Definition 0.0.1. A surface $f : \mathcal{O} \to \mathbb{R}^3$ is parametrized by *lines of curvature coordinates* if

$$f_{x_1} \cdot f_{x_2} = 0, \quad f_{x_1,x_2} \cdot \nu = 0.$$

Or equivalently, f_{x_1}, f_{x_2} are orthogonal and are eigenvectors of the shape operator $-d\nu$, or the first and second fundamental forms in this coordinate system is

$$\mathrm{I} = g_{11} dx_1^2 + g_{22} dx_2^2, \quad \mathrm{II} = \ell_{11} dx_1^2 + \ell_{22} dx_2^2.$$

• The Gauss-Codazzi equations in line of curvature coordinates

Suppose $f(x_1, x_2)$ is a surface parametrized by lines of curvature coordinates. Write

$$g_{11} = A_1^2, \quad g_{22} = A_2^2.$$

Then

$$e_1 = \frac{f_{x_1}}{A_1}, \quad e_2 = \frac{f_{x_2}}{A_2}, \quad e_3 = \nu$$

form an orthonormal frame. Instead of substituting g_{ij}, ℓ_{ij} into the nine Gauss-Codazzi equation we derived in [], we will use this orthonormal frame to derive the equation again. The advantage of this is that we will automatically get 3 equations instead of 9 equations and the computation is much simpler too. Since Gauss-Codazzi equation comes the compatibility condition, we take partial derivatives of the frame (e_1, e_2, e_3) and write $(e_i)_{x_1}$ and $(e_i)_{x_2}$ as linear combinations of the basis $\{e_1, e_2, e_3\}$:

$$(e_i)_{x_1} = \sum_j P^1_{ji} e_j, \quad (e_i)_{x_2} = \sum_j P^2_{ji} e_j.$$

Write these equations in matrix form to get

$$\begin{cases} (e_1, e_2, e_3)_{x_1} = (e_1, e_2, e_3) P^1, \\ (e_1, e_2, e_3)_{x_2} = (e_1, e_2, e_3) P^2. \end{cases}$$

The compatibility condition is

$$P^1_{x_2} - P^2_{x_1} = [P^1, P^2]. \tag{0.0.1}$$ fb

Use the dot product of $\mathbb{R}^3$, it is easy to see that

$$P^1_{ji} = (e_i)_{x_1} \cdot e_j, \quad P^2_{ji} = (e_i)_{x_2} \cdot e_j.$$

Since $e_i \cdot e_j = 0$,

$$P^k_{ij} + P^k_{ji} = 0,$$

i.e., P^1, P^2 must be skew-symmetric. Next we claim that $[P^1, P^2]$ is also skew-symmetric. This is because

$$[P^1, P^2]^t = (P^1P^2 - P^2P^1)^t = (P^2)^t(P^2)^t - (P^1)^t(P^2)^t = (-P^2)(-P^1) - (-P^1)(-P^2) = -[P^1, P^2]$$

Note that we just proved that both the left and right hand sides of ?? are skew-symmetric. Thus to write down equation (??) explicitly, we need only compute the 12, 13, and 23 entries of P^1, P^2 and equate these entries of (??). We carry out these computation next:

$$\begin{aligned} P^1_{21} = (e_1)_{x_1} \cdot e_2 &= \left(\frac{f_{x_1}}{A_1}\right)_{x_1} \cdot \frac{f_{x_2}}{A_2} \\ &= \left(\frac{f_{x_1x_1}}{A_1} - \frac{f_{x_1}(A_1)_{x_1}}{A_1^2}\right) \cdot \frac{f_{x_2}}{A_2}. \end{aligned}$$

But $f_{x_1} \cdot f_{x_2} = 0$ and

$$\begin{aligned} f_{x_1x_1} \cdot f_{x_2} &= (f_{x_1} \cdot f_{x_2})_{x_1} - f_{x_1} \cdot f_{x_1x_2} \\ &= 0 - \frac{1}{2}(f_{x_1} \cdot f_{x_1})_{x_2} = -\frac{1}{2}(A_1^2)_{x_2} \\ &= -A_1(A_1)_{x_2}. \end{aligned}$$

