Project 8 Memo

For problem 4

Choose the domain $[a_0, a_1] \times [b_0, b_1]$ for different a values as follows:

(1) $a = 1$, choose $[a_0, a_1] \times [b_0, b_1] = [-4, -0.1] \times [0, 6.3]$, i.e., $t_1 \in [-4, -0.1]$ and $t_2 \in [0, 6.3]$,
(2) $a = 0.8$, choose $[0.1, 4] \times [0.46, 20]$,
(3) $a = 0.6$, choose $[-3, 0.1] \times [0.21, 12]$,
(4) $a = 0.4$, choose $[-3, 0.1] \times [0.13, 12]$,
(5) $a = 0.2$, choose $[-4, 0.1] \times [0.09, 8]$.

If the domain is $[a_0, b_0] \times [a_1, b_1]$, then we choose the inital data as follows: the initial point, and initial frame v_1, v_2, v_3:

$$\begin{cases} X(a_0, b_0) &= (0, 0, 0), \\ v_1(a_0, b_0) &= (\sqrt{g_{11}(a_0, b_0)}, 0, 0), \\ v_2(a_0, b_0) &= (0, \sqrt{g_{22}(a_0, b_0)}, 0), \\ v_3(a_0, b_0) &= (0, 0, 1) \end{cases}$$

Look at the formula of the first fundamental form, we see that $g_{11}g_{22} = 0$ exactly when the function

$$C(t_1, t_2) = 1,$$

i.e., when

$$(a + a^{-1})t_1 + (a - a^{-1})t_2 = 0,$$

or equivalently

$$\frac{t_2}{t_1} = \frac{a + a^{-1}}{a^{-1} - a}.$$

So if you choose any rectangular region domain that does not meet this line, then $g_{11}g_{22}$ will stay positive and your program should work.

For Problem 5

You need to choose your rectangular domain away from where $B(t_1, t_2) = \pm 1$, which can not be solved explicitly. So it will be difficult to get a good rectangular region for the domain. However, there is another way:

An alternative better method: use orthonormal frame

We have seen in the last lecture that for surfaces with $g_{12} = \ell_{12} = 0$, the Gauss-Codazzi equation has only three equations. Moreover, if you modify your program using the orthonormal frame, then even when $g_{11}g_{22} = 0$ somewhere, you will be able to solve the Gauss-Codazzi equation. To be more specific, write a program to solve the following system

$$\begin{cases} (e_1, e_2, e_3)_{x_1} = (e_1, e_2, e_3) \begin{pmatrix} 0 & -q_{x_2} & -\sin q \\ q_{x_2} & 0 & 0 \\ \sin q & 0 & 0 \end{pmatrix}, \\ (e_1, e_2, e_3)_{x_2} = (e_1, e_2, e_3) \begin{pmatrix} 0 & -q_{x_1} & 0 \\ q_{x_1} & 0 & \cos q \\ 0 & -\cos q & 0 \end{pmatrix}, \end{cases} \tag{0.0.1}$$

Then solve

$$\begin{cases} X_{x_1} = \cos q \ e_1, \\ X_{x_2} = \sin q \ e_2. \end{cases} \tag{0.0.2}$$

In this case, you can choose any domain and initial data and the program won't break down.

Note that if $q = 2\tan^{-1} B$, then

$$q_{x_i} = \frac{2B_{x_i}}{1+B^2}, \quad \cos q = \frac{1-B^2}{1+B^2}, \quad \sin q = \frac{2B}{1+B^2}.$$

If you have finished the final Matlab project 8 for general coordinates, then you can modify a little bit what you have, to write a program for the above simpler systems (0.0.1). (0.0.2), and the resulting X is your surface.

Use this new program to do problem 5, you may use inital data

$$X(a_0, b_0) = (0, 0, 0),$$

$$e_1(a_0, b_0) = (1, 0, 0), \quad e_2(a_0, b_0) = (0, 1, 0), \quad e_3(a_0, b_0) = (0, 0, 1),$$

and the domain $[a_0, a_1] \times [b_0, b_1]$ can be chosen to be

(1) $[-8, 8] \times [-10, 10]$ if $\alpha = 0.6$,
(2) $[-8, 8] \times [-16, 16]$ if $\alpha = 0.8$.