Lecture 12
Curves in 3-Space

Some Definitions and Notations. In this section $\alpha : [0, L] \to \mathbf{R}^3$ will denote a curve in $\mathbf{R}^3$ that is parametrized by arclength and is of class C^k, $k \geq 2$. We recall that the unit tangent to α at s is defined by $\overrightarrow{t}(s) := \alpha'(s)$. (That it is a unit vector is just the definition of α being parametrized by arclength.) The *curvature* of α is the non-negative real-valued function $k : [0, L] \to \mathbf{R}$ defined by $k(s) := \|\alpha''(s)\|$, and we call α a *Frenet curve* if its curvature function is strictly positive. For a Frenet curve, we define its normal vector $\overrightarrow{n}(s)$at s by $\overrightarrow{n}(s) := \frac{\alpha''(s)}{k(s)}$. Since $\overrightarrow{t}(s)$ has constant length one, it is orthogonal to its derivative $\overrightarrow{t}'(s) = \alpha''(s)$, hence:

12.0.1 First Frenet Equation. *If $\alpha : [0, L] \to \mathbf{R}^3$ is a Frenet curve then its normal vector $\overrightarrow{n}(s)$ is a unit vector that is orthogonal to $\overrightarrow{t}(s)$, and $\overrightarrow{t}'(s) = k(s)\overrightarrow{n}(s)$.*

For the remainder of this section we will assume that α is a Frenet curve.

12.0.2 Definition. We define the *binormal vector* to α at s by $\overrightarrow{b}(s) := \overrightarrow{t}(s) \times \overrightarrow{n}(s)$. The ordered triple of unit vectors $f(s) := (\overrightarrow{t}(s), \overrightarrow{n}(s), \overrightarrow{b}(s))$ is called the *Frenet frame* of α at $\alpha(s)$, and the mapping $s \mapsto f(s)$ of $[0, L] \to \left(\mathbf{R}^3\right)^3$ is called the *Frenet framing* of the curve α. The plane spanned by $\overrightarrow{t}(s)$ and $\overrightarrow{n}(s)$ is called the *osculating plane* to α at s, and the plane spanned by $\overrightarrow{n}(s)$ and $\overrightarrow{b}(s)$ is called the *normal plane* to α at $\alpha(s)$.

12.1 Quick Review of the Vector Product

We recall that for $u = (u_1, u_2, u_3)$ and $v = (v_1, v_2, v_3)$ in $\mathbf{R}^3$, we define their *vector-product* $u \times v \in \mathbf{R}^3$ by $u \times v := (u_2v_3 - v_2u_3, u_3v_1 - u_1v_3, u_1v_2 - v_1u_2)$.

12.1.1 Remark. Symbolically we can write this definition as:

$$u \times v = \det \begin{pmatrix} u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \\ e_1 & e_2 & e_3 \end{pmatrix},$$

where e_1, e_2, e_3 is the standard basis for $\mathbf{R}^3$. What this means is that we get a correct result if we use Cramer's rule to expand this "determinant" by minors of the third row. From this formula the following facts are immediate:

12.1.2 Proposition. *The vector product is bilinear and skew-symmetric. That, is it is linear in each argument and changes sign when the arguments are interchanged. The vector product of u and v vanishes if and only if u and v are linearly dependent, and $(u \times v) \cdot w$ is the determinant of the matrix with rows u,v, and w. In particular, $(u \times v) \cdot u$ and $(u \times v) \cdot v$ are both zero, i.e., $u \times v$ is orthogonal to both u and v.*

12.1.3 Lemma. $(u \times v) \cdot (x \times y) = \det \begin{pmatrix} u \cdot x & v \cdot x \\ u \cdot y & v \cdot y \end{pmatrix}$.

PROOF. Either direct verification, or use linearity in all four variables to reduce the result to the case that all of the variables are equal to one of the standard orthonormal basis vectors e_i, in which case it is obvious.

