LECTURE NOTES ON CURVES AND SURFACES IN $\mathbf{R}^3$

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1 Curves

Definition 1 *A smooth parametrized curve in $\mathbf{R}^3$ is a map $\boldsymbol{\gamma} : I \to \mathbf{R}^3$ from an open interval $I \subseteq \mathbf{R}$ such that*

- $\boldsymbol{\gamma}(t) = (x(t), y(t), z(t))$ *has derivatives of all orders*
- $\boldsymbol{\gamma}'(t) \neq 0$ *for $t \in I$*

Examples:

1) A straight line:

$$\boldsymbol{\gamma}(t) = \mathbf{a} + t\mathbf{b}, \qquad (\mathbf{b} \neq 0).$$

2) The same straight line:

$$\boldsymbol{\gamma}(u) = \mathbf{a} + \tan u\, \mathbf{b}, \qquad (\mathbf{b} \neq 0)$$

but with a different parametrization.

3) A circle of radius r in the (x, y) plane:

$$\boldsymbol{\gamma}(t) = r(\cos t\, \mathbf{i} + \sin t\, \mathbf{j}).$$

Geometrically, we are interested in the curve which is the image of $\boldsymbol{\gamma}$. The parametrization is not unique. We are allowed to change a parametrization by a function $f : I_1 \to I_2$ which has derivatives of all orders and has an inverse with the same property. The chain rule gives

$$\frac{d}{dt}\boldsymbol{\gamma}(f(t)) = \boldsymbol{\gamma}'(f(t))f'(t).$$

Since f has a differentiable inverse, $f'(t) \neq 0$, so both parts of Definition 1 are consistent with a change of parametrization.

Definition 2 *Let $\boldsymbol{\gamma}$ be a smooth parametrized curve and $t_0 \in I$. Then the arc length $s(t)$ of $\boldsymbol{\gamma}$ from t_0 to t is*

$$s(t) = \int_{t_0}^{t} |\boldsymbol{\gamma}'(u)|du.$$

Note that if $u = f(v)$, then

$$|\boldsymbol{\gamma}'(u)|du = |\boldsymbol{\gamma}'(f(v))|f'(v)dv = |\frac{d}{dv}\boldsymbol{\gamma}(f(v))|dv$$

if $f'(v) > 0$, which means that the integral in Definition 2 is independent of the choice of parametrization. Arc length $s(t)$ is itself a parametrization since

$$\frac{ds}{dt} = |\boldsymbol{\gamma}'(t)| \neq 0$$

by the definition of s. We shall often use it as a *theoretical convenience*. In concrete problems it is rarely worth calculating it.

Definition 3 *Let $\boldsymbol{\gamma} : I \to \mathbf{R}^3$ be a smooth curve parametrized by arc length. Then*

$$\mathbf{t}(s) = \frac{d\boldsymbol{\gamma}}{ds}$$

is the unit tangent vector at the point s.

Note that by definition of arc length

$$\frac{ds}{dt} = |\boldsymbol{\gamma}'(t)|$$

so if $t = s$, $\boldsymbol{\gamma}'(s)$ certainly is a unit vector.

Definition 4 *The curvature* κ *of* $\boldsymbol{\gamma}(s)$ *is*

$$\kappa(s) = |\mathbf{t}'|.$$

Example: Consider the circle of radius r in the x, y plane:

$$\boldsymbol{\gamma}(t) = r(\cos t\,\mathbf{i} + \sin t\,\mathbf{j}).$$

Then $\boldsymbol{\gamma}'(t) = r(-\sin t\,\mathbf{i} + \cos t\,\mathbf{j})$ and $\boldsymbol{\gamma}' \cdot \boldsymbol{\gamma}' = r^2$ which means that

$$\frac{ds}{dt} = r$$

and so arc length $s = rt$. Thus

$$\mathbf{t} = -\sin t\,\mathbf{i} + \cos t\,\mathbf{j}$$

and

$$\mathbf{t}' = \frac{1}{r}(-\cos t\,\mathbf{i} - \sin t\,\mathbf{j})$$

giving

$$\kappa = \frac{1}{r}.$$

So for a circle the curvature κ is the inverse of the radius.

Definition 5 *If* $\kappa(s) \neq 0$*, the normal to* $\boldsymbol{\gamma}$ *at* s *is the unit vector* $\mathbf{n}$ *determined by*

$$\mathbf{t}' = \kappa\mathbf{n}.$$

Since $\mathbf{t}$ is a unit vector, $\mathbf{t} \cdot \mathbf{t} = 1$ so differentiating gives $\mathbf{t}' \cdot \mathbf{t} = 0$ and we see that

$$\mathbf{t} \cdot \mathbf{n} = 0.$$

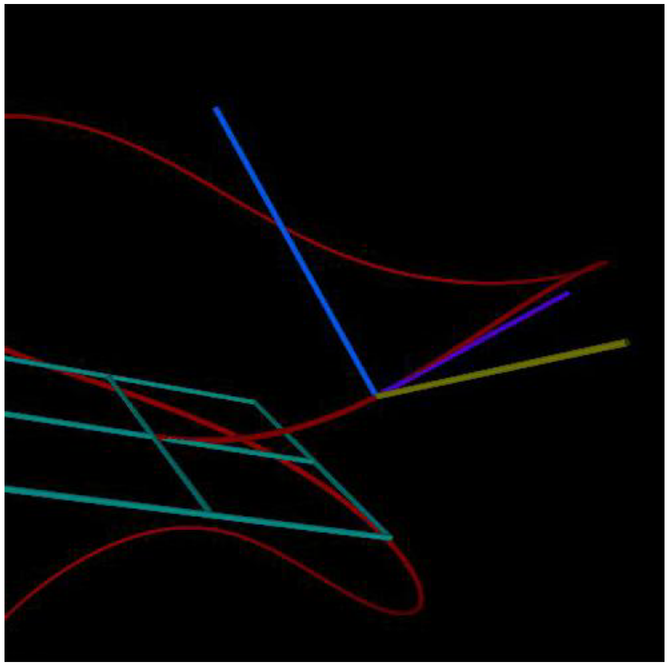
Definition 6 *The binormal* $\mathbf{b}$ *at* s *is the unit vector defined by*

$$\mathbf{b} = \mathbf{t} \wedge \mathbf{n}$$

The triple of unit vectors

$$(\mathbf{t}(s), \mathbf{n}(s), \mathbf{b}(s))$$

defines an oriented orthonormal basis of $\mathbf{R}^3$ at each point of the curve:



Since $\mathbf{n}$ and $\mathbf{b}$ are unit vectors, $\mathbf{n}'$ is orthogonal to $\mathbf{n}$ and $\mathbf{b}'$ is orthogonal to $\mathbf{b}$, but also

$$\mathbf{b}' = (\mathbf{t}\wedge\mathbf{n})' = \mathbf{t}'\wedge\mathbf{n} + \mathbf{t}\wedge\mathbf{n}'.$$

By definition of the normal $\mathbf{t}' = \kappa\mathbf{n}$ so this gives $\mathbf{b}' = \mathbf{t}\wedge\mathbf{n}'$ which means that $\mathbf{b}'$ is orthogonal to $\mathbf{t}$. Since it is also orthogonal to $\mathbf{b}$ it is parallel to $\mathbf{n} = \mathbf{b}\wedge\mathbf{t}$. This gives us