Substitute this equation into the computation of P^1_{21} to get

$$P^1_{21} = -\frac{(A_1)_{x_2}}{A_2}.$$

Next we compute:

$$\begin{aligned} P^1_{31} = (e_1)_{x_1} \cdot e_3 &= -e_1 \cdot (e_3)_{x_1} \\ &= -\frac{f_{x_1}}{A_1} \cdot (\nu)_{x_1} = -\frac{\ell_{11}}{A_2}. \end{aligned}$$

The entries P^1_{23}, P^2_{ij} can be computed similarly and we get

$$\begin{cases} P^1 = \begin{pmatrix} 0 & \frac{(A_1)_{x_2}}{A_2} & -\frac{\ell_{11}}{A_1} \\ -\frac{(A_1)_{x_2}}{A_2} & 0 & 0 \\ \frac{\ell_{11}}{A_1} & 0 & 0 \end{pmatrix}, \qquad P^2 = \begin{pmatrix} 0 & -\frac{(A_2)_{x_1}}{A_1} & 0 \\ \frac{(A_2)_{x_1}}{A_1} & 0 & -\frac{\ell_{22}}{A_2} \\ 0 & \frac{\ell_{22}}{A_2} & 0 \end{pmatrix}. \end{cases} \tag{0.0.2}$$

cg

Substitute (??) into (??) and equate the $12, 13, 23$ entries to get the Gauss-Codazzi equation. We summarize this in the following theorem:

Theorem 0.0.2. *Suppose $f(x_1, x_2)$ is a local line of curvature coordinate system on an embedded surface M in $\mathbb{R}^3$ with* I, II *as in Theorem*

cm
??. *Then the Gauss-Codazzi equation is*

$$\left(\frac{(A_1)_{x_2}}{A_2}\right)_{x_2} + \left(\frac{(A_2)_{x_1}}{A_1}\right)_{x_1} = -\frac{\ell_{11}\ell_{22}}{A_1A_2}, \tag{0.0.3}$$
$\boxed{\texttt{by:1}}$

$$\left(\frac{\ell_{11}}{A_1}\right)_{x_2} = \frac{\ell_{22}(A_1)_{x_2}}{A_2^2} \tag{0.0.4}$$
$\boxed{\texttt{by:2}}$

$$\left(\frac{\ell_{22}}{A_2}\right)_{x_1} = \frac{\ell_{11}(A_2)_{x_1}}{A_1^2}. \tag{0.0.5}$$
$\boxed{\texttt{by:3}}$

Does line of curvature coordinate system always exist? To answer this, we need the following Theorem, which can be proved using the Frobenius Theorem:

$\boxed{\texttt{bx}}$ **Theorem 0.0.3.** *Suppose $f : \mathcal{O} \to \mathbb{R}^3$ be a surface, $x_0 \in \mathcal{O}$, and $Y_1, Y_2 : \mathcal{O} \to \mathbb{R}^3$ smooth maps so that $Y_1(x_0), Y_2(x_0)$ are linearly independent and tangent to $M = f(\mathcal{O})$ at $f(x_0)$. Then there exist open subset $\mathcal{O}_0$ of $\mathcal{O}$ containing x_0, open subset $\mathcal{O}_1$ of $\mathbb{R}^2$, and a diffeomorphism $h : \mathcal{O}_1 \to \mathcal{O}_0$ so that $(f \circ h)_{y_1}$ and $(f \circ h)_{y_2}$ are parallel to $Y_1 \circ h$ and $Y_2 \circ h$.*

The above theorem says that if we have two linearly independent vector fields Y_1, Y_2 on a surface, then we can find a local coordinate system $\phi(y_1, y_2)$ so that ϕ_{y_1}, ϕ_{y_2} are parallel to Y_1, Y_2 respectively.

It follows from the calculation of eigenvectors that if the shape operator of a surface $f : \mathcal{O} \to \mathbb{R}^3$ has distinct eigenvalues at every $p \in \mathcal{O}$ then the two eigenvectors as a $\mathbb{R}^3$ valued maps can be chosen to be smooth. Thus as a consequence of Theorem bx ??

