Lecture 8
Ordinary Differential Equations (aka ODE)

8.1 The Initial Value Problem for an ODE.

Suppose we know the wind velocity at every point of space and at every instant of time. A puff of smoke drifts by, and at a certain moment we observe the precise location of a particular smoke particle. Can we then predict where that particle will be at all future times? By making this metaphorical question precise we will be led to the concept of an initial value problem for an ordinary differential equation.

We will interpret "space" to mean $\mathbf{R}^n$, or more generally an inner-product space V, and an "instant of time" will be represented by a real number t. Thus, knowing the wind velocity at every point of space and at all instants of time means that we have a function $X : V \times \mathbf{R} \to V$ that associates to each (v, t) in $V \times \mathbf{R}$ a vector $X(v, t)$ in V representing the wind velocity at v at time t. Such a mapping is called a *time-dependent vector field on* V. We will always be working with such X that are at least continuous, and usually X will even be continuously differentiable. In case $X(v, t)$ does not actually depend on t then we call X a *time-independent vector field on* V, or simply a vector field on V. Note that this is the same as giving a map $X : V \to V$.

How should we model the path taken by the smoke particle? An ideal smoke particle is characterized by the fact that it "goes with the flow", i.e., it is carried along by the wind, meaning that if $x(t)$ is its location at a time t, then its velocity at time t will be the wind velocity at that point and time, namely $X(x(t), t)$. But the velocity of the particle at time t is $x'(t) = \frac{dx}{dt}$, so the path of a smoke particle will be a differentiable curve $x : (a, b) \to V$ such that $x'(t) = X(x(t), t)$ for all $t \in (a, b)$. Such a curve is called a *solution curve* of the time-dependent vector field X and we also say that"x satisfies the ordinary differential equation $\frac{dx}{dt} = X(x, t)$".

Usually we will be interested in solution curves of a differential equation $\frac{dx}{dt} = X(x, t)$ that satisfy a particular *initial condition*. This means that we have singled out some special time t_0 (often, $t_0 = 0$), and some specific point $v_0 \in V$, and we look for a solution of the ODE that satisfies $x(t_0) = v_0$. (In our smoke particle metaphor, this corresponds to observing the particle at v_0 as our clock reads t_0.) The pair of equations $\frac{dx}{dt} = X(x, t)$ and $x(t_0) = v_0$ is called an "initial value problem" (abbreviated as IVP) for the ODE $\frac{dx}{dt} = X$. The reason that it is so important is that the so-called Existence and Uniqueness Theorem for ODE says (more or less) that, under reasonable assumptions on X, the initial value problem has a "unique" solution.

8.1.1 Remark. If the vector field X is time-independent, then the ODE $\frac{dx}{dt} = X$ is often called *autonomous*.

8.1.2 Remark. In the case $V = \mathbf{R}^n$, $X(x, t) = (X_1(x, t), \dots, X_n(x, t))$ so that written out in full, the ODE $\frac{dx}{dt} = X$ looks like $\frac{dx_i}{dt} = X_i(x_1(t), \dots, x_n(t), t)$, $i = 1, \dots n$. In this form it is usually referred to as a "system of ordinary differential equations".

8.1—Example 1. X a constant vector field, i. e., $X(v,t) = u$, where u is some fixed element of V. The solution with initial condition $x(t_0) = v_0$ is clearly the straight line $x(t) := v_0 + (t - t_0)u$.

8.1—Example 2. X is the "identity" vector field, $X(v,t) = v$. The solution with initial condition $x(t_0) = v_0$ is clearly $x(t) := e^{(t-t_0)}v_0$. (Later we will see how to generalize this to an arbitrary linear vector field, i.e., one of the form $X(v,t) = Tv$ where $T : V \to V$ is a continuous linear map.)

8.1—Example 3. A vector field that is "space-independent", i. e., $X(v,t) = f(t)$ where $f : \mathbf{R} \to V$ is continuous. The solution with initial condition $x(t_0) = v_0$ is $x(t) = v_0 + \int_{t_o}^t f(s)\,ds$.

8.2 The Local Existence and Uniqueness Theorem.

In what follows, $X : V \times \mathbf{R} \to V$ is a time dependent vector field on V, $v_0 \in V$, $I = [a,b]$ is an interval of real numbers, and $t_0 \in I$. We define a map $F = F^X_{t_0,v_0}$ of $C(I,V)$ to itself by: $F(\sigma)(t) := v_0 + \int_{t_0}^t X(\sigma(s),s)\,ds$.

The following proposition follows immediately from the definitions and the Fundamental Theorem of Integral Calculus.

