Lecture 11
Differentiable Parametric Curves

11.1 Definitions and Examples.

11.1.1 Definition. A *differentiable parametric curve* in $\mathbf{R}^n$ of class C^k $(k \geq 1)$ is a C^k map $t \mapsto \alpha(t) = (\alpha_1(t), \ldots, \alpha_n(t))$ of some interval I (called the domain of α) into $\mathbf{R}^n$. We call α *regular* if its velocity vector, $\alpha'(t)$, is non-zero for all t in I. The image of the mapping α is often referred to as the *trace* of the curve α.

Some Conventions To avoid endless repetition, in all that follows we will assume that α is a regular, differentiable parametric curve in $\mathbf{R}^n$ of class C^k $(k \geq 2)$, and we will abbreviate this by referring to α as a "parametric curve", or just as a "curve". Frequently we will take the domain I of α to be the closed interval $[a, b]$. In case $n = 2$ we will often write $\alpha(t) = (x(t), y(t))$ and similarly when $n = 3$ we will often write $\alpha(t) = (x(t), y(t), z(t))$.

11.1—Example 1. A Straight Line. Let $x^0, v^0 \in \mathbf{R}^n$ with $v_0 \neq 0$. Then $\alpha(t) = x^0 + tv^0$ is a straight line in $\mathbf{R}^n$ with constant velocity v^0. The domain is all of $\mathbf{R}$.

11.1—Example 2. Let $r > 0$, $n = 2$, take $I = [0, 2\pi]$ and define $\alpha(t) = (r\cos(t), r\sin(t))$. The trace of α is the set of (x, y) satisfying $x^2 + y^2 = r^2$, so α is a parameterization of the circle of radius r centered at the origin.

11.1—Example 3. Let $r, b > 0$, $n = 3$, $I = \mathbf{R}$, and define $\alpha(t) = (r\cos(t), r\sin(t), bt)$. The trace of α is a helix of radius r and pitch $\frac{2\pi}{b}$.

11.2 Reparametrizaion by Arclength.

Suppose $\phi : [c, d] \to \mathbf{R}$ is C^k and $\phi'(t) > 0$ for all $t \in [c, d]$. Then ϕ is monotonic and so a one-to-one map of $[c, d]$ onto $[a, b]$ where $a = \phi(c)$ and $b = \phi(d)$. Moreover, by the Inverse Function Theorem, $\psi = \phi^{-1}$ is C^k and $\psi'(\phi(t)) = 1/\phi'(t)$. If $\alpha : [a, b] \to \mathbf{R}^n$ is a parametric curve, then $\tilde{\alpha} := \alpha \circ \phi : [c, d] \to \mathbf{R}^n$ is a parametric curve with the same trace as α. In this setting, ϕ is called a parameter change and $\tilde{\alpha}$ is called a reparametrization of α. Since α and $\tilde{\alpha}$ have the same trace, in some naive sense at least, they represent the same "curve".

Of course for many purposes, the way a curve is parametric is of crucial importance—for example, reparametrizing a solution of an ODE will nearly always result in a non-solution. However in geometric considerations it is natural to concentrate on the trace and regard two parametric curves that differ by a change of parameterization as representing the same object. Formally speaking, differing by a change of parameterization is an equivalence relation on the set of parametric curves, and we regard the corresponding equivalence classes as being the primary objects of study in differential geometry. This raises a problem.

Whenever we define some property of parametric curves, then we should check that it is independent of the choice of parameterization. As we shall now see, there is an elegant way to avoid this complication. Namely, among all the parameterizations of a parametric curve α there is one that is the"most natural" choice, namely parameterization by arclength, and in our theoretical study of curves and their properties we will usually pefer this one over others and define properties of a curve using this parameterization.

Recall that in elementary Calculus, if $\alpha : [a,b] \to \mathbf{R}^n$ is a parametric curve, then its length L is defined as $L := \int_a^b \|\alpha'(t)\|\, dt$. More generally, the arclength along α is the function $s : [a,b] \to [0,L]$ defined by $s(t) = \int_a^t \|\alpha'(\tau)\|\, d\tau$. Since $s'(t) = \|\alpha'(t)\| > 0$, as remarked above it has a C^k inverse $t : [0,L] \to [a,b]$, and $\tilde{\alpha} : [0,L] \to \mathbf{R}^n$ defined by $\tilde{\alpha}(s) = \alpha(t(s))$ is a reparameterization of α called its parameterization by arclength. Note that by definition, the length of $\tilde{\alpha}$ between 0 and s is s, so the name is well-chosen.