$\boxed{\texttt{cm}}$ **Theorem 0.0.4.** *If $f : \mathcal{O} \to \mathbb{R}^3$ is a surface in $\mathbb{R}^3$ and $p_0 \in M$ is not umbilic, then there exist an open subset $\mathcal{O}_0$ of $\mathcal{O}$ containing p_0, an open subset $\mathcal{O}_1$ of $\mathbb{R}^2$, and a diffeomorphism $h : \mathcal{O}_1 \to \mathcal{O}_0$ such that $F = f \circ h : \mathcal{O}_1 \to \mathbb{R}^3$ is parametrized by line of curvature coordinates, i.e.,*

$$g_{12} = F_{x_1} \cdot F_{x_2} = 0, \quad \ell_{12} = F_{x_1x_2} \cdot \nu = 0,$$

or equivalently,

$$\mathrm{I} = A_1^2\ dx_1^2 + A_2^2\ dx_2^2, \quad \mathrm{II} = \ell_{11}\ dx_1^2 + \ell_{22}\ dx_2^2.$$

•**Surfaces in $\mathbb{R}^3$ with $K = -1$ and the sine-Gordon equation**

Suppose the Gaussian curvature K of a surface is identically equal to -1. Recall that we have proved

$$K = \lambda_1\lambda_2,$$

where λ_1, λ_2 are the eigenvalues of the shape operator $-d\nu$. This implies that λ_1, λ_2 must be distinct. If two real numbers are reciprocal of each other, then we may assume

$$\lambda_1 = \tan q, \quad \lambda_2 = -\cot q,$$

(i.e., $q = \arctan \lambda_1$). By Theorem cm ??, we may assume that this surface is parametrized by line of curvature coordinates $F : \mathcal{O} \to \mathbb{R}^3$, i.e.,

$$\mathrm{I} = A_1^2 dx_1^2 + A_2^2 dx_2^2, \quad \mathrm{II} = \ell_1 dx_1^2 + \ell_2 dx_2^2.$$

The Gauss-Codazzi equation is a system of 3 equations. What we want to do below is to use the condition that $K = -1$ to show that we can reparemetrize the surface with

$$\begin{cases} \tilde{x}_1 = c_1(x_1), \\ \tilde{x}_2 = c_2(x_2), \end{cases}$$

for some one variable functions c_1, c_2 so that the Gauss-Codazzi equation will be just one single equation in q. The method we are going to use is one of the standard method in differential geometry that we use the Codazzi equations and the geometric condition we put on the surface to get the best coordinate system on a surface so that the G-C equation becomes much simpler.

Since both I and II are diagonalized, the principal curvatures

$$\lambda_1 = \frac{\ell_{11}}{A_1^2}, \quad \lambda_2 = \frac{\ell_{22}}{A_2^2}.$$

So we have

$$\lambda_1 = \frac{\ell_{11}}{A_1^2} = \tan q, \qquad \lambda_2 = \frac{\ell_{22}}{A_2^2} = -\cot q,$$

i.e.,

$$\frac{\ell_{11}}{A_1} = A_1 \tan q, \quad \frac{\ell_{22}}{A_2} = -A_2 \cot q. \tag{0.0.6}$$ co

Since $A_1, A_2, \ell_{11}, \ell_{22}$ satisfy the Gauss-Codazzi equation, substitue (co ??) to the Codazzi equations (by:2 ??) and (by:3 ??) to get

$$\begin{aligned} (A_1 \tan q)_{x_2} &= -\cot q \ (A_1)_{x_2}, \\ (-A_2 \cot q)_{x_1} &= \tan q \ (A_2)_{x_1}. \end{aligned}$$