Proposition. *A necessary and sufficient condition for $\sigma : I \to V$ to be a solution of the initial value problem $\frac{dx}{dt} = X(x,t)$ and $x(t_0) = v_0$ is that σ be a fixed point of $F^X_{t_0,v_0}$.*

This immediately suggests using successive approximations as a strategy for solving the initial value problem. Start say with the constant curve $x_0 : I \to V$ given by $x_0(t) = v_0$, and define $x_n : I \to V$ inductively by $x_{n+1} := F^X_{t_0,v_0}(x_n)$, and attempt to show that x_n converges to a fixed point of $F^X_{t_0,v_0}$, perhaps by showing that the Contraction Principle applies. As we shall now see, this simple idea actually works for a very general class of ODE.

▷ **8.2—Exercise 1.** Carry out the above strategy for the case of the time-independent vector field $X(v) := v$ with $t_0 = 0$. We saw above that the solution in this case is $x(t) = e^t v_0$. Show by induction that $x_n(t) = P_n(t)v_0$, where P_n is the n-th order Taylor polynomial for e^t, i.e., $P_n(t) = \sum_{k=0}^n \frac{t^k}{k!}$.

Local Existence and Uniqueness Theorem For ODE. *Let $X : V \times \mathbf{R} \to V$ be a C^1 time-dependent vector field on V, $p \in V$, and $t_0 \in \mathbf{R}$. There are positive constants ϵ and δ depending on X, p, and t_0 such that if $I = [t_0 - \delta, t_0 + \delta]$, then for each $v_0 \in V$ with $\|v_0 - p\| < \epsilon$ the differential equation $\sigma'(t) = X(\sigma(t),t)$ has a unique solution $\sigma : I \to V$ satisfying $\sigma(t_0) = v_0$.*

PROOF. If $\epsilon > 0$, then using the technique explained earlier we can find a Lipschitz constant M for X restricted to the set of $(x,t) \in V \times \mathbf{R}$ such that $\|x - p\| \leq 2\epsilon$ and $|t - t_0| \leq \epsilon$. Let B be the maximum value of $\|X(x,t)\|$ on this same set, and choose $\delta > 0$ so that $K = M\delta < 1$ and $B\delta < \epsilon$, and define Y to be the set of σ in $C(I,V)$ such that

$\|\sigma(t) - p\| \leq 2\epsilon$ for all $|t| \leq \delta$. It is easy to see that Y is closed in $C(I, V)$, hence a complete metric space. The theorem will follow from the Banach Contraction Principle if we can show that for $\|v_0\| < \epsilon$, $F = F^X_{t_0, v_0}$ maps Y to itself and has K as a Lipschitz bound.

If $\sigma \in Y$ then $\|F(\sigma)(t) - p\| \leq \|v_0 - p\| + \int_0^t \|X(\sigma(s), s)\| \, ds \leq \epsilon + \delta B \leq 2\epsilon$, so F maps Y to itself. And if $\sigma_1, \sigma_2 \in X$ then $\|X(\sigma_1(t), t) - X(\sigma_2(t), t)\| \leq M \|\sigma_1(t) - \sigma_2(t)\|$, so

$$\begin{aligned}\|F(\sigma_1)(t) - F(\sigma_2)(t)\| &\leq \int_0^t \|X(\sigma_1(s), s) - X(\sigma_2(s), s)\| \, ds \\ &\leq \int_0^t M \|\sigma_1(s) - \sigma_2(s)\| \, ds \\ &\leq \int_0^t M\rho(\sigma_1, \sigma_2) \, ds \\ &\leq \delta M \rho(\sigma_1, \sigma_2) \leq K\rho(\sigma_1, \sigma_2).\end{aligned}$$

and it follows that $\rho(F(\sigma_1), F(\sigma_2) \leq K\rho(\sigma_1, \sigma_2)$. ∎

▷ **8.2—Exercise 2.** Show that continuity of V is **not** sufficient to guarantee uniqueness for an IVP. Hint: The classic example (with $V = \mathbf{R}$) is the initial value problem $\frac{dx}{dt} = \sqrt{x}$, and $x(0) = 0$. (Note that this is C^1, except at the point $x = 0$.) Show that for each $T > 0$, we get a distinct solution $x_T(t)$ of this IVP by defining $x_T(t) = 0$ for $t < T$ and $x_T(t) = \frac{1}{4}(t - T)^2$ for $t \geq T$.

8.2.1 Remark. The existence and uniqueness theorems tell us that for a given initial condition we can solve our initial value problem (uniquely) for a short time interval. The next question we will take up is for just how long we can "follow a smoke particle". You might guess for each initial condition p in V we should have a solution $x_p : \mathbf{R} \to V$ with $x_p(t_0) = p$. But such global in time existence is too much to expect in general. For example, take $V = \mathbf{R}$ and consider the differential equation $\frac{dx}{dt} = x^2$ with the initial condition $x(0) = x_0$. An easy calculation shows that the unique solution is $x(t) = \frac{x_0}{1 - tx_0}$. Note that x_0, this solution "blows up" at time $T = \frac{1}{x_0}$, and by the uniqueness theorem, no solution can exist for a time greater than T.