Definition 7 *The torsion τ of a curve $\boldsymbol{\gamma}$ at s is defined by*

$$\mathbf{b}' = -\tau\mathbf{n}.$$

As s varies along the curve, the orthogonal frame $(\mathbf{t}, \mathbf{n}, \mathbf{b})$ varies. The Serret-Frenet formula says how:

Theorem 1 *(Serret-Frenet)*

$$\frac{d}{ds}\begin{pmatrix}\mathbf{t}\\ \mathbf{n}\\ \mathbf{b}\end{pmatrix} = \begin{pmatrix} & \kappa\mathbf{n} & \\ -\kappa\mathbf{t} & & +\tau\mathbf{b}\\ & -\tau\mathbf{n} & \end{pmatrix}$$

Writing it this way helps you to remember the formula, which is of the form

$$\frac{d}{ds}\begin{pmatrix}\mathbf{t}\\ \mathbf{n}\\ \mathbf{b}\end{pmatrix} = A\begin{pmatrix}\mathbf{t}\\ \mathbf{n}\\ \mathbf{b}\end{pmatrix}$$

for the skew-symmetric matrix

$$A = \begin{pmatrix} 0 & \kappa & 0\\ -\kappa & 0 & \tau\\ 0 & -\tau & 0\end{pmatrix}$$

The matrix is also one way of proving the theorem:

Proof: Since $\{\mathbf{t}, \mathbf{n}, \mathbf{b}\}$ is an orthonormal basis, the matrix P with rows $\mathbf{t}, \mathbf{n}, \mathbf{b}$ is orthogonal, that is

$$PP^T = P^TP = I \tag{1}$$

and we want to find P'.

By orthogonality P' satisfies $P' = P'P^TP$ which with $A = P'P^T$ can be written

$$P' = AP.$$

Differentiating (1) gives

$$P'P^T + PP'^T = A + A^T = 0$$

so A is skew symmetric. By the definitions of curvature and torsion $\mathbf{t}' = \kappa\mathbf{n}$ and $\mathbf{b}' = -\tau\mathbf{n}$ so skew symmetry gives the missing line

$$\mathbf{n}' = -\kappa\mathbf{t} + \tau\mathbf{b}$$

in P'. □

The meaning of curvature and torsion is best expressed in the following proposition:

Proposition 2 *Let* $\boldsymbol{\gamma}$ *be a smooth curve in* $\mathbf{R}^3$*, then*

- $\kappa \equiv 0$ *if and only if the curve is part of a straight line*
- $\tau \equiv 0$ *if and only if the curve is lies in a plane.*

Proof: If the curvature $\kappa \equiv 0$ then from Definition 4 we have $\mathbf{t}' = 0$, so

$$\frac{d\boldsymbol{\gamma}}{ds} = \mathbf{t} = \mathbf{a}$$

for a constant vector $\mathbf{a}$. Integrating,

$$\boldsymbol{\gamma}(s) = s\mathbf{a} + \mathbf{c}$$

which is a line. The converse is clear.

If the torsion $\tau \equiv 0$, then from Definition 7, $\mathbf{b}$ is a constant. Thus

$$\frac{d}{ds}(\boldsymbol{\gamma} \cdot \mathbf{b}) = \mathbf{t} \cdot \mathbf{b} = 0$$

by orthogonality. This means

$$\boldsymbol{\gamma} \cdot \mathbf{b} = k$$

for a constant k so that the curve lies in the plane $\mathbf{r} \cdot \mathbf{b} = k$. Conversely, if $\boldsymbol{\gamma}$ lies in a plane, then $\mathbf{t}$ and $\mathbf{t}' = \kappa\mathbf{n}$ are parallel to the plane. Then $\mathbf{b} = \mathbf{t}\wedge\mathbf{n}$ is orthogonal to the plane and hence constant, $\mathbf{b}' = 0$ and $\tau = 0$. □

Since you don't always want to work out arc length for a curve here is an exercise to calculate the curvature and torsion for a general parametrization:

Exercise 3 *Let $\boldsymbol{\gamma}(t)$ be a curve in $\mathbf{R}^3$ with an arbitrary parametrization. Show that*

$$\kappa = \frac{|\boldsymbol{\gamma}'\wedge\boldsymbol{\gamma}''|}{|\boldsymbol{\gamma}'|^3} \qquad \tau = \frac{(\boldsymbol{\gamma}'\wedge\boldsymbol{\gamma}'') \cdot \boldsymbol{\gamma}'''}{|\boldsymbol{\gamma}'\wedge\boldsymbol{\gamma}''|^2}$$

And here is an application of Serret-Frenet:

Exercise 4 *Let $\boldsymbol{\gamma}(t)$ be a curve in $\mathbf{R}^3$. If $\tau = 0$ and κ is a constant, show that $\boldsymbol{\gamma}$ is part of a circle.*

Answer: Since $\tau = 0$ the curve lies in a plane by Proposition 2. Also Serret-Frenet gives

$$\mathbf{n}' = -\kappa\mathbf{t} + \tau\mathbf{b} = -\kappa\mathbf{t} = -\kappa\boldsymbol{\gamma}'.$$

Since κ is constant, integrate to get

$$\frac{1}{\kappa}\mathbf{n} + \boldsymbol{\gamma} = \mathbf{c}$$

and hence, since $\mathbf{n}$ is a unit vector,

$$\|\boldsymbol{\gamma} - \mathbf{c}\| = \frac{1}{\kappa}.$$

Thus $\boldsymbol{\gamma}$ is part of the intersection of a plane and a sphere, and so is a circle. □

2 Surfaces

Definition 8 *A smooth parametrized surface in* $\mathbf{R}^3$ *is a map* $\mathbf{r} : U \to \mathbf{R}^3$ *from an open set* $U \subseteq \mathbf{R}^2$ *such that*

- $\mathbf{r}(u, v) = (x(u, v), y(u, v), z(u, v))$ *has derivatives of all orders*
- *at each point* $\mathbf{r}_u = \partial\mathbf{r}/\partial u$ *and* $\mathbf{r}_v = \partial\mathbf{r}/\partial v$ *are linearly independent.*

Examples:

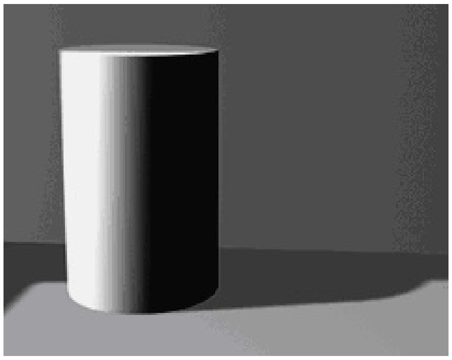
1) A plane:

$$\mathbf{r}(u, v) = \mathbf{a} + u\mathbf{b} + v\mathbf{c}$$

for constant vectors $\mathbf{a}, \mathbf{b}, \mathbf{c}$ where $\mathbf{b}, \mathbf{c}$ are linearly independent.