12.1.4 Proposition. *The norm of $u \times v$ is the area of the parallelogram spanned by u and v. That is, if θ is the angle between u and v then $\|u \times v\| = \|u\| \|v\| \sin(\theta)$.*

PROOF. $\|u \times v\|^2 = (u \times v) \cdot (u \times v) = \det \begin{pmatrix} u \cdot u & v \cdot u \\ u \cdot v & v \cdot v \end{pmatrix} = \|u\|^2 \|v\|^2 - \|u\|^2 \|v\|^2 \cos^2(\theta)$ $= \|u\|^2 \|v\|^2 (1 - \cos^2(\theta))$. ∎

▷ **12.1—Exercise 1.** Show that the "triple vector product" $(u \times v) \times w$ is given by the formula $(u \cdot w)v - (v \cdot w)u$. Hint. Since it is orthogonal to $u \times w$ it must be of the form $f(w)v + g(w)u$, where, since the result is linear in w, f and g have the form $a \cdot w$ and $b \cdot w$, so we have $(u \times v) \times w = (a \cdot w)v + (b \cdot w)u$, for some a and b.

▷ **12.1—Exercise 2.** If $u(t)$ and $v(t)$ are smooth curves in $\mathbf{R}^3$, show that $(u(t) \times v(t))' = u'(t) \times v(t) + (u(t) \times v'(t)$

12.2 The Frenet Formulas

We now return to the Frenet Frame $\overrightarrow{t}(s)$, $\overrightarrow{n}(s)$, $\overrightarrow{b}(s)$ for α. Recall that the binormal vector was defined by $\overrightarrow{b}(s) := \overrightarrow{t}(s) \times \overrightarrow{n}(s)$. In particular, as the vector product of orthogonal unit vectors it is a unit vector and so its derivative orthogonal $\overrightarrow{b}'(s)$ is orthogonal to $\overrightarrow{b}(s)$ and so it is a linear combination of $\overrightarrow{t}(s)$ and $\overrightarrow{n}(s)$. But in fact $\overrightarrow{b}'(s)$ is orthogonal to $\overrightarrow{t}(s)$ also, and hence it is a multiple of $\overrightarrow{n}(s)$. To see this we calculate:

$$\begin{aligned} \overrightarrow{b}'(s) &= (\overrightarrow{t}(s) \times \overrightarrow{n}(s))' \\ &= \overrightarrow{t}'(s) \times \overrightarrow{n}(s) + \overrightarrow{t}(s) \times \overrightarrow{n}'(s) \\ &= k(s)\overrightarrow{n}(s) \times \overrightarrow{n}(s) + \overrightarrow{t}(s) \times \overrightarrow{n}'(s) \\ &= \overrightarrow{t}(s) \times \overrightarrow{n}'(s) \end{aligned}$$

since $\overrightarrow{n}(s) \times \overrightarrow{n}(s) = 0$. Thus $\overrightarrow{b}'(s)$ is orthogonal to both $\overrightarrow{b}(s)$ and $\overrightarrow{t}(s)$ and so:

12.2.1 Proposition. *$\overrightarrow{b}'(s)$ is a multiple of $\overrightarrow{n}(s)$.*

12.2.2 Definition. We define the *torsion* of α at s to be the real number $\tau(s)$ such the $\overrightarrow{b}'(s) = \tau(s)\overrightarrow{n}(s)$.

12.2.3 Remark. Since $\overrightarrow{b}(s)$ is the normal to the osculating plane to the curve α at s, the torsion measures the rate at which the curve is twisting out of its osculating plane. Note for example, that if $\tau(s)$ is identically zero, then $\overrightarrow{b}(s)$ is a constant b_0, and so the osculating plane is fixed, and it follows easily that $\alpha(s)$ lies in a plane parallel to its osculating plane. In fact, since $(\alpha(s)\cdot b_0)' = \overrightarrow{t}(s)\cdot b_0 = 0$, $(\alpha(s)-\alpha(0))\cdot b_0 = 0$, which says that α lies in the plane Π orthogonal to b_0 that contains $\alpha(0)$.