You may object that a particle of smoke will never go off to infinity in a finite amount of time! Perhaps the smoke metaphor isn't so good after all. The answer is that a real, physical wind field has bounded velocity, and it isn't hard to show that in this case we do indeed have global in time existence.

We are now going to make a simplification, and restrict attention to time-**independent** vector fields (which we shall simply call vector fields). That may sound like a tremendous loss of generality, but in fact it is no loss of generality at all!

▷ **8.2—Exercise 3.** Let $X(v, t)$ be a time-dependent vector field in V, and define an associated time independent vector field $\tilde{X}$ in $V \times \mathbf{R}$ by $\tilde{X}(y) = (X(y), 1)$. Show that $y(t) = (x(t), f(t))$ is a solution of the differential equation $\frac{dy}{dt} = \tilde{X}(y)$ if and only if $f(t) = t + c$ and $x(t)$ is a solution of $\frac{dx}{dt} = X(x, t + c)$. Deduce that if $y(t) = (x(t), f(t))$ solves the IVP $\frac{dy}{dt} = \tilde{X}(y)$, $y(t_0) = (x_0, t_0)$, then $x(t)$ solves $\frac{dx}{dt} = X(x, t)$, $x(t_0) = x_0$.

This may seem like a swindle—we don't seem to have done much beyond coalescing the original time variable t with the space variables, i.e., we have switched from a space + time description to a space-time descrition. But there is another important difference, namely $\tilde{X}$ takes values in $V \times \mathbf{R}$. In any case, this is a true reduction of the non-autonomous case to the autonomous case, and it is important, since autonomous differential equations have special properties that make them easier to study. Here is one such special property of autonomous systems.

8.2.2 Proposition. *If $x : (a, b) \to V$ is any solution of the autonomous differentiable equation $\frac{dx}{dt} = X(x)$ and $t_0 \in \mathbf{R}$, then $y : (a + t_0, b + t_0) \to V$ defined by $y(t) = x(t - t_0)$ is also a solution of the same equation.*

▷ **8.2—Exercise 4.** Prove the above Proposition.

Consequently, when considering the IVP for an autonomous differentiable equation we can assume that $t_0 = 0$. For if $x(t)$ is a solution with $x(0) = p$, then $x(t - t_0)$ will be a solution with $x(t_0) = p$.

8.2.3 Remark. There is another trick that allows us to reduce the study of higher order differential equations to the case of first order equations. Consider the second order differential equation: $\frac{d^2x}{dt^2} = f(x, \frac{dx}{dt}, t)$. Introduce a new variable v, (the velocity) and consider the following related system of first order equations: $\frac{dx}{dt} = v$ and $\frac{dv}{dt} = f(x, v, t)$. It is pretty obvious there is a close relation between curves $x(t)$ satisfying $x''(t) = f(x(t), x'(t), t)$ and pairs of curves $x(t)$, $v(t)$ satisfying $x'(t) = v(t)$ and $v'(t) = f(x(t), v(t), t)$.

▷ **8.2—Exercise 5.** Define the notion of an initial value problem for the above second order differential equation, and write a careful statement of the relation between solutions of this initial value problem and the initial value problem for the related system of first order differential equations.

We will now look more closely at the uniqueness question for solutions of an initial value problem. The answer is summed up succinctly in the following result.

8.2.4 Maximal Solution Theorem. *Let $\frac{dx}{dt} = X(x)$ be an autonomous differential equation in V and p any point of V. Among all solutions $x(t)$ of the equation that satisfy the initial condition $x(0) = p$, there is a maximum one, σ_p, in the sense that any solution of this IVP is the restriction of σ_p to some interval containing zero.*

▷ **8.2—Exercise 6.** If you know about connectedness you should be able to prove this very easily. First, using the local uniqueness theorem, show that any two solutions agree on the intersection of their domains. Then define σ_p to be the union of all solutions.

Henceforth whenever we are considering some autonomous differential equation, σ_p will denote this maximal solution curve with initial condition p. The interval on which σ_p is defined will be denoted by $(\alpha(p), \omega(p))$, where of course $\alpha(p)$ is either $-\infty$ or a negative real number, and $\omega(p)$ is either ∞ or a positive real number.

We have seen that the maximal solution need **not** be defined on all of $\mathbf{R}$, and it is important to know just how the solution "blows up" as t approaches a finite endpoint of its interval of definition. *A priori* it might seem that the solution could remain in some bounded region, but it is an important fact that this is impossible—if $\omega(p)$ is finite then the reason the solution cannot be continued past $\omega(p)$ is simply that $\sigma(t)$ escapes to infinity (in the sense that $\|\sigma_p(t)\| \to \infty$) as t approaches $\omega(p)$.