2) A cylinder:

$$\mathbf{r}(u, v) = a(\cos v\,\mathbf{i} + \sin v\,\mathbf{j}) + u\mathbf{k}$$

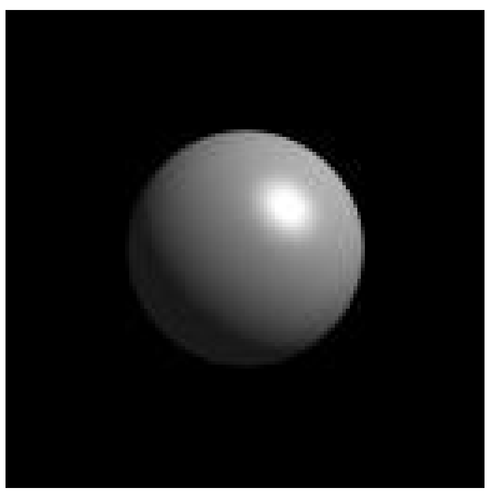


3) A cone:

$$\mathbf{r}(u,v) = au\cos v\,\mathbf{i} + au\sin v\,\mathbf{j} + u\mathbf{k}$$

4) A sphere:

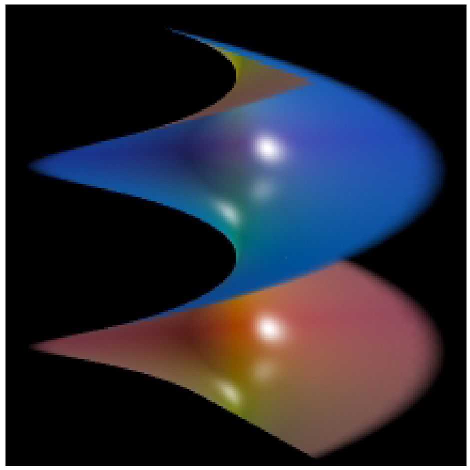
$$\mathbf{r}(u,v) = a\sin u\sin v\,\mathbf{i} + a\cos u\sin v\,\mathbf{j} + a\cos v\,\mathbf{k}$$



Note that the condition on the partial derivatives of $\mathbf{r}$ does not hold for all values of (u,v) – we need to use different parametrizations sometimes.

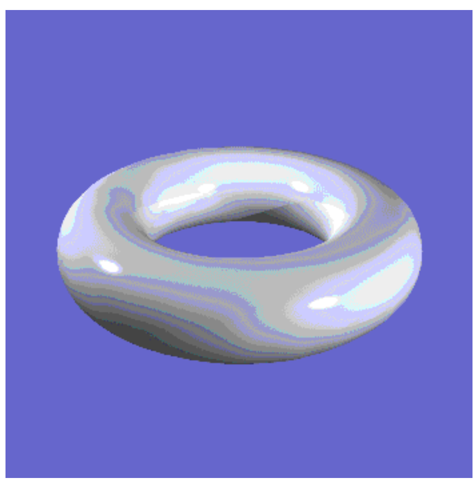
5) A helicoid:

$$\mathbf{r}(u,v) = au\cos v\,\mathbf{i} + au\sin v\,\mathbf{j} + v\mathbf{k}$$



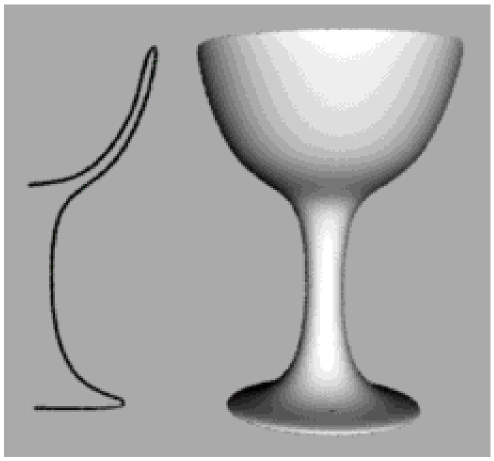
6) A torus:

$$\mathbf{r}(u,v) = (a + b\cos u)(\cos v\,\mathbf{i} + \sin v\,\mathbf{j}) + b\sin u\,\mathbf{k}$$



7) A surface of revolution:

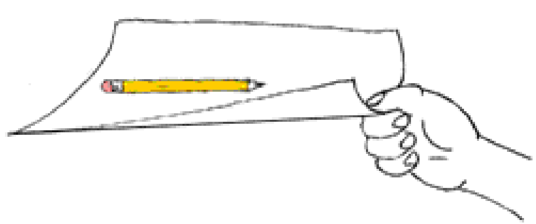
$$\mathbf{r}(u, v) = f(u)(\cos v\,\mathbf{i} + \sin v\,\mathbf{j}) + u\mathbf{k}$$



8) A developable surface: take a curve $\boldsymbol{\gamma}(u)$ parametrized by arc length and set

$$\mathbf{r}(u, v) = \boldsymbol{\gamma}(u) + v\mathbf{t}(u).$$

This is the surface formed by bending a piece of paper:



Here $\mathbf{r}_u = \mathbf{t} + v\kappa\mathbf{n}$ and $\mathbf{r}_v = \mathbf{t}$ so we need $\kappa \neq 0$ and $v \neq 0$ to satisfy the second condition for a parametrized surface.

A change of parametrization of a surface is the composition

$$\mathbf{r} \circ f : U' \to \mathbf{R}^3$$

where $f : U' \to U$ is a *diffeomorphism* – an invertible map such that f and f^{-1} have derivatives of all orders. Note that if

$$f(x, y) = (u(x, y), v(x, y))$$

then by the chain rule

$$\begin{aligned} (\mathbf{r} \circ f)_x &= \mathbf{r}_u u_x + \mathbf{r}_v v_x \\ (\mathbf{r} \circ f)_y &= \mathbf{r}_u u_y + \mathbf{r}_v v_y \end{aligned}$$

so

$$\begin{pmatrix} (\mathbf{r} \circ f)_x \\ (\mathbf{r} \circ f)_y \end{pmatrix} = \begin{pmatrix} u_x & v_x \\ u_y & v_y \end{pmatrix} \begin{pmatrix} \mathbf{r}_u \\ \mathbf{r}_v \end{pmatrix}.$$

Since f has a differentiable inverse, the Jacobian matrix is invertible, so $(\mathbf{r} \circ f)_x$ and $(\mathbf{r} \circ f)_y$ are linearly independent if $\mathbf{r}_u, \mathbf{r}_v$ are.

Example: The (x, y) plane

$$\mathbf{r}(x, y) = x\mathbf{i} + y\mathbf{j}$$

has a different parametrization in polar coordinates

$$\mathbf{r} \circ f(r, \theta) = r\cos\theta\,\mathbf{i} + r\sin\theta\,\mathbf{j}.$$

Unlike curves, there is no natural parametrization like arc length for a surface.

Definition 9 *The tangent plane (or* tangent space*) of a surface at the point $P = (u, v)$ is the vector space spanned by $\mathbf{r}_u, \mathbf{r}_v$.*

Note that this space is independent of parametrization. One should think of the origin of the vector space as the point P.