▷ **11.2—Exercise 1.** Show that a parametric curve α is parametrized by arclength if and only if $\|\alpha'(t)\|$ is identically equal to 1.

11.2.1 Remark. From now on we will usually assume that α is parametrized by its arclength. It is traditional to signify this by using s as the variable name when dealing with such paths.

Notation If $\alpha(s)$ is a curve parametrized by arclength, then we will denote its unit tangent vector at s by $\overrightarrow{t}(s) := \alpha'(s)$.

11.2.2 Remark. In $\mathbf{R}^2$ there is an important orthogonal transformation that we shall denote by $R_{\frac{\pi}{2}}$. It is defined by $R_{\frac{\pi}{2}}(x,y) = (y,-x)$, and geometrically speaking it rotates any vector v through 90 degrees into a vector orthogonal to v. If α is a curve in $\mathbf{R}^2$ parametrized by arclength, then we define its normal at s by $\overrightarrow{n}(s) = R_{\frac{\pi}{2}} \overrightarrow{t}(s)$.

11.2.3 Remark. In $\mathbf{R}^n$ for $n > 2$, there is no natural way to assign a normal to a curve at every point that works in complete generality. To convince yourself of this, just think of the case of a straight line in $\mathbf{R}^3$—there is a whole circle of directions at each point that are normal to the line, and no way to prefer any one of them. However, at a point where $\alpha''(s) \neq 0$, we define the unit normal $\overrightarrow{n}(s)$ to α at s by $\overrightarrow{n}(s) := \frac{\alpha''(s)}{\|\alpha''(s)\|}$. (Recall that since $\|\alpha'(s)\|$ is identically equal to one, it follows that its derivative $\alpha''(s)$ is orthogonal to $\alpha'(s)$.)

▷ **11.2—Exercise 2.** Show that when the straight line $\alpha(t) = x^0 + tv^0$ is reparametrized by arclength the result is $\tilde{\alpha}(s) = x^0 + su$ where $u = \frac{v^0}{\|v^0\|}$

▷ **11.2—Exercise 3.** Consider the circle of radius r, $\alpha(t) = (r\cos(t), r\sin(t))$, with $I = [0, 2\pi]$. Show that $s(t) = rt$, so that $t(s) = s/r$, and deduce that reparameterization by arclength gives $\tilde{\alpha}(s) = (r\cos(s/r), r\sin(sr))$, and $\overrightarrow{t}(s) = (-\sin(s/r), \cos(s/r))$.

What is the Curvature of a Curve?

How should we define the "curvature" of a curve? For a plane curve, $\alpha(s) = (x(s), y(s))$

there is a natural and intuitive definition—namely the rate at which its unit tangent vector $\alpha'(s) = (x'(s), y'(s))$ is "turning". Now since $\alpha'(s)$ is a unit vector, we can write it as $\alpha'(s) = (\cos(\theta(s)), \sin(\theta(s)))$ where $\theta(s)$ is the angle $\alpha'(s)$ makes with the x-axis. Thus we can define the curvature $k(s)$ of α at s as $\theta'(s)$—the rate of change of this angle with respect to arclength. Notice that $\alpha''(s) = \theta'(s)(-\sin((\theta(s)), \cos((\theta(s))) = k(s) R_{\frac{\pi}{2}} \overrightarrow{t}(s) = k(s)\overrightarrow{n}(s)$, so we make the following definition:

11.2.4 Definition. If α is a curve in $\mathbf{R}^2$ that is parametrized by arclength, then its *curvature* at $\alpha(s)$ is defined to be the scalar $k(s)$ such that $\alpha''(s) = k(s)\overrightarrow{n}(s)$.

Note that in the plane, $\mathbf{R}^2$, the curvature $k(s)$ can be either positive or negative. Its absolute value is of course given by $|k(s)| = \|\alpha''(s)\|$.

11.2.5 Definition. If α is a curve in $\mathbf{R}^n$, $n > 2$ that is parametrized by arclength, then its *curvature* at $\alpha(s)$ is defined to be $k(s) := \|\alpha''(s)\|$.