Now that we have computed $\overrightarrow{t}'(s)$ and $\overrightarrow{b}'(s)$, it is easy to compute $\overrightarrow{n}'(s)$. In fact, since $\overrightarrow{b}(s) = \overrightarrow{t}(s)\times\overrightarrow{n}(s)$, it follows that $n(s) = b(s)\times t(s)$, so

$$\begin{aligned}\overrightarrow{n}'(s) &= (\overrightarrow{b}(s)\times\overrightarrow{t}(s))' \\ &= \overrightarrow{b}(s)\times\overrightarrow{t}'(s) + \overrightarrow{b}'(s)\times\overrightarrow{t}(s) \\ &= \overrightarrow{b}(s)\times(k(s)n(s)) + \tau(s)\overrightarrow{n}(s)\times\overrightarrow{t}(s) \\ &= -k(s)\overrightarrow{t}(s) - \tau(s)\overrightarrow{b}(s)\end{aligned}$$

The equations that express the derivative of the Frenet frame in terms of the frame itself are referred to as the Frenet Equations. Let's rewrite them as a single matrix equation:

Frenet Equations

$$\begin{pmatrix}\overrightarrow{t}(s)\\ \overrightarrow{n}(s)\\ \overrightarrow{b}(s)\end{pmatrix}' = \begin{pmatrix}0 & k(s) & 0\\ -k(s) & 0 & -\tau(s)\\ 0 & \tau(s) & 0\end{pmatrix}\begin{pmatrix}\overrightarrow{t}(s)\\ \overrightarrow{n}(s)\\ \overrightarrow{b}(s)\end{pmatrix}$$

12.2.4 Remark. We can write this symbolicly as $f' = Af$ where $f = (\overrightarrow{t}(s), \overrightarrow{n}(s), \overrightarrow{b}(s))$ is the Frenet frame, and A is a 3×3 matrix with entries 0, $\pm k(s)$ and $\pm\tau(s)$. The fact that this matrix is skew-symmetric is, as we shall see next is just a reflection of the fact that $(\overrightarrow{t}(s), \overrightarrow{n}(s), \overrightarrow{b}(s))$ are orthonormal.

12.2.5 Proposition. *Let $f_i : [0, L] \to \mathbf{R}^n$, $i = 1,\ldots,n$, be C^1 maps and suppose that the $f_i(t)$ are orthonormal for all t. Let $a_{ij}(t)$ be the $n\times n$ matrix that expresses the $f_i'(t)$ as a linear combination of the $f_i(t)$, i.e., $f_i'(t) = \sum_{j=1}^n a_{ij}(t)f_j(t)$ (so that $a_{ij}(t) = f_i'(t)\cdot f_j(t)$). Then $a_{ij}(t)$ is skew-symmetric. Conversely, if $t \mapsto a_{ij}(t)$ is a continuous map of $[0, L]$ into the skew-adjoint $n\times n$ matrices and $\phi_1,\ldots,\phi_n$ is any orthonormal frame for $\mathbf{R}^n$, then there are unique differentiable maps $f_i : [0, L] \to \mathbf{R}^n$ such that $f_i(0) = \phi_i$ and $f_i'(t) = \sum_{j=1}^n a_{ij}(t)f_j(t)$, and these $f_i(t)$ are orthonormal for all t.*