Definition 10 *The vectors*

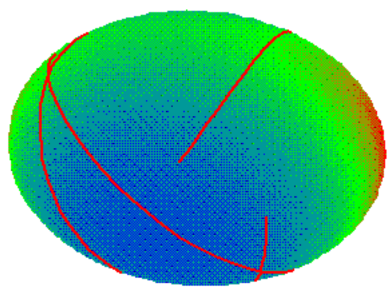
$$\pm\frac{\mathbf{r}_u \wedge \mathbf{r}_v}{|\mathbf{r}_u \wedge \mathbf{r}_v|}$$

are the two unit normals ("inward and outward") to the surface at (u, v).

3 The first fundamental form

Definition 11 *A smooth curve lying in the surface S is a map $t \mapsto (u(t), v(t))$ with derivatives of all orders such that $\boldsymbol{\gamma}(t) = \mathbf{r}(u(t), v(t))$ is a parametrized curve.*

Recall that to be a parametrized curve means that $u(t), v(t)$ have derivatives of all orders and $\boldsymbol{\gamma}' = \mathbf{r}_u u' + \mathbf{r}_v v' \neq 0$. The definition of a surface implies that $\mathbf{r}_u, \mathbf{r}_v$ are linearly independent, so this condition is equivalent to $(u', v') \neq 0$.



Recall the Definition 2 for the arclength from $t = a$ to $t = b$ of a curve in $\mathbf{R}^3$ and apply it to a curve on the surface S:

$$\begin{aligned}\int_a^b |\boldsymbol{\gamma}'(t)| dt &= \int_a^b \sqrt{\boldsymbol{\gamma}' \cdot \boldsymbol{\gamma}'} dt \\ &= \int_a^b \sqrt{(\mathbf{r}_u u' + \mathbf{r}_v v') \cdot (\mathbf{r}_u u' + \mathbf{r}_v v')} dt \\ &= \int_a^b \sqrt{E u'^2 + 2F u' v' + G v'^2} dt\end{aligned}$$

where

$$E = \mathbf{r}_u \cdot \mathbf{r}_u, \quad F = \mathbf{r}_u \cdot \mathbf{r}_v, \quad F = \mathbf{r}_v \cdot \mathbf{r}_v.$$

Definition 12 *the first fundamental form of a parametrized surface is the expression*

$$E du^2 + 2F du dv + G dv^2$$

where $E = \mathbf{r}_u \cdot \mathbf{r}_u, F = \mathbf{r}_u \cdot \mathbf{r}_v, G = \mathbf{r}_v \cdot \mathbf{r}_v$.

The first fundamental form is just the quadratic form

$$Q(\mathbf{v}, \mathbf{v}) = \mathbf{v} \cdot \mathbf{v}$$

on the tangent space written in terms of the basis $\mathbf{r}_u, \mathbf{r}_v$. It is represented in this basis by the symmetric matrix

$$\begin{pmatrix} E & F \\ F & G \end{pmatrix}.$$

So why do we write it as $Edu^2 + 2Fdudv + Gdv^2$? At this stage it is not worth worrying about what exactly du^2 is, instead let's see how the terminology helps to manipulate the formulas.

For example, to find the length of a curve $u(t), v(t)$ on the surface, we calculate

$$\int \sqrt{E\left(\frac{du}{dt}\right)^2 + 2F\frac{du}{dt}\frac{dv}{dt} + G\left(\frac{dv}{dt}\right)^2}\,dt$$

– divide the first fundamental form by dt^2 and multiply its square root by dt.

Furthermore if we change the parametrization of the surface via $u(x, y), v(x, y)$ and try to find the length of the curve $(x(t), y(t))$ then from first principles we would calculate

$$u' = u_x x' + u_y y' \qquad v' = v_x x' + v_y y'$$

by the chain rule and then

$$\begin{aligned} Eu'^2 + 2Fu'v' + Gv'^2 &= E(u_x x' + u_y y')^2 + 2F(u_x x' + u_y y')(v_x x' + \dots \\ &= (Eu_x^2 + 2Fu_x v_x + Gv_x^2)x'^2 + \dots \end{aligned}$$

which is heavy going. Instead, using du, dv etc. we just write

$$\begin{aligned} du &= u_x dx + u_y dy \\ dv &= v_x dx + v_y dy \end{aligned}$$

and substitute in

$$Edu^2 + 2Fdudv + Gdv^2.$$

Example: For the plane

$$\mathbf{r}(x, y) = x\mathbf{i} + y\mathbf{j}$$

we have $\mathbf{r}_x = \mathbf{i}, \mathbf{r}_y = \mathbf{j}$ and so the first fundamental form is

$$dx^2 + dy^2.$$

Now change to polar coordinates $x = r\cos\theta, y = r\sin\theta$. We have

$$\begin{aligned} dx &= dr\cos\theta - r\sin\theta d\theta \\ dy &= dr\sin\theta + r\cos\theta d\theta \end{aligned}$$

so that

$$dx^2 + dy^2 = (dr\cos\theta - r\sin\theta d\theta)^2 + (dr\sin\theta + r\cos\theta)^2 = dr^2 + r^2 d\theta^2$$

Here are some examples of first fundamental forms:

Examples:

1. The cylinder

$$\mathbf{r}(u,v) = a(\cos v\,\mathbf{i} + \sin v\,\mathbf{j}) + u\mathbf{k}.$$

We get

$$\mathbf{r}_u = \mathbf{k}, \quad \mathbf{r}_v = a(-\sin v\,\mathbf{i} + \cos v\,\mathbf{j})$$

so

$$E = \mathbf{r}_u \cdot \mathbf{r}_u = 1, \quad F = \mathbf{r}_u \cdot \mathbf{r}_v = 0, \quad G = \mathbf{r}_v \cdot \mathbf{r}_v = a^2$$

giving

$$\boxed{du^2 + a^2 dv^2}$$

2. The cone

$$\mathbf{r}(u,v) = a(u\cos v\,\mathbf{i} + u\sin v\,\mathbf{j}) + u\mathbf{k}.$$

Here

$$\mathbf{r}_u = a(\cos v\,\mathbf{i} + \sin v\,\mathbf{j}) + \mathbf{k}, \quad \mathbf{r}_v = a(-u\sin v\,\mathbf{i} + u\cos v)$$

so

$$E = \mathbf{r}_u \cdot \mathbf{r}_u = 1 + a^2, \quad F = \mathbf{r}_u \cdot \mathbf{r}_v = 0, \quad G = \mathbf{r}_v \cdot \mathbf{r}_v = a^2 u^2$$

giving

$$\boxed{(1+a^2)du^2 + a^2u^2dv^2}$$

3. The sphere

$$\mathbf{r}(u,v) = a\sin u\sin v\,\mathbf{i} + a\cos u\sin v\,\mathbf{j} + a\cos v\,\mathbf{k}$$

gives

$$\mathbf{r}_u = a\cos u \sin v\,\mathbf{i} - a\sin u\sin v\,\mathbf{j}, \quad \mathbf{r}_v = a\sin u\cos v\,\mathbf{i} + a\cos u\cos v\,\mathbf{j} - a\sin v\,\mathbf{k}$$

so that

$$E = \mathbf{r}_u\cdot\mathbf{r}_u = a^2\sin^2 v, \quad F = \mathbf{r}_u\cdot\mathbf{r}_v = 0, \quad G = \mathbf{r}_v\cdot\mathbf{r}_v = a^2$$