8.2.5 No Bounded Escape Theorem. *If $\omega(p) < \infty$ then $\lim_{t\to\omega(p)} \|\sigma_p(t)\| = \infty$, and similarly, if $\alpha(p) > -\infty$ then $\lim_{t\to\alpha(p)} \|\sigma_p(t)\| = \infty$.*

$\triangleright$ **8.2—Exercise 7.** Prove the No Bounded Escape Theorem.
(Hint: If $\lim_{t\to\omega(p)} \|\sigma(p)\| \neq \infty$, then by Bolzano-Weierstrass there would be a sequence t_k converging to $\omega(p)$ from below, such that $\sigma_p(t_k) \to q$. Then use the local existence theorem around q to show that you could extend the solution beyond $\omega(p)$. Here is where we get to use the fact there is a neighborhood O of q such that a solution exists with any initial condition q' in O and defined **on the whole interval** $(-\epsilon, \epsilon)$. For k sufficiently large, we will have both $\sigma_p(t_k)$ in O and $t_k > \omega - \epsilon$, which quickly leads to a contradiction.)

Here is another interesting and important special properties of autonomous systems.

$\triangleright$ **8.2—Exercise 8.** Show that the images of the σ_p partition V into disjoint smooth curves (the "streamlines" of smoke particles). These curves are referred to as the *orbits* of the ODE. (Hint: If $x(t)$ and $\xi(t)$ are two solutions of the same autonomous ODE and if $x(t_0) = \xi(t_1)$ then show that $x(t_0 + s) = \xi(t_1 + s)$.)

$\triangleright$ **8.2—Exercise 9.** Show that if the vector field $X : V \to V$ is C^k then each maximal solution $\sigma_p : (\alpha_p, \omega_p) \to V$ is C^{k+1}. (Hint: To begin with we know that σ_p is differentiable and hence continuous, so $t \mapsto X(\sigma_p(t))$ is at least continuous. Now use the relation $\sigma_p'(t) = X(\sigma_p(t))$ to argue by induction.)

8.2.6 Remark. In the next section we will discuss the smoothness of $\sigma_p(t)$ as a function of p and t jointly, and see that it is of class C^k.

8.2.7 Definition. A C^k vector field $X : V \to V$ (and also the autonomous differential equation $\frac{dx}{dt} = X(x)$) is called *complete* if $\alpha(p) = -\infty$ and $\omega(p) = \infty$ for all p in V. In this case, for each $t \in \mathbf{R}$ we define a map $\phi_t : V \to V$ by $\phi_t(p) = \sigma_p(t)$. The mapping $t \mapsto \phi_t$ is called the *flow* generated by the differential equation $\frac{dx}{dt} = X(x)$.

8.2.8 Remark. Using our smoke particle metaphor, the meaning of ϕ_t can be explained as follows: if a puff of smoke occupies a region U at a given time, then t units of time later it will occupy the region $\phi_t(U)$. Note that ϕ_0 is clearly the identity mapping of $\mathbf{R}^n$

$\triangleright$ **8.2—Exercise 10.** Show that the ϕ_t satisfy $\phi_{t_1+t_2} = \phi_{t_1}\phi_{t_2}$, so that in particular $\phi_{-t} = \phi_t^{-1}$. By the next section, each ϕ_t is actually a "diffeomorphism" of V, i.e., both it and its inverse are C^1. So the flow generated by a complete, autonomous C^1 vector field is a homomorphism of the additive group of real numbers into the group of diffeomorphisms of V.

8.3 Smoothness of Solutions of ODE.

As we saw above, it is an easy exercise to show that each solution of the differential equation $\frac{dx}{dt} = X(x)$ will be C^{k+1} if X is C^k. But in practice it is also important to know how solutions of an ODE depend on initial conditions. Also, if we have a family of $X(x, \alpha)$ of vector fields depending on some parameters $\alpha = (\alpha_1, \ldots, \alpha_k) \in \mathbf{R}^k$, we would like to know how the solutions of the corresponding IVP depend on these parameters. The following theorem answers all these questions. We will not give the proof here. It can be found in any good textbook on ODE. (One excellent source is *Differential Equations, Dynamical Systems, and Linear Algebra* by Morris Hirsch and Stephen Smale. See in particular Chapter 15.)

Theorem. *Let $X : V \times \mathbf{R}^k \to V$ be a C^k map. We regard $X(x, \alpha)$ as a family of vector fields on V depending on the parameter $\alpha \in \mathbf{R}^k$, and denote by $t \mapsto \sigma_p^\alpha(t)$ the maximal solution of the ODE $\frac{dx}{dt} = X(x, \alpha)$ with $\sigma_p^\alpha(0) = p$. Then the map $(p, t, \alpha) \mapsto \sigma_p^\alpha(t)$ is of class C^k.*