PROOF. Differentiating $f_i(t)\cdot f_j(t) = \delta_{ij}$ gives $f_i'(t)\cdot f_j(t) + f_i(t)\cdot f_j'(t) = 0$ proving the first statement. Conversely, if we are given $a_{ij}(t)$ and the ϕ_i, then by the existence and uniqueness theorem for ODE, we can solve the IVP $f_i'(t) = \sum_{j=1}^n a_{ij}(t)f_j(t)$ and $f_i(0) = \phi_i$ uniquely on the interval $[0, L]$, and we only have to show that $s_{ij}(t) := f_i(t)\cdot f_j(t)$ is identically equal to δ_{ij}. We note that $s_{ij}(t)$ satisfies the IVP $s_{ij}(0) = \delta_{ij}$ and

$$
\begin{aligned}
\frac{d}{dt}\left(s_{ij}(t)\right) &= f_i'(t)\cdot f_j(t) + f_i(t)\cdot f_j'(t) \\
&= \sum_{k=1}^{n} a_{ik}(t) f_k(t)\cdot f_j(t) + \sum_{k=1}^{n} a_{jk}(t) f_i(t) f_k(t) \\
&= \sum_{k=1}^{n} a_{ik}(t) s_{kj}(t) + \sum_{k=1}^{n} a_{jk}(t) s_{ij}(t)
\end{aligned}
$$

Since the $a_{ij}(t)$ are skew symmetric it is clear that $s_{ij}(t) = \delta_{ij}$ is also a solution, so the converse follows from the uniquenes of the solution to this IVP. ∎

12.3 The Fundamental Theorem of Space Curves

We are now in a position to generalize our theory of plane curves to an analogous theory of space curves. We first make a number of simple observations.

Observation 1. The Frenet framing associated to a space curve is invariant under orthogonal transformations in the sense that if α and $\tilde{\alpha}$ are space curves and g is an element of the orthogonal group $\boldsymbol{O}(\mathbf{R}^3)$ such that $\tilde{\alpha} = g \circ \alpha$, then g maps the Frenet frame of α at s to the Frenet frame of $\tilde{\alpha}$ at s.

Observation 2. The curvature and torsion functions of a plane curve are likewise invariant under orthogonal transformation and also under translation.

▷ **12.3—Exercise 1.** Prove the validity of these two observations. Hint; They depend on little more than the definitions of the quantities involved and the fact that orthogonal transformations preserve lengths of curves.

We now in a position to prove the following analogue of the Fundamental Theorem of Plane Curves.

Fundamental Theorem of Space Curves. *Two space curves are congruent if and only if they have the same curvature and torsion functions. Moreover any pair of continuous functions $k : [0, L] \to (0, \infty)$ and $\tau : [0, L] \to \mathbf{R}$ can be realized as the curvature and torsion functions of some space curve.*

PROOF. Given k and τ, it follows from the preceding Proposition that we can find $\mathbf{R}^3$ valued functions $\overrightarrow{t}(t), \overrightarrow{n}(t), \overrightarrow{b}(t)$ defined on $[0, L]$ that are orthonormal and satisfy the Frenet Equations. Then, just as in the planar case, we can define a curve $\alpha(s) := \int_0^s \overrightarrow{t}(t)\, dt$. Since $\overrightarrow{t}(t)$ is a unit vector, α is parametrized by arclength and $\overrightarrow{t}$ is clearly its unit tangent vector, so it is a consequence of the Frenet Equations that $\overrightarrow{n}(t)$, and $\overrightarrow{b}(t)$ are its normal and binormal and k and τ its curvature and torsion.

Now suppose $\tilde{\alpha}$ is a second curve with the same curvature and torsion as α. If we translate α by $\tilde{\alpha}(0) - \alpha(0)$ and then rotate it by the rotation carrying the Frenet frames of α at 0 into that of $\tilde{\alpha}$ at 0, then we get a curve congruent to α that has the same initial point and initial Frenet frame as $\tilde{\alpha}$, so it will sufice to show that if α and $\tilde{\alpha}$ have the *same* initial point and Frenet frame, then they are identical. But from the uniqueness of the solution of the IVP for the Frenet equations it now follows that α and $\tilde{\alpha}$ have the same Frenet frame at all $s \in [0, L]$, and it the follows that both $\alpha(s)$ and $\tilde{\alpha}(s)$ equal $\int_0^s \overrightarrow{t}(t)\, dt$. ∎

Matlab Project # 7.