▷ **11.2—Exercise 4.** Show that if $\alpha(t) = (x(t), y(t))$ is a plane parametrized curve that is not necessarily parametrized by arclength, then its curvature at $\alpha(t)$ is given by the following formula. Hint: $\theta = \tan^{-1}(y'/x')$.

$$\frac{x'(t)y''(t) - y'(t)x''(t)}{\left(x'(t)^2 + y'(t)^2\right)^{\frac{3}{2}}}.$$

11.2.6 Remark. Recall that when $n > 2$ at points where $k(s) := \|\alpha''(s)\| > 0$ the normal $\overrightarrow{n}(s)$ was defined by by $\overrightarrow{n}(s) := \frac{\alpha''(s)}{\|\alpha''(s)\|}$, so the equality $\alpha''(s) = k(s)\overrightarrow{n}(s)$ holds in this case too. But note the subtle difference; for a plane curve the curvature can be positive or negative, while in higher dimensions it is (by definition) always positive.

▷ **11.2—Exercise 5.** Show that a straight line has curvature zero, and that a circle of radius r has constant curvature $1/r$.

▷ **11.2—Exercise 6.** Show that straight lines are the only curves with zero curvature, but show that curves with positive constant curvature are not necessarily circles. (Hint: Show that a helix has constant curvature.) However, in $\mathbf{R}^2$, show that a curve of constant positive curvature k must be a circle of radius $1/k$.

Osculating Circles and Evolutes

At any point $\alpha(s)$ of a plane curve α there are clearly circles of any radius that are tangent to α at this point. In fact, just move out a distance r from $\alpha(s)$ along the normal $\overrightarrow{n}(s)$ and construct the circle with radius r centered at that point. However there is one special tangent circle that is "more tangent" than the others, in the sense that it has "second order contact" with α. This is the so-called *osculating circle* at $\alpha(s)$ and is the circle having the same curvature as α at this point, namely $k(s)$. Recalling that a circle of radius r has curvature $1/r$, we see that the radius of the osculating circle is $1/k(s)$, and its center, called the *center of curvature* of α at $\alpha(s)$ is clearly the point $c(s) := \alpha(s) - (1/k(s))\overrightarrow{n}(s)$.

As the point $\alpha(s)$ varies over the curve α, the corresponding centers of curvature trace out a curve called the *evolute* of the original curve α.

11.3 The Fundamental Theorem of Plane Curves

Recall that $\boldsymbol{Euc}(\mathbf{R}^n)$ denotes the Euclidean group of $\mathbf{R}^n$, i.e., all the distance preserving maps of $\mathbf{R}^n$ to itself. We have seen that every element of $\boldsymbol{Euc}(\mathbf{R}^n)$ can be written as the composition of a translation and an orthogonal transformation.

11.3.1 Definition. Two parametric curves α_1 and α_2 in $\mathbf{R}^n$ are called *congruent* if there is an element $g \in \boldsymbol{Euc}(n)$ such that $\alpha_2 = g \circ \alpha_1$.

11.3.2 Proposition. *The curvature function of a parametrized curve is invariant under congruence. That is, if two parametrized curves in $\mathbf{R}^n$ are congruent, then their curvature functions are identical.*

▷ **11.3—Exercise 1.** Prove this Proposition.

It is a remarkable fact that for plane curves the converse is true, so remarkable that it goes by the name The Fundamental Theorem of Plane Curves. The Fundamental Theorem actually says more—any continuous function $k(s)$ is the curvature function for some parametrized plane curve, and this curve is uniquely determined up to congruence. In fact, we will get an explicit formula below for the curve in terms of k.

11.3.3 Proposition. *Let $k : [0, L] \to \mathbf{R}$ be continuous and let $\alpha(s) = (x(t), y(t))$ be a parametrized plane curve that is parametrized by arclength. A neccessary and sufficient condition for α to have k as its curvature function is that θ, x, y be a solution on $[0, L]$ of the following system of first order ODE:*

$$\frac{d\theta}{ds} = k(s),$$
$$\frac{dx}{ds} = \cos(\theta),$$
$$\frac{dy}{ds} = \sin(\theta).$$

▷ **11.3—Exercise 2.** Prove this Proposition.