and so we get the first fundamental form

$$\boxed{a^2dv^2 + a^2\sin^2 v du^2}$$

4. A surface of revolution

$$\mathbf{r}(u,v) = f(u)(\cos v\,\mathbf{i} + \sin v\,\mathbf{j}) + u\mathbf{k}$$

has

$$\mathbf{r}_u = f'(u)(\cos v\,\mathbf{i} + \sin v\,\mathbf{j}) + \mathbf{k}, \quad \mathbf{r}_v = f(u)(-\sin v\,\mathbf{i} + \cos v\,\mathbf{j})$$

so that

$$E = \mathbf{r}_u\cdot\mathbf{r}_u = 1 + f'(u)^2, \quad F = \mathbf{r}_u\cdot\mathbf{r}_v = 0, \quad G = \mathbf{r}_v\cdot\mathbf{r}_v = f(u)^2$$

gives

$$\boxed{(1+f(u)'^2)du^2 + f(u)^2dv^2}$$

5. A developable surface

$$\mathbf{r}(u,v) = \boldsymbol{\gamma}(u) + v\mathbf{t}(u).$$

here the curve is parametrized by arc length $u = s$ so that

$$\mathbf{r}_u = \mathbf{t} + v\kappa\mathbf{n}, \quad \mathbf{r}_v = \mathbf{t}$$

and this gives

$$\boxed{(1+v^2\kappa^2)du^2 + 2dudv + dv^2}$$

We introduced the first fundamental form to measure lengths of curves on a surface but it does more besides. Firstly if two curves $\boldsymbol{\gamma}_1, \boldsymbol{\gamma}_2$ on the surface intersect, the angle θ between them is given by

$$\cos\theta = \frac{\boldsymbol{\gamma}_1'\cdot\boldsymbol{\gamma}_2'}{|\boldsymbol{\gamma}_1'||\boldsymbol{\gamma}_2'|} \tag{2}$$

But $\boldsymbol{\gamma}_i' = \mathbf{r}_u u_i' + \mathbf{r}_v v_i'$ so

$$\begin{aligned}\boldsymbol{\gamma}_i' \cdot \boldsymbol{\gamma}_j' &= (\mathbf{r}_u u_i' + \mathbf{r}_v v_i') \cdot (\mathbf{r}_u u_j' + \mathbf{r}_v v_j') \\ &= E u_i' u_j' + F(u_i' v_j' + u_j' v_i') + G v_i' v_j'\end{aligned}$$

and each term in (2) can be expressed in terms of the curves and the coefficients of the first fundamental form.

We can also define *area* using the first fundamental form:

Definition 13 *The area of the domain $\mathbf{r}(U) \subset \mathbf{R}^3$ in a surface is defined by*

$$\int_U |\mathbf{r}_u \wedge \mathbf{r}_v| du dv = \int_U \sqrt{EG - F^2} du dv.$$

The second form of the formula comes from the identity

$$|\mathbf{r}_u \wedge \mathbf{r}_v|^2 = (\mathbf{r}_u \cdot \mathbf{r}_u)(\mathbf{r}_v \cdot \mathbf{r}_v) - (\mathbf{r}_u \cdot \mathbf{r}_v)^2 = EG - F^2.$$

Note that the definition of area is independent of parametrization for if

$$\mathbf{r}_x = \mathbf{r}_u u_x + \mathbf{r}_v v_x, \quad \mathbf{r}_y = \mathbf{r}_u u_y + \mathbf{r}_v v_y$$

then

$$\mathbf{r}_x \wedge \mathbf{r}_y = (u_x v_y - v_x u_y) \mathbf{r}_u \wedge \mathbf{r}_v$$

so that

$$\int_U |\mathbf{r}_x \wedge \mathbf{r}_y| dx dy = \int_U |\mathbf{r}_u \wedge \mathbf{r}_v| |u_x v_y - v_x u_y| dx dy = \int_U |\mathbf{r}_u \wedge \mathbf{r}_v| du dv$$

using the formula for change of variables in multiple integration.

Example: Consider a surface of revolution

$$(1 + f'(u)^2) du^2 + f(u)^2 dv^2$$

and the area between $u = a, u = b$. We have

$$EG - F^2 = f(u)^2 (1 + f'(u)^2)$$

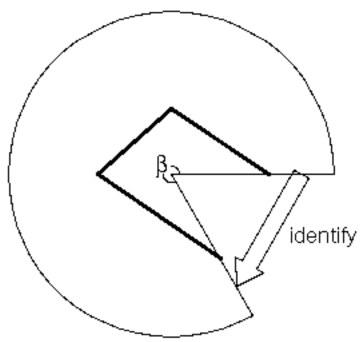
so the area is

$$\int_a^b f(u) \sqrt{1 + f'(u)^2} du dv = 2\pi \int_a^b f(u) \sqrt{1 + f'(u)^2} du.$$

4 Isometric surfaces

Definition 14 *Two surfaces S, S' defined by $\mathbf{r} : U \to \mathbf{R}^3$ and $\mathbf{r}' : U' \to \mathbf{R}^3$ are isometric if there is an invertible map $f : U \to U'$ such that f and f^{-1} have derivatives of all orders, and which maps curves in S to curves in S' of the same length.*

A practical example of this is to take a piece of paper and bend it: the lengths of curves in the paper do not change. The cone and plane are isometric this way:



Analytically this is how to tell if two surfaces are isometric;

Theorem 5 *The surfaces S and S' are isometric if and only if there exist parametrizations $\mathbf{r} : U \to \mathbf{R}^3$ and $\mathbf{r}' : U \to \mathbf{R}^3$ with the same first fundamental form.*

Proof: Suppose such a parametrization exists, then the identity map is an isometry since the first fundamental form determines the length of curves.

Conversely, suppose S, S' are isometric using the function $f : U \to U'$. Then

$$\mathbf{r}' \circ f : U \to \mathbf{R}^3, \qquad \mathbf{r} : U \to \mathbf{R}^3$$

are parametrizations using the same open set U, so the first fundamental forms are

$$\tilde{E}du^2 + 2\tilde{F}dudv + \tilde{G}dv^2, \quad Edu^2 + 2Fdudv + Gdv^2$$

and since f is an isometry

$$\int_I \sqrt{\tilde{E}u'^2 + 2\tilde{F}u'v' + \tilde{G}v'^2}dt = \int_I \sqrt{Eu'^2 + 2Fu'v' + Gv'^2}dt$$

for all curves $t \mapsto (u(t), v(t))$ and *all intervals*. Since

$$\frac{d}{dt}\int_a^{a+t} h(u)du = h(t)$$

this means that

$$\sqrt{\tilde{E}u'^2 + 2\tilde{F}u'v' + \tilde{G}v'^2} = \sqrt{Eu'^2 + 2Fu'v' + Gv'^2}$$

for all $u(t), v(t)$. So, choosing u, v appropriately:

$$\begin{aligned} u = t, v = a &\Rightarrow \tilde{E} = E \\ u = a, v = t &\Rightarrow \tilde{G} = G \\ u = t, v = t &\Rightarrow \tilde{F} = F \end{aligned}$$

and we have the same first fundamental form as required. □

Examples:

1) The cone has first fundamental form

$$(1 + a^2)du^2 + a^2u^2dv^2.$$

Put

$$r = \sqrt{1 + a^2}u$$

then we get

$$dr^2 + \left(\frac{a^2}{1 + a^2}\right) r^2dv^2$$

and now put

$$\theta = \sqrt{\frac{a^2}{1 + a^2}}v$$

to get the plane in polar coordinates

$$dr^2 + r^2d\theta^2.$$

Note that as $0 \leq v \leq 2\pi$, $0 \leq \theta \leq \beta$ where

$$\beta = \sqrt{\frac{a^2}{1 + a^2}}2\pi < 2\pi$$

as in the picture.