Compute directly to get

$$(A_1)_{x_2} \tan q + A_1 \sec^2 q \ q_{x_2} = -\cot q \ (A_1)_{x_2},$$

which implies that

$$(\tan q + \cot q)(A_1)_{x_2} = -A_1 (\sec^2 q) \ q_{x_2}.$$

So we obtain

$$\frac{(A_1)_{x_2}}{A_1} = -\frac{\sin q}{\cos q}\, q_{x_2}.$$

Use a similar computation to get

$$\frac{(A_2)_{x_1}}{A_2} = \frac{\cos q}{\sin q}\, q_{x_1}.$$

In other words, we have

$$(\log A_1)_{x_2} = (\log\cos q)_{x_2}, \quad (\log A_2)_{x_1} = (\log\sin q)_{x_1}.$$

Hence there exist $c_1(x_1)$ and $c_2(x_2)$ so that

$$\log A_1 = \log\cos q + c_1(x_1), \quad \log A_2 = \log\sin q + c_2(x_2),$$

i.e.,

$$A_1 = e^{c_1(x_1)}\cos q, \quad A_2 = e^{c_2(x_2)}\sin q.$$

Because I is positive definite, A_1, A_2 never vanishes. So we may assume both $\sin q$ and $\cos q$ are positive, i.e., $q \in (0, \frac{\pi}{2})$. Now change coordinates to $(\tilde{x}_1(x_1), \tilde{x}_2(x_2))$ so that

$$\frac{d\tilde{x}_1}{dx_1} = e^{c_1(x_1)}, \quad \frac{d\tilde{x}_2}{dx_2} = e^{c_2(x_2)}.$$

Since

$$\frac{\partial f}{\partial \tilde{x}_1} = \frac{\partial f}{\partial x_1}\frac{\partial x_1}{\partial \tilde{x}_1} = \frac{\partial f}{\partial x_1} e^{-c_1(x_1)},$$

$||f_{\tilde{x}_1}|| = \cos q$. Similar calculation implies that $||f_{\tilde{x}_2}|| = \sin q$. Since $\tilde{x}_i$ is a function of $\tilde{x}_i$ alone, $f_{\tilde{x}_i}$ is parallel to f_{x_i}. So $(\tilde{x}_1, \tilde{x}_2)$ is also a line of curvature coordinate system and the coefficients of II in $(\tilde{x}_1, \tilde{x}_2)$ coordinate system is

$$\tilde{\ell}_{11} = \tan q\cos^2 q = \sin q\cos q, \quad \tilde{\ell}_{22} = -\cot q\sin^2 q = -\sin q\cos q.$$

Therefore, we have proved part of the following Theorem:

cq **Theorem 0.0.5.** *Let M^2 be a surface in $\mathbb{R}^3$ with $K = -1$. Then locally there exists line of curvature coordinates x_1, x_2 so that*

$$\mathrm{I} = \cos^2 q\ dx_1^2 + \sin^2 q\ dx_2^2, \quad \mathrm{II} = \sin q\cos q\ (dx_1^2 - dx_2^2), \tag{0.0.7}$$ ch

where $2q$ is the angle between two asymptotic directions. Moreover, the Gauss-Codazzi equation is the sine-Gordon equation

$$q_{x_1x_1} - q_{x_2x_2} = \sin q\cos q. \tag{0.0.8}$$ bz

Proof. It remains to compute the Gauss-Codazzi equation. Let

$$e_1 = \frac{f_{x_1}}{\cos q}, \quad e_2 = \frac{f_{x_2}}{\sin q}, \quad e_3 = N = \frac{f_{x_1} \times f_{x_2}}{||f_{x_1} \times f_{x_2}||}.$$

Write

$$(e_1, e_2, e_3)_{x_1} = (e_1, e_2, e_3)P, \quad (e_1, e_2, e_3)_{x_2} = (e_1, e_2, e_3)Q.$$

We have given formulas for P, Q in (cg ??) when (x_1, x_2) is a line of curvature coordinate system. Since $A_1 = \cos q$, $A_2 = \sin q$, and $\ell_{11} = -\ell_{22} = \sin q \cos q$, we have

$$P = \begin{pmatrix} 0 & -q_{x_2} & -\sin q \\ q_{x_2} & 0 & 0 \\ \sin q & 0 & 0 \end{pmatrix}, \qquad Q = \begin{pmatrix} 0 & -q_{x_1} & 0 \\ q_{x_1} & 0 & \cos q \\ 0 & -\cos q & 0 \end{pmatrix}. \tag{0.0.9}$$ cp