The first ODE of the above system integrates immediately to give $\theta(\sigma) = \theta_0 + \int_0^\sigma k(\tau)\, d\tau$. If we now substitute this into each of the remaining equations and integrate again we obtain the following important corollary.

11.3.4 Corollary. *If $k : [0, L] \to \mathbf{R}$ is a continuous function, then the set of plane curves $\alpha(s) = (x(s), y(s))$ that are parametrized by arclength and have k as their curvature* *function are just those of the form:*

$$x(s) := x_0 + \int_0^s \cos\left(\theta_0 + \int_0^\sigma k(\tau)\,d\tau\right),$$
$$y(s) := y_0 + \int_0^s \sin\left(\theta_0 + \int_0^\sigma k(\tau)\,d\tau\right).$$

▷ **11.3—Exercise 3.** Use this corollary to rederive the fact that straight lines and circles are the only plane curves with constant curvature.

11.3.5 Remark. Note the geometric meaning of the constants of integration x_0, y_0 and θ_0. The initial point $\alpha(0)$ of the curve α is (x_0, y_0),while θ_0 is the angle that the initial tangent direction $\alpha'(0)$ makes with the x-axis. If in particular we take all three constants to be zero and call the resulting curve α_0, then we get the general solution from α_0 by first rotating by θ_0 and then translating by (x_0, y_0). This proves:

11.3.6 Fundamental Theorem of Plane Curves. *Two plane curves are congruent if and only if they have the same curvature function. Moreover any continuous function $k : [0, L] \to \mathbf{R}$ can be realized as the curvature function of a plane curve.*

Why the Fancy Name?

"Fundamental Theorem" sounds rather imposing— what's the big deal? Well, if you think about it, we have made remarkable progress in our understanding of curve theory in the past few pages—progress that actually required many years of hard work—and that progress is summed up in the Fundamental Theorem. There are two major insights involved in this progress. The first is that, from the viewpoint of geometry, we should consider curves that differ by parameterization as "the same", and that we can avoid the ambiguity of description this involves by choosing parameterization by arclength. (The lack of any analogous "canonical parameterization" for a surface will make our study of surface thory considerably more complicated.) The second insight is that, from the geometric viewpoiint again, congruent curves should also be regarded as "the same", and if we accept this then the simplest geometric description of a plane curve—one that avoids all redundency—is just its curvature function.

11.4 Sixth Matlab Project.

The Matlab project below is concerned in part with the visualization and animation of curves. Before getting into the details of the project, I would like to make a few general remarks on the subject of mathematical visualization that you should keep in mind while working on this project—or for that matter when you have any programming task that involves visualization and animation of mathematical objects.

1) How should you choose an error tolerance?

First, an important principle concerning the handling of errors in any computer graphics context. Books on numerical analysis tell you how to estimate errors and how to keep them below a certain tolerance, but they cannot tell you what that tolerance should be—that must depend on how the numbers are going to be used. Beginners often assume they should aim for the highest accuracy their programming system can provide—for example fourteen decimal places for Matlab. But that will often be far more than is required for the task at hand, and as you have already seen, certain algorithms may require a very long time to attain that accuracy. The degree of one's patience hardly seems to be the best way to go about choosing an error tolerance.

In fact, there is often is a more rational way to choose appropriate error bounds. For example, in financial calculations it makes no sense to compute values with an error less than half the smallest denomination of the monetary unit involved. And when making physical calculations, it is useless to calculate to an accuracy much greater than can be measured with the most precise measuring instruments available. Similarly, in carpentry there is little point to calculating the length of a board to a tolerance less than the width of the blade that will make the cut.

This same principle governs in mathematical visualization. My approach is to choose a tolerance that is "about half a pixel", since any higher accuracy won't be visible anyway. It is usually fairly easy to estimate the size of a pixel. There are roughly 100 pixels per inch, so for example if you are are graphing in a six inch square window, and the axes go from minus one to one, then six hundred pixels equals two length units, so half a pixel accuracy means a tolerance of $\frac{1}{600}$ or roughly 0.002.

1) How should you represent a curve?