2) A developable surface has first fundamental form

$$(1+v^2\kappa^2)du^2+2dudv+dv^2.$$

Note that the curve $\boldsymbol{\gamma}$ that defines the surface only appears here through its curvature. The torsion is absent, so we can't tell whether this curve lies in the plane or not. If the curve does lie in the plane then the developable surface

$$\mathbf{r}(u,v)=\boldsymbol{\gamma}(u)+v\mathbf{t}(u)$$

is the plane.

What we have to show then is that given any function $f(s)$ there exists a plane curve with curvature $\kappa(s)=f(s)$, for then the first fundamental form is the same as that of the plane. This means solving a differential equation – if the curve is $s\mapsto x(s)\mathbf{i}+y(s)\mathbf{j}$ then

$$\mathbf{t}=x'\mathbf{i}+y'\mathbf{j}$$

and

$$\mathbf{n}=\pm(y'\mathbf{i}-x'\mathbf{j}).$$

The curvature satisfies $\mathbf{t}'=\kappa\mathbf{n}$ so given f we need to solve

$$x''\mathbf{i}+y''\mathbf{j}=f(y'\mathbf{i}-x'\mathbf{j})$$

or equivalently the first order system:

$$\begin{aligned}x' &= p\\ y' &= q\\ p' &= fq\\ q' &= -fp\end{aligned}$$

and the usual Cauchy existence theorem says we can do that.

Note that for any solution to this equation

$$x'x''+y'y''=f(x'y'-y'x')=0$$

so that $(x'^2+y'^2)'=0$ and $x'^2+y'^2=const$ which we can take to be 1. In other words the curve we get by solving the equation really is parametrized by arc length and hence its curvature κ is f.

5 The second fundamental form

The first fundamental form describes the intrinsic geometry of a surface – the experience of an insect crawling around it. The second fundamental form relates to the way the surface sits in $\mathbf{R}^3$.

First take a surface $\mathbf{r}(u, v)$ and push it inwards a distance t along its normal to get a one-parameter family of surfaces:

$$\mathbf{R}(u, v, t) = \mathbf{r}(u, v) - t\mathbf{n}(u, v)$$

with

$$\mathbf{R}_u = \mathbf{r}_u - t\mathbf{n}_u, \quad \mathbf{R}_v = \mathbf{r}_v - t\mathbf{n}_v.$$

We now have a first fundamental form $Edu^2 + 2Fdudv + Gdv^2$ depending on t and we calculate

$$\frac{1}{2}\frac{\partial}{\partial t}(Edu^2 + 2Fdudv + Gdv^2)|_{t=0} = -(\mathbf{r}_u \cdot \mathbf{n}_u du^2 + (\mathbf{r}_u \cdot \mathbf{n}_v + \mathbf{r}_v \cdot \mathbf{n}_u)dudv + \mathbf{r}_v \cdot \mathbf{n}_v dv^2).$$

The right hand side is the second fundamental form. From this point of view it is clearly the same type of object as the first fundamental form — a quadratic form on the tangent space.

In fact it is useful to give a slightly different expression. Since $\mathbf{n}$ is orthogonal to $\mathbf{r}_u$ and $\mathbf{r}_v$,

$$0 = (\mathbf{r}_u \cdot \mathbf{n})_u = \mathbf{r}_{uu} \cdot \mathbf{n} + \mathbf{r}_u \cdot \mathbf{n}_u$$

and similarly

$$\mathbf{r}_{uv} \cdot \mathbf{n} + \mathbf{r}_u \cdot \mathbf{n}_v = 0, \quad \mathbf{r}_{vu} \cdot \mathbf{n} + \mathbf{r}_v \cdot \mathbf{n}_u = 0$$

and since $\mathbf{r}_{uv} = \mathbf{r}_{vu}$ we have $\mathbf{r}_u \cdot \mathbf{n}_v = \mathbf{r}_v \cdot \mathbf{n}_u$. We then define:

Definition 15 *The second fundamental form of a surface is the expression*

$$Ldu^2 + 2Mdudv + Ndv^2$$

where $L = \mathbf{r}_{uu} \cdot \mathbf{n}, M = \mathbf{r}_{uv} \cdot \mathbf{n}, N = \mathbf{r}_{vv} \cdot \mathbf{n}$.

Examples:

1) The plane

$$\mathbf{r}(u, v) = \mathbf{a} + u\mathbf{b} + v\mathbf{c}$$

has $\mathbf{r}_{uu} = \mathbf{r}_{uv} = \mathbf{r}_{vv} = 0$ so the second fundamental form vanishes.

2) The sphere of radius a: here with the origin at the centre, $\mathbf{r} = a\mathbf{n}$ so

$$\mathbf{r}_u \cdot \mathbf{n}_u = a^{-1}\mathbf{r}_u \cdot \mathbf{r}_u, \quad \mathbf{r}_u \cdot \mathbf{n}_v = a^{-1}\mathbf{r}_u \cdot \mathbf{r}_v, \quad \mathbf{r}_v \cdot \mathbf{n}_v = a^{-1}\mathbf{r}_v \cdot \mathbf{r}_v$$

and

$$Ldu^2 + 2Mdudv + Ndv^2 = a^{-1}(Edu^2 + 2Fdudv + Gdv^2).$$

The plane is characterised by the vanishing of the second fundamental form:

Proposition 6 *If the second fundamental form of a surface vanishes, it is part of a plane.*

Proof: If the second fundamental form vanishes,

$$\mathbf{r}_u \cdot \mathbf{n}_u = 0 = \mathbf{r}_v \cdot \mathbf{n}_u = \mathbf{r}_u \cdot \mathbf{n}_v = \mathbf{r}_v \cdot \mathbf{n}_v$$

so that

$$\mathbf{n}_u = \mathbf{n}_v = 0$$

since $\mathbf{n}_u, \mathbf{n}_v$ are orthogonal to $\mathbf{n}$ and hence linear combinations of $\mathbf{r}_u, \mathbf{r}_v$. Thus $\mathbf{n}$ is constant. This means

$$(\mathbf{r} \cdot \mathbf{n})_u = \mathbf{r}_u \cdot \mathbf{n} = 0, \quad (\mathbf{r} \cdot \mathbf{n})_v = \mathbf{r}_v \cdot \mathbf{n} = 0$$

and so

$$\mathbf{r} \cdot \mathbf{n} = \mathit{const}$$

which is the equation of a plane. □

Consider now a surface given as the graph of a function $z = f(x, y)$:

$$\mathbf{r}(x, y) = x\mathbf{i} + y\mathbf{j} + f(x, y)\mathbf{k}.$$

Here

$$\mathbf{r}_x = \mathbf{i} + f_x\mathbf{k}, \qquad \mathbf{r}_y = \mathbf{j} + f_y\mathbf{k}$$

and so

$$\mathbf{r}_{xx} = f_{xx}\mathbf{k}, \quad \mathbf{r}_{xy} = f_{xy}\mathbf{k}, \quad \mathbf{r}_{yy} = f_{yy}\mathbf{k}.$$