As you have probably guessed, your assignment for the seventh Matlab project is to implement the Fundamental Theorem of Space Curves. That is, given a (positive) curvature function $k : [0, L] \to \mathbf{R}$, and a torsion function $\tau : [0, L] \to \mathbf{R}$, construct and plot a space curve x that has k as its curvature function and τ as its torsion function. To make the solution unique, take the initial point of x to be the origin and its initial tangent direction to be the direction of the positive x-axis. You should also use plot3 to plot the curve. See if you can create an animation that moves the Frenet Frame along the curve. For uniformity, name your M-File SpaceCurveFT, and let it start out:

```
function       x = SpaceCurveFT(k,tau,L)
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12.4 Surface Theory: Basic Definitions and Examples

The theory of curves in the plane, three-space, and higher dimensions is deep and rich in detail, and we have barely scratched the surface. However I would like to save enough time to cover at least the basics of surface theory, so we will now leave the theory of curves.

How should we define a surface? As with curves, there are many possible answers, and we will select not the most general definition but one that is both intuitive and leads quickly to a good theory. The simplest curve is just an interval in the line, and we defined other curves to be maps of an interval into $\mathbf{R}^n$ with non-vanishing derivative, The natural two dimensional analog of an interval is a connected open set or *domain* $\mathcal{O}$ in $\mathbf{R}^2$.

12.4.1 Definition. A C^k *parametric surface* in $\mathbf{R}^3$ $(k \geq 3)$ is a C^k map $\mathcal{F} : \mathcal{O} \to \mathbf{R}^3$ (where $\mathcal{O}$ is a domain in $\mathbf{R}^2$) such that its differential, $D\mathcal{F}_p$, has rank 2 at all points $p \in \mathcal{O}$.

Of course, intuitively speaking, it is the image of $\mathcal{F}$ that constitutes the surface, but as with curves we will allow ourselves a looseness of language and not always distinguish carefully between $\mathcal{F}$ and its image.

Notation. Just as it is traditional to use t as the parameter of a curve (or s if the parameter is arclength), it is traditional to use u and v to denote the parameters of a parametric surface, so a surface is given by a mapping $(u, v) \mapsto \mathcal{F}(u, v) = (\mathcal{F}_1(u, v), \mathcal{F}_2(u, v), \mathcal{F}_2(u, v))$. Instead of the $\mathcal{F}_i(u, v)$ it is also traditional to use $(x(u, v).y(u.v), z(u, v))$ to denote the three components of $\mathcal{F}(u, v)$, and we will often use this notation without explicit mention when it is clear what surface is under consideration.

▷ **12.4—Exercise 1.** Show that if $p_0 = (u_0, v_0)$, then the condition that $D\mathcal{F}_p$ has rank two is equuivalent to $\frac{\partial \mathcal{F}(u_0,v_0)}{\partial u}$ and $\frac{\partial \mathcal{F}(u_0,v_0)}{\partial v}$ being linearly independent.

