▷Project 2. Implement the Trapezoidal Rule and Simpson's Rule in Matlab

1.1 Review of Trapezoidal and Simpson's Rules.

One usually cannot find anti-derivatives in closed form, so it is important to be able to "evaluate an integral numerically"—meaning approximate it with arbitrary precision. In fact, this is so important that there are whole books devoted the study of numerical integration methods (aka quadrature rules). We will consider only two such methods, the Trapezoidal Rule and Simpson's Rule. In what follows, we will assume that the integrand f is always at least continuous.

1.1.1 Definition. By a *quadrature rule* we mean a function M that assigns to each continuous function $f : [a, b] \to V$ (mapping a closed interval $[a, b]$ into an inner-product space V) a vector $M(f, a, b) \in V$—which is supposed to be an approximation of the integral, $\int_a^b f(t)\,dt$. A particular quadrature rule M is usually given by specifying a linear combination of the values of f at certain points of he interval $[a, b]$; that is, it has the general form $M(f, a, b) := \sum_{i=1}^n w_i f(t_i)$, where the points $t_i \in [a, b]$ are called the *nodes* of M and the scalars w_i are called its *weights*. The *error* of M for a particular f and $[a, b]$ is defined as $\mathrm{Err}(M, f, a, b) := \left\| \int_a^b f(t)\,dt - M(f, a, b) \right\|$.

1.1—Example 1. The Trapezoidal Rule: $M^T(f, a, b) := \frac{b-a}{2}[f(a) + f(b)]$.
In this case, there are two nodes, namely the two endpoints of the interval, and they have equal weights, namely half the length of the interval. Later we shall see the origin of this rule (and explain its name).

1.1—Example 2. Simpson's Rule: $M^S(f, a, b) := \frac{b-a}{6}[f(a) + 4f(\frac{a+b}{2}) + f(b)]$.
So now the nodes are the two endpoints, as before, and in addition the midpoint of the interval. And the weights are $\frac{b-a}{6}$ for the two endpoints and $\frac{2(b-a)}{3}$ for the midpoint.

1.1.2 Remark. Notice that in both examples the weights add up to one. This is no accident; any "reasonable" quadrature rule should have a zero error for a constant function, and this easily implies that the weights must add to one.

1.1.3 Proposition. *If $f : [a, b] \to V$ has two continuous derivatives, and $\|f''(t)\| < C$ for all $t \in [a, b]$ then $\mathrm{Err}(M^T, f, a, b) \le C\frac{(b-a)^3}{12}$. Similarly, if $f : [a, b] \to V$ has four continuous derivatives, and $\|f''''(t)\| < C$ for all $t \in [a, b]$ then $\mathrm{Err}(M^S, f, a, b) \le C\frac{(b-a)^5}{90}$*

1.1.4 Remark. The proof of this proposition is not difficult—it depends only the Mean Value Theorem—but it can be found in any numerical analysis text and will not be repeated here.

1.1.5 Definition. If M is a quadrature rule then we define a sequence M_n of *derived* quadrature rules by $M_n(f, a, b) := \sum_{i=0}^{n-1} M(f, a + ih, a + (i + 1)h)$ where $h = \frac{b-a}{n}$. We say that the rule M is *convergent* for f on $[a, b]$ if the sequence $M_n(f, a, b)$ converges to $\int_a^b f(t)\,dt$.

In other words, to estimate the integral $\int_a^b f(t)\,dt$ using the n-th derived rule M_n, we simply divide the interval $[a,b]$ of integration into n equal sub-intervals, estimate the integral on each sub-interval using M, and then add these estimates to get the estimate of the integral on the whole interval.

1.1.6 Remark. We next note an interesting relation between the errors of M and of M_n. Namely, with the notation just used in the above definition, we see that by the additivity of the integral, $\int_a^b f(t)\,dt = \sum_{i=0}^{n-1} \int_{a+ih}^{a+(i+1)h} f(t)\,dt$, hence from the definition of M_n and the triangle inequality, we have $\mathrm{Err}(M_n, f, a, b) \le \sum_{i=0}^{n-1} \mathrm{Err}(M, f, a+ih, a+(i+1)h)$. We can now use this together with Proposition 7.4.3 to prove the following important result:

1.1.7 Theorem. *If $f : [a,b] \to V$ has two continuous derivatives, and $\|f''(t)\| < C$ for all $t \in [a,b]$ then $\mathrm{Err}(M_n^T, f, a, b) \le C\frac{(b-a)^2}{12n^2}$. Similarly, if $f : [a,b] \to V$ has four continuous derivatives, and $\|f''''(t)\| < C$ for all $t \in [a,b]$ then $\mathrm{Err}(M_n^S, f, a, b) \le C\frac{(b-a)^4}{90n^4}$*

1.1.8 Remark. This shows that both the Trapezoidal Rule and Simpson's Rule are convergent for any reasonably smooth function. But it also shows that Simpson's Rule is far superior to the Trapezoidal Rule. For just fifty per cent more "effort" (measured by the number of evaluations of f) one gets a far more accurate result.

The second Matlab project is to develop Matlab code to implement the Trapezoidal Rule and Simpson's Rule, and then to do some experimentation with your software, checking that the error estimates of theorem 7.4.7 are satisfied for some test cases where the function f has a known anti-derivative and so can be evaluated exactly. In more detail:

1) Write a Matlab function M-file defining a function TrapezoidalRule(f,a,b,n). This should return the value of $M_n^T(f, a, b)$. Here of course the parameters a and b represent real numbers and the parameter n a positive integer. But what about the parameter f, i.e., what should it be legal to substitute for f when the TrapezoidalRule(f,a,b,n) is called? Answer: f should represent a function of a real variable whose values are arrays (of some fixed size) of real numbers. The function that you are permitted to substitute for f should either be a built-in Matlab function (such as sin) or an inline function in the Matlab Workspace, or a function that is defined in some other M-File.
2) Write a second Matlab function M-file defining a function SimpsonsRule(f,a,b,n) that returns $M_n^S(f, a, b)$.
3) Recall that $\int_0^t \frac{dx}{1+x^2} = \arctan(t)$, so that in particular $\int_0^1 \frac{4\,dx}{1+x^2} = 4\arctan(1) = \pi$. Using the error estimates for the Trapezoidal Rule and Simpson's Rule, calculate how large n should be to calculate π correct to d decimal places from this formula using Trapezoidal and Simpson. Set format long in Matlab and get the value of π to fifteen decimal places by simply typing pi. Then use your Trapezoidal and Simpson functions from parts 1) and 2) to see how large you actually have to choose n to calculate π to 5, 10, and 15 decimal places.
4) Be prepared to discuss your solutions in the Computer Lab.