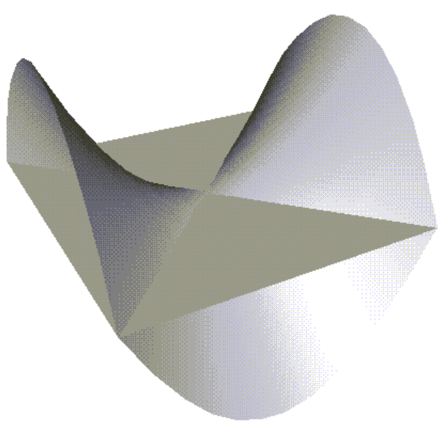
At a critical point of f, $f_x = f_y = 0$ and so the normal is $\mathbf{k}$. The second fundamental form is then the *Hessian* of the function at this point:

$$\begin{pmatrix} L & M \\ M & N \end{pmatrix} = \begin{pmatrix} f_{xx} & f_{xy} \\ f_{xy} & f_{yy} \end{pmatrix}.$$

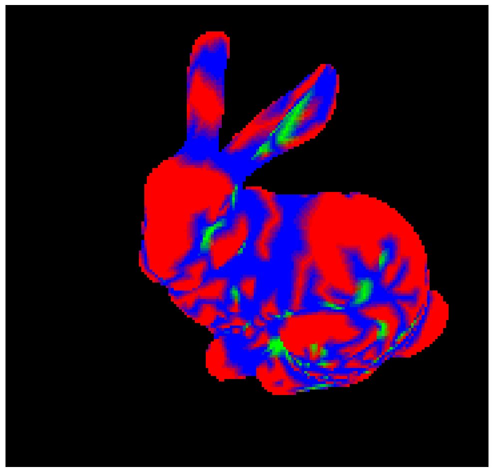
We can use this to qualitatively describe the behaviour of the second fundamental form at different points on the surface. For any point P parametrize the surface by its projection on the tangent plane and then $f(x, y)$ is the height above the plane. Now use the theory of critical points of functions of two variables.

If $f_{xx}f_{yy} - f_{xy}^2 > 0$ then the critical point is a local maximum if the matrix is negative definite and a local minimum if it is positive definite. For the surface the difference is only in the choice of normal so the local picture of the surface is like the sphere – it lies on one side of the tangent plane at the point P.

If on the other hand $f_{xx}f_{yy} - f_{xy}^2 < 0$ we have a saddle point and the surface lies on both sides of the tangent plane:



A general surface has points of both types, like this rabbit:



This descriptive approach can be tightened by studying the the notion of curvature for a surface. First recall a bit of linear algebra:

> We have two quadratic forms on the tangent space, the first and second fundamental forms. The first is positive definite, the second not always so. More generally suppose $B(x, y)$ is a positive definite symmetric bilinear form on a vector space V and $A(x, y)$ is another symmetric bilinear form. Then there is a unique linear transformation
>
> $$\alpha : V \to V$$
>
> such that
>
> $$A(x, y) = B(\alpha x, y).$$
>
> Moreover,
>
> $$B(x, \alpha y) = B(\alpha y, x) = A(y, x) = A(x, y) = B(\alpha x, y)$$
>
> so that α is self-adjoint. We know then that it has real eigenvalues and an orthonormal basis of eigenvectors.
>
> Concretely, what is α? Choose a basis $x_1, \ldots, x_n$ and write the symmetric matrices
>
> $$A_{ij} = A(x_i, x_j), \qquad B_{ij} = B(x_i, x_j)$$

then

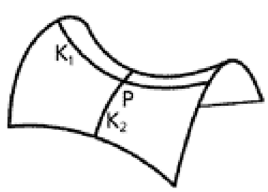
$$\begin{aligned} A_{ji} = A_{ij} = A(x_i, x_j) &= B(\alpha x_i, x_j) \\ &= B(\sum_k \alpha_{ki} x_k, x_j) \\ &= \sum_k \alpha_{ki} B(x_k, x_j) \\ &= \sum_k \alpha_{ki} B_{kj} = \sum_k B_{jk} \alpha_{ki} \end{aligned}$$

and since B is positive definite, its matrix is invertible. Thus

$$\alpha_{ij} = (B^{-1}A)_{ij}.$$

Definition 16 *For a point P on the surface, let T_P be the tangent space and $\alpha : T_P \to T_P$ the linear transformation defined by the second fundamental form. The eigenvalues κ_1, κ_2 of α are called the principal curvatures of the surface at P and the corresponding eigenvectors the principal directions.*

As we have seen, the principal directions, eigenvectors of a self-adjoint transformation, are orthogonal:



To calculate the principal curvatures we therefore want the eigenvalues of $\alpha = B^{-1}A$, the roots of

$$0 = \det(\lambda - \alpha) = \det(\lambda - B^{-1}A) = (\det B)^{-1} \det(\lambda B - A)$$

and so the roots of $\det(\lambda B - A) = 0$.

Since B is positive definite, $\det B > 0$ and the sign of $\det A$ is the same as the sign of

$$\det(B^{-1}A) = \det \alpha = \kappa_1 \kappa_2.$$

This is called the Gaussian curvature: it is positive for the sphere and negative near a saddle.

Examples:

1) For the plane the second fundamental form is zero so both principal curvatures vanish and any direction is a principal direction.

2) For a sphere of radius a we saw that the second fundamental form was a^{-1} times the first, so

$$\alpha = a^{-1} I$$

and the principal curvatures are both equal to a^{-1}. Again any direction is a principal direction.

3) For a developable surface

$$\mathbf{r} = \boldsymbol{\gamma}(u) + v\mathbf{t}(u) \tag{3}$$

we have

$$\mathbf{r}_u = \mathbf{t} + v\kappa\mathbf{n}, \quad \mathbf{r}_v = \mathbf{t}$$

where $\mathbf{n}$ is here the normal of the curve. The normal to the surface is therefore $\mathbf{b}$, the binormal to the curve. We find

$$\mathbf{r}_{uu} = \kappa\mathbf{n} + v\kappa'\mathbf{n} + v\kappa(-\kappa\mathbf{t} + \tau\mathbf{b})$$

using Serret-Frenet and

$$\mathbf{r}_{vu} = \kappa\mathbf{n}, \quad \mathbf{r}_{vv} = 0.$$

Thus the second fundamental form is

$$v\kappa\tau du^2.$$

To find the principal curvatures and directions we need the eigenvalues and eigenvectors of

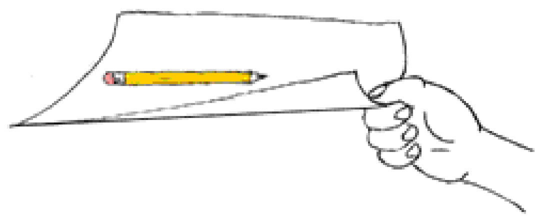
$$\begin{pmatrix} E & F \\ F & G \end{pmatrix}^{-1} \begin{pmatrix} L & M \\ M & N \end{pmatrix} = \begin{pmatrix} 1+v^2\kappa^2 & 1 \\ 1 & 1 \end{pmatrix}^{-1} \begin{pmatrix} v\kappa\tau & 0 \\ 0 & 0 \end{pmatrix}$$

using the first fundamental form $(1+v^2\kappa^2)du^2 + 2dudv + dv^2$.

