cos(t_1), x(t_2)\sin(t_1), z(t_2))$. Show that the First Fundamental Form coefficients are:

$$g_{11} = x(t_2)^2, \quad g_{12} = 0, \quad g_{22} = 1,$$

and that the Second Fundamental Form coefficients are:

$$\ell_{11} = -x(t_2)z'(t_2), \quad \ell_{12} = 0, \quad \ell_{22} = x''(t_2)z'(t_2) - x'(t_2)z''(t_2).$$

Show also that the principle curvatures are

$$x''(t_2)z'(t_2) - x'(t_2)z''(t_2) \ \text{ and } \quad -\frac{z'(t_2)}{x(t_2)},$$

(so the Gaussian curvature is $K = -\frac{x''(t_2)z'(t_2)^2 - x'(t_2)z'(t_2)z''(t_2)}{x(t_2)} = -\frac{x''(t_2)}{x(t_2)}$).

▷ **Problem 3.** Let $q : \mathbf{R}^2 \to \mathbf{R}$ be a C^2 function and define symmetric 2×2 matrix -valued functions g and ℓ on $\mathbf{R}^2$ by:

$$g_{11} = \cos^2(q), \quad g_{12} = 0, \quad g_{22} = \sin^2(q),$$

and:

$$\ell_{11} = -\ell_{22} = \sin(q)\cos(q), \quad \ell_{12} = 0.$$

a) Derive explicit expressions in terms of q for the 3×3 matrices P^1, P^2, and also for their commutator $[P^1, P^2] := P^1 P^2 - P^2 P^1$.

b) Prove that the Gauss-Codazzi equation

$$(P^1)_{t_2} - (P^2)_{t_1} = [P^1, P^2]$$

is satisfied if and only if q satisfies the so-called sine-Gordon equation (SGE)

$$q_{t_1 t_1} - q_{t_2 t_2} = \sin q \cos q.$$

(Hint: Check the the Gauss-Codazzi equation entry by entry. This gives 9 equations, most of which are automatically satisfied just because g_{12} and ℓ_{12} vanish, and the rest reduce to SGE.

c) Now we know that if q satisfies the SGE, then, by the Fundamental Theorem of Surfaces, there exists a surface in $\mathbf{R}^3$, unique up to rigid motion with First and Second Fundamental Forms respectively:

$$\cos^2 q \ dt_1^2 + \sin^2 q \ dt_2^2, \quad \text{and} \sin q \cos q \ (dt_1^2 - dt_2^2).$$

Use the formula $K = \frac{\det(\ell_{ij})}{\det(g_{ij})}$ to prove that the Gaussian curvature of this surface has the constant value -1.

Note that if we find solutions q of the SGE, then g_{ij}, ℓ_{ij} given in Problem 3 satisfy the Gauss-Codazzi equation, so your program can draw the corresponding $K = -1$ surfaces. The next two Exercises give two family of examples of solutions of the SGE.

▷ **Problem 4.** Let a be a non-zero constant, and

$$q(t_1, t_2) = 2 \ \text{arctan} \ C(t_1, t_2),$$

where

$$C(t_1, t_2) = \exp\left(\frac{1}{2}(a + a^{-1})t_1 + \frac{1}{2}(a - a^{-1})t_2\right).$$

(i) Prove that $\cos q = \frac{1-C^2}{1+C^2}$ and $\sin q = \frac{2C}{1+C^2}$. (Hint: by definition of q, $\tan \frac{q}{2} = C$).

(ii) Prove that q satisfies the SGE.

(iii) Use your program to prove that for the case when $a = 0.2, 0.4, 0.6, 0.8$ and 1

$$g_{11} = \cos^2 q = \left(\frac{1 - C^2}{1 + C^2}\right)^2, \quad g_{22} = \sin^2 q = \left(\frac{2C}{1 + C^2}\right)^2, \quad g_{12} = 0,$$

and

$$\ell_{11} = -\ell_{22} = \sin q \cos q = \frac{2C(1 - C^2)}{(1 + C^2)^2}, \quad \ell_{12} = 0$$

satisfy the Gauss-Codazzi equation, and draw the corresponding surfaces. (Note that all these surfaces are of curvature -1, and when $a = 1$ you should get the pseudosphere and for $a \neq 1$ you get a family of Dini surfaces),

▷ **Problem 5.** Let $0 < \alpha < 1$, and let

$$B(t_1, t_2) = \frac{\sqrt{1-\alpha^2}\sin(\alpha\ t_2)}{\alpha\cosh(\sqrt{1-\alpha^2}\ t_1)},$$

$$q(t_1, t_2) = 2\arctan B(t_1, t_2),$$

and

$$g_{11} = \cos^2 q = \left(\frac{1-B^2}{1+B^2}\right)^2, \quad g_{22} = \sin^2 q = \left(\frac{2B}{1+B^2}\right)^2, \quad g_{12} = 0,$$

$$\ell_{11} = -\ell_{22} = \sin q\cos q = \frac{2B(1-B^2)}{(1+B^2)^2}, \quad \ell_{12} = 0.$$

By direct calculation you can check that,

$$q_{t_1t_1} - q_{t_2t_2} = \frac{2B(1-B^2)}{(1+B^2)^2} = \sin q\cos q,$$

i.e., q satisfies the SGE. So g_{ij} and ℓ_{ij} should give a surface with $K = -1$ for each constant $0 < \alpha < 1$, and in this exercise, we ask you to use your program to check the Gauss-Codazzi equation numerically and then draw the corresponding surfaces: Use your program to check that the above g_{ij}, ℓ_{ij} satisfy the Gauss-Codazzi equation when the constant $\alpha = 0.4, 0.6$ and 0.8, and draw the corresponding surface. (Note that $B(t_1, t_2)$ is periodic in t_2, so to get beautiful picture you should draw your surface with domain $t_2 \in [0, \frac{2\pi}{\alpha}]$ and $t_1 \in [-2, 2]$. Also you can do some experiments to see what happen if you change α to other rational numbers between 0 and 1).