A direct computation shows that Codazzi equations are satified automatically, i.e., the 13 and 23 entries of $P_{x_2} - Q_{x_1}$ are equal to the 13 and 23 entries of $[P, Q]$ respectively. The Gauss equation (equating the 12 entry) gives

$$-q_{x_2 x_2} + q_{x_1 x_1} = \sin q \cos q.$$

Since $\mathrm{II} = \sin q \cos q \, (dx_1^2 - dx_2^2)$, $f_{x_1} \pm f_{x_2}$ are asymptotic directions. Use I to see that $f_{x_1} \pm f_{x_2}$ are unit vectors. Since

$$(f_{x_1} + f_{x_2}) \cdot (f_{x_1} - f_{x_2}) = \cos^2 q - \sin^2 q = \cos(2q),$$

the angle between the asymptotic directions $f_{x_1} + f_{x_2}$ and $f_{x_1} - f_{x_2}$ is $2q$. $\square$

A consequence of the proof of the above theorem is

cr **Corollary 0.0.6.** *Let P, Q be as in (cp ??). Then system*

$$(e_1, e_2, e_3)_{x_1} = (e_1, e_2, e_3)P, \quad (e_1, e_2, e_3)_{x_2} = (e_1, e_2, e_3)Q$$

is solvable if and only if q satisfies the SGE (bz ??).

Let $O(3)$ denote the space of all orthogonal 3×3 matrices, and $o(3)$ the space of all skew-symmetric 3×3 matrices. A map $g : \mathbb{R}^2 \to O(3)$ is smooth if $i \circ g : \mathbb{R}^2 \to gl(3)$ is smooth, where $i : O(3) \to gl(3)$ is the inclusion.

It follows from the Fundamental Theorem of surfaces and Corollary cr ?? that The converse of Theorem cq ?? is true :

Theorem 0.0.7. *Let $q : \mathcal{O} \to \mathbb{R}$ be a solution of the SGE (bz ??), $p_0 \in \mathbb{R}^3$, $(x_1^0, x_2^0) \in \mathcal{O}$, and u_1, u_2, u_3 an o.n. basis. Let $P, Q : \mathcal{O} \to o(3)$ be the maps defined by (cp ??). Then there exists an open subset $\mathcal{O}_1$ of*

(x_1^0, x_2^0) in $\mathcal{O}$ and a unique solution $(f, e_1, e_2, e_3) : \mathcal{O}_1 \to \mathbb{R}^3 \times O(3)$ for the following system

$$\begin{cases} (e_1, e_2, e_3)_{x_1} = (e_1, e_2, e_3)P, \\ (e_1, e_2, e_3)_{x_2} = (e_1, e_2, e_3)Q, \\ f_{x_1} = \cos q \; e_1, \\ f_{x_2} = \sin q \; e_2, \\ f(x_1^0, x_2^0) = p_0, \quad e_1(x_1^0, x_2^0) = u_1, \quad e_2(x_1^0, x_2^0) = u_2. \end{cases} \tag{0.0.10}$$

Moreover, if $\sin q \cos q > 0$ on $\mathcal{O}_1$, then $f(\mathcal{O}_1)$ is an immersed surface with $K = -1$ and the two fundamental forms are of the form (??).

In other words, there is a 1-1 correspondence between solutions q of the SGE (??) with $\mathrm{Im}(q) \subset (0, \frac{\pi}{2})$ and local surfaces of $\mathbb{R}^3$ with $K = -1$ up to rigid motion.