Mathematically a curve in $\mathbf{R}^n$ is given by a map of an interval $[a, b]$ into $\mathbf{R}^n$. We can only represent the curve on a computer screen when $n = 2$ or $n = 3$. Let's consider the case of plane curves ($n = 2$) first. If $\alpha(t) = (x(t), y(t))$ then for any N we can divide the interval $[a, b]$ into N equal subintervals of length $h = \frac{b-a}{N}$, namely $[t_k, t_{k+1}]$, where $t_k = a + kh$ and $k = 0, \ldots, N(-1$. We associate to α and N an approximating "N-gon" α_N (i.e., a polygon with N sides) with vertices $v_k := (x(t_k), y(t_k))$. It is some α_N with N suitably large) that actually gets drawn on the computer screen when we want to display α. This reduces the actual drawing problem to that of drawing a straight line segment, and the latter is of course built into every computer system at a very low level.

In Matlab the code for plotting the curve α, or rather the polygon α_{30} would be:

```
N = 30
h = (b-a)/N;
t = a:h:b ;
plot(x(t),y(t)), axis equal;
```

To plot a curve $\alpha(t) = (x(t), y(t), z(t))$ in $\mathbf{R}^3$ is really no more difficult. In Matlab the only change is that the last line gets replaced by:

```
plot3(x(t),y(t),z(t)), axis equal;
```

only now one has to be more careful about interpreting just what it is that one sees on the screen in this case. The answer is that one again is seeing a certain polygon in the plane, but now it is the projection of the polygon in $\mathbf{R}^3$ with vertices at $v_k := (x(t_k), y(t_k), z(t_k))$. (The projection can be chosen to be either an orthographic projection in some direction or else a perspective projection from some point.)

1) How do you create animations?

Viisualization can be a powerful tool for gaining insight into the nature of complex mathematical objects, and frequently those insights can be further enhanced by careful use of animation. Remember that time is essentially another dimension, so animations allow us to pack a lot more information onto a computer screen in a format that the human brain can easily assimilate. The number of ways that animation can be used are far to numerous to catalog here, but in addition to obvious ones, such as rotating a three dimensional object, one should also mention ”morphing”. Mathematical objects frequently depend on several parameters (e.g., think of the family of ellipses: $x = a\cos(\theta)$, $y = b\sin(\theta)$). Morphing refers to moving along a curve in the space of parameters and creating frames of an animation as you go.

All animation techniques use the same basic technique—namely showing a succession of “frames” on the screen in rapid succession. If the number of frames per second is fast enough, and the change between frames is small enough, then the phenomenon of “persistence of vision” creates the illusion that one is seeing a continuous process evolve. Computer games have become very popular in recent years, and they depend so heavily on high quality animation that the video hardware in personal computers has improved very rapidly. Still, there are many different methods (and tricks) involved in creating good animations, and rather than try to cover them here we will have some special lectures on various animation techniques, with particular emphasis on how to implement these techniques in Matlab.

Matlab Project # 6.

Your assignment for the sixth project is to implement the Fundamental Theorem of Plane Curves using Matlab. That is, given a curvature function $k : [0, L] \to \mathbf{R}$, construct and plot a plane curve $x : [0, L] \to \mathbf{R}^2$ that has k as its curvature function. To make the solution unique, take the initial point of x to be the origin and its initial tangent direction to be the direction of the positive x-axis. You should also put in an option to plot the evolute of the curve as well as the curve itself. Finally see if you can build an animation that plots the osculating circle at a point that moves along the curve x. For uniformity, name your M-File PlaneCurveFT, and let it start out:

```
function      x = PlaneCurveFT(k,L,option)
```

If option is not given (i.e., nargin = 2) or if option = 0, then just plot the curve x. If option = 1, then plot x and, after a pause, plot its evolute in red. Finally, if option = 2, then plot x and its evolute, and then animate the osculating circle (in blue) along the curve, also drawing the radius from the center of curvature.

[To find the curve x, you first integrate k to get $\overrightarrow{t} = x'$, and then integrate $\overrightarrow{t}$. The curvature, k, will be given as a Matlab function, so you can use the version of Simpson's Rule previously discussed for the first integration. But $\overrightarrow{t}$ will not be in the form of a Matlab function that you can substitute into that version of Simpson's Rule, so you will need to develop a slightly modified version of Simpson's. where the input is a matrix that gives the values of the integrand at the nodes rather than the integrand as a function.]