12.4.2 Definition. If $\mathcal{F} : \mathcal{O} \to \mathbf{R}^3$ is a parametric surface in $\mathbf{R}^3$ and $p \in \mathcal{O}$, the *tangent space to* $\mathcal{F}$ *at* p is defined to be the image of the linear map $D\mathcal{F}_p : \mathbf{R}^2 \to \mathbf{R}^3$, and we denote it by $T\mathcal{F}_p$. We note that $T\mathcal{F}_p$ is by assumption a two-dimensional linear subspace of $\mathbf{R}^3$ and that $\frac{\partial \mathcal{F}(u_0,v_0)}{\partial u}$ and $\frac{\partial \mathcal{F}(u_0,v_0)}{\partial v}$ is clearly a basis. This is called the basis for $T\mathcal{F}_p$ defined by the parameters u, v. We define the *normal vector to* $\mathcal{F}$ *at* p to be the unit vector $\overrightarrow{\nu}(p)$ obtained by normalizing $\frac{\partial \mathcal{F}(u_0,v_0)}{\partial u} \times \frac{\partial \mathcal{F}(u_0,v_0)}{\partial v}$. The map $\overrightarrow{\nu} : \mathcal{O} \to \mathbf{S}^2$, $p \mapsto \overrightarrow{\nu}(p)$ of $\mathcal{O}$ to the unit sphere $\mathbf{S}^2 \subseteq \mathbf{R}^3$ is called the *Gauss map* of the surface $\mathcal{F}$.

12.4.3 Remark. You will probably guess that the "curvature" of the surface $\mathcal{F}$ (whatever it means) will somehow be measured by the rate at which the normal $\overrightarrow{\nu}(p)$ varies with p.

▷ **12.4—Exercise 2.** Show that the tangent space to $\mathcal{F}$ at p and the tangent space to $\mathbf{S}^2$ at $\overrightarrow{\nu}(p)$ are the same.

12.4.4 Remark. It is natural to try to use what we have learned about curves to help us investigate surfaces, and this approach turns out to be very effective. It $(u(t), v(t))$ is a smooth parametric curve in the domain $\mathcal{O}$ of the of the surface $\mathcal{F}$, then $\alpha(t) := \mathcal{F}(u(t), v(t))$ is a parametric curve in $\mathbf{R}^3$, and we shall call such curve a parametric curve on the surface $\mathcal{F}$. If we put $u_0 = u(t_0)$, $v_0 = v(t_0)$ and $p = (u_0, v_0)$, then the by the chain-rule, the tangent vector, $\alpha'(t_0)$, to $\alpha(t)$ at t_0 is $D\mathcal{F}_p(u'(t_0), v'(t_0))$, an element of the tangent space $T\mathcal{F}_p$ to $\mathcal{F}$ at p. In terms of the basis defined by the parameters u, v we have $\alpha'(t_0) = u'(t_0)\frac{\partial \mathcal{F}(u_0,v_0)}{\partial u} + v'(t_0)\frac{\partial \mathcal{F}(u_0,v_0)}{\partial v}$.

▷ **12.4—Exercise 3.** Show that every element of $T\mathcal{F}_p$ is the tangent vector to some curve on the surface $\mathcal{F}$ as above.

12.4.5 Remark. There is a special two-parameter "net" of curves on a surface $\mathcal{F}$ defined by taking the images of the straight lines in the domain $\mathcal{O}$ that are parallel to the u, v axes. Through each point $p_0 = (u_0, v_0)$ there are two such lines, $t \mapsto (u_0 + t, v_0)$ and $t \mapsto (u_0, v_0 + t)$, called the u-gridline through p and the v-gridline through p, and to visualize the surface, one plots the images under $\mathcal{F}$ of a more or less dense collection of these "gridlines".

▷ **12.4—Exercise 4.** Show that the tangent vectors to the u and v gridlines through p are just the elements of the basis for $T\mathcal{F}_p$ defined by the parameters u, v.

There are two types of surfaces that everyone learns about early in their mathematical training—graphs and surfaces of revolution.

12.4—Example 1. Graphs of Functions. Given a real-valued function $f : \mathcal{O} \to \mathbf{R}$ we get a parametric surface $\mathcal{F} : \mathcal{O} \to \mathbf{R}^3$ called the graph of f by $\mathcal{F}(u, v) := (u, v, f(u, v))$.

12.4—Example 2. Surfaces of Revolution. Let $t \mapsto \alpha(t) = (x(t), z(t))$ be a curve in the x, z-plane that does not meet the z-axis—i.e., $x(t) > 0$ for all t in the domain (a, b) of α, and let $\mathcal{O} = (0, 2\pi) \times (a, b)$. We define a surface $\mathcal{F} : \mathcal{O} \to \mathbf{R}^3$, called the surface of revolution (about the z-axis) defined from the curve α, by $\mathcal{F}(u, v) := (x(v)\cos(u), x(v)\sin(u), z(v))$.