▷Project 3. Implement the Method of Successive Approximations

1.1 Review of The Contraction Principle.

If X is any set and $f : X \to X$ a mapping of X to itself, then for each positive integer n we define a mapping $f^{(n)} X \to X$ by composing f with itself n times. That is, $f^{(1)}(x) = f(x)$, $f^{(2)}(x) = f(f(x))$, $f^{(3)}(x) = f(f(f(x)))$, etc. To be more formal, we define the sequence $f^{(n)}$ inductively by: $f^{(1)} := f$ and $f^{(n+1)} := f \circ f^{(n)}$.

Elsewhere you have verified the following facts:

1) $f^{(n)} \circ f^{(k)} = f^{(n+k)}$.
2) If X is a metric space and that f satisfies a Lipschitz condition with constant K then $f^{(n)}$ satisfies a Lipschitz condition with constant K^n.
3) Assuming again that X is a metric space and that $f : X \to X$ is a contraction mapping, i.e., that f satisfies a Lipschitz condition with constant $K < 1$, we have the so-called Fundamental Inequality For Contraction Mappings, namely, for all $x_1, x_2 \in X$,

$$\rho(x_1, x_2) \le \frac{1}{1-K}\Big(\rho(x_1, f(x_1)) + \rho(x_2, f(x_2))\Big).$$

4) With the same assumptions, if x is **any** point of X then

$$\rho(f^{(n)}(x), f^{(m)}(x)) \le \left(\frac{K^n + K^m}{1-K}\right)\rho(x, f(x)),$$

5) If $f : X \to X$ is a contraction mapping and p is the unique fixed point of f, then for any x in X, $\rho(f^{(n)}(x), p) \le \left(\frac{K^n}{1-K}\right)\rho(x, f(x))$

Remark. The sequence $\{f^{(n)}(x)\}$ is usually referred to as *the sequence of iterates of x under f*, and the process of locating the fixed point p of a contraction mapping f by taking the limit of a sequence of iterates of f goes by the name *"the method of successive approximations"*. To make this into a rigorous algorithm, we must have a "stopping rule". That is, since we cannot keep iterating f forever, we must know when to stop. One rather rough approach is to keep on iterating until successive iterates are "close enough", but a better method is provided by the previous problem. Suppose we decide to be satisfied with the approximation $f^{(n)}(x)$ if we can be sure that $\rho(f^{(n)}(x), p) \le \epsilon$ where ϵ is some "tolerance" given in advance. We first compute $f(x)$, then $\rho(f(x), x)$, and then solve $\left(\frac{K^n}{1-K}\right)\rho(x, f(x)) = \epsilon$ for n and iterate $n-1$ more times to get our acceptable approximation $f^{(n)}(x)$ to p.

6) Solve $\left(\frac{K^n}{1-K}\right)\rho(x, f(x)) = \epsilon$ for n in terms of ϵ, K, and $\rho(x, f(x))$.

Third Matlab Project.

Write an Matlab M-file that implements the Successive Approximations algorithm. Name it SuccessiveApprox.m, and use it to define a Matlab function SuccessiveApprox(f, K, x, eps). Assume that $f : \mathbf{R}^n \to \mathbf{R}^n$ is known to be a contraction mapping with contraction constant K, that $x \in \mathbf{R}^n$, and you want to compute iterates of x until you are within eps of the fixed point p of f. Use a subfunction to compute the number of times n you need to iterate f starting from x to get within eps of p, and then use a loop and feval to iterate applying f to x the appropriate number of times.