The principal curvatures are then $0, \tau/v\kappa$ and the principal directions

$$\begin{pmatrix} 0 \\ 1 \end{pmatrix}, \quad \begin{pmatrix} 1 \\ -1 \end{pmatrix}.$$

The first of these, corresponding to the zero eigenvalue, is $\mathbf{r}_v = \mathbf{t}(u)$ — the direction of the line $\mathbf{r} = \boldsymbol{\gamma}(u) + v\mathbf{t}(u)$ for fixed u, and this lies entirely in the surface (3). The pencil in the picture is just such a line:



6 Geodesics

Geodesics on a surface are curves which are the analogues of straight lines in the plane. Lines can be thought of in two ways:

- shortest curves
- straightest curves

The first point of view says that a straight line minimizes the distance between any two of its points. Conceptually this leads to the idea of stretching a string between two points on a surface until it tightens, and this certainly is one approach to geodesics. The second approach is however easier and we shall follow it here. A line is straightest because its tangent vector doesn't change – it is constant along the line. We generalize this to a curve on a surface by insisting that the component of $\mathbf{t}'$ tangential to the surface should vanish. Or....

Definition 17 *A geodesic on a surface S is a curve $\boldsymbol{\gamma}(s)$ on S such that $\mathbf{t}'$ is normal to the surface.*

The general problem of finding geodesics on a surface is very complicated. The case of the ellipsoid is a famous example, needing what are called *hyperelliptic functions* to solve it. But there are cheap ways to find some of them, as in these examples:

Examples:

1) The normal to a curve in the plane is parallel to the plane, so the condition that $\mathbf{t}'$ is normal to the plane means $\mathbf{t}' = 0$ which integrates to the equation of a straight line just as in Proposition 2. Geodesics in the plane really are straight lines, then.

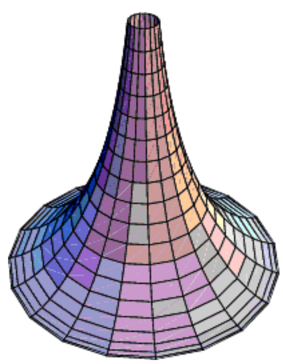
2) Take the unit sphere and a plane section through the origin. This intersects the sphere in a great circle, and the normal to the sphere along this circle is the position vector which lies in the plane and is normal to the circle. Thus the normal to the circle is the normal to the surface, and so a great circle is a geodesic.

3) Similarly, any plane of symmetry intersects a surface in a geodesic, because the normal to the surface at such a point must be invariant under reflection in the plane of symmetry and hence lie in that plane. It is orthogonal to the tangent vector of the curve of intersection and so is the normal to the curve.

A useful class of examples is provided by a surface of revolution

$$\mathbf{r}(u, v) = f(u)(\cos v\, \mathbf{i} + \sin v\, \mathbf{j}) + u\mathbf{k}$$

The reflection $(x, y, z) \mapsto (x, -y, z)$ maps the surface to itself, as, by symmetry, does any reflection in a plane containing the z-axis. So the *meridians* $v = const.$ are geodesics:



To find the geodesics in general we need to solve a nonlinear system of ordinary differential equations:

Proposition 7 *A curve $\boldsymbol{\gamma}(s) = (u(s), v(s))$ on a surface parametrized by arc length is a geodesic if and only if*

$$\begin{aligned} \frac{d}{ds}(Eu' + Fv') &= \frac{1}{2}(E_u u'^2 + 2F_u u'v' + G_u v'^2) \\ \frac{d}{ds}(Fu' + Gv') &= \frac{1}{2}(E_v u'^2 + 2F_v u'v' + G_v v'^2) \end{aligned}$$

Proof: We have for the curve $\boldsymbol{\gamma}$

$$\mathbf{t} = \mathbf{r}_u u' + \mathbf{r}_v v'$$

and it is a geodesic if and only if $\mathbf{t}'$ is normal i.e.

$$\mathbf{t}' \cdot \mathbf{r}_u = \mathbf{t}' \cdot \mathbf{r}_v = 0.$$

Now

$$\mathbf{t}' \cdot \mathbf{r}_u = (\mathbf{t} \cdot \mathbf{r}_u)' - \mathbf{t} \cdot \mathbf{r}_u'$$

so the first equation is

$$(\mathbf{t} \cdot \mathbf{r}_u)' = \mathbf{t} \cdot \mathbf{r}_u'.$$

The left hand side is

$$\frac{d}{ds}((\mathbf{r}_u u' + \mathbf{r}_v v') \cdot \mathbf{r}_u) = \frac{d}{ds}(Eu' + Fv')$$

an the right hand side is

$$\begin{aligned}
\mathbf{t} \cdot (\mathbf{r}_{uu} u' + \mathbf{r}_{uv} v') &= \mathbf{r}_u \cdot \mathbf{r}_{uu} u'^2 + (\mathbf{r}_v \cdot \mathbf{r}_{uu} + \mathbf{r}_u \cdot \mathbf{r}_{uv}) u'v' + \mathbf{r}_v \cdot \mathbf{r}_{uv} v'^2 \\
&= \frac{1}{2} E_u u'^2 + (\mathbf{r}_v \cdot \mathbf{r}_u)_u u'v' + \frac{1}{2} G_u v'^2 \\
&= \frac{1}{2}(E_u u'^2 + 2F_u u'v' + G_u v'^2)
\end{aligned}$$

The other equation follows similarly. □

A corollary of the theorem is that geodesics only depend on the first fundamental form, so that an isometry takes geodesics to geodesics.

Examples:

1) The plane: $E = 1, F = 0, G = 1$ in Cartesian coordinates, so the geodesic equations are

$$x'' = 0 = y''$$

which gives straight lines

$$x = \alpha_1 s + \beta_1, \quad y = \alpha_2 s + \beta_2.$$

2) The cylinder

$$\mathbf{r}(u, v) = a(\cos v\, \mathbf{i} + \sin v\, \mathbf{j}) + u\mathbf{k}$$

has first fundamental form

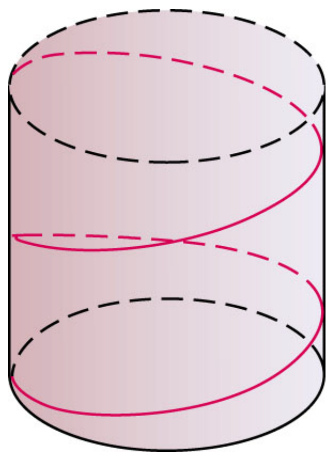
$$du^2 + a^2 dv^2 = du^2 + d(av)^2.$$

This is isometric to the plane so the geodesics are of the form

$$u = \alpha_1 s + \beta_1, \quad v = \alpha_2 s + \beta_2$$

which gives a helix

$$\boldsymbol{\