Let $f(x_1, x_2)$ be a local line of curvature coordinate system given in Theorem ?? of an embedded surface M in $\mathbb{R}^3$ with $K = -1$, and q the corresponding solution of the SGE (??). The SGE is a non-linear wave equation, and (x_1, x_2) is the space-time coordinate. We have proved that $f_{x_1} \pm f_{x_2}$ are asymptotic directions. If we make a change of coordinates $x_1 = s + t$, $x_2 = s - t$, then $f_s = f_{x_1} + f_{x_2}$ and $f_t = f_{x_1} - f_{x_2}$. A direct computation shows that the two fundamental forms written in (s, t) coordinates are

$$\begin{cases} \mathrm{I} = ds^2 + 2\cos 2q \; ds dt + dt^2, \\ \mathrm{II} = \sin 2q \; ds \; dt, \end{cases} \tag{0.0.11}$$ ej

and the SGE (??) becomes

$$q_{st} = \sin q \cos q. \tag{0.0.12}$$ dh

A local coordinate system (x, y) on a surface $f(x, y)$ in $\mathbb{R}^3$ is called an *asymptotic coordinate system* if f_s, f_t are parallel to the asymptotic direction, i.e., $\ell_{11} = \ell_{22} = 0$. Note that the (s, t) coordinate constructed above for $K = -1$ surfaces in $\mathbb{R}^3$ is an asymptotic coordinate system and s, t are arc length parameter. This coordinate system is called the *Tchbyshef coordinate system for $K = -1$ surfaces in $\mathbb{R}^3$.*

• **Bäcklund transformation for the SGE**

ff **Theorem 0.0.8.** *Given a smooth function $q : \mathbb{R}^2 \to \mathbb{R}$ and a non-zero real constant r, the following system of first order PDE is solvable for $q^* : \mathbb{R}^2 \to \mathbb{R}$:*

$$\begin{cases} (q^*)_s = -q_s + \mu \sin(q^* - q), \\ (q^*)_t = q_t + \frac{1}{\mu} \sin(q^* + q). \end{cases} \tag{0.0.13}$$ dl

if and only if q satisfies the SGE (??). Moreover, if q is a solution of the SGE (??), then the solution q^* of (??) is again a solution of the SGE (??).

Proof. If (??) has a C^2 solution q^*, then the mixed derivatives must be equal. Compute directly to see that

$$\begin{cases} ((q^*)_s)_t & = -q_{st} + r\cos(q^* - q)(q^* - q)_t \\ & = -q_{st} + r\cos(q^* - q)\left(\frac{1}{r}\ \sin(q^* + q)\right), \end{cases}$$

so we get

$$((q^*)_s)_t = -q_{st} + \cos(q^* - q)\sin(q^* + q). \tag{0.0.14}$$ fd

A similar computation implies that

$$((q^*)_t)_s = -q_{ts} + \cos(q^* + q)\sin(q^* - q). \tag{0.0.15}$$ fe

Since both q and q^* are C^2, $q_{st} = q_{ts}$ and $q^*_{st} = q^*_{ts}$. Subtract (??) from (??) to get

$$2q_{st} = \sin(q^* + q)\cos(q^* - q) - \sin(q^* - q)\cos(q^* + q) = \sin(2q).$$

Add (??) and (??) to get

$$2q^*_{st} = \sin(q^* + q)\cos(q^* - q) + \sin(q^* - q)\cos(q^* + q) = \sin(2q*).$$

In the above computation we use the addition formulas

$$\sin(A \pm B) = \sin A\cos B \pm \cos A \sin B.$$

This completes proof of the theorem. □

This theorem says that if we know one solution of the SGE the we can solve the first order system (??), which is just solving two ODEs, to construct a family of solutions (one for each real constant r). Note that $q = 0$ is a trivial solution of the SGE. Theorem ?? implies that system (??) can be solved for q^* with $q = 0$:

$$\begin{cases} q^*_s = r\sin q^*, \\ q^*_t = \frac{1}{r}\sin q^*. \end{cases} \tag{0.0.16}$$ fg

To solve this, we note that the ODE $dy/dx = \sin y$ can be solved easily using integration:

$$\int \frac{dy}{\sin y} = \int dx,$$

which implies

$$y(x) = 2\arctan(e^x).$$

Hence we find solutions for (??):

$$q^*(s,t) = 2\arctan\left(e^{rs + \frac{1}{r}\ t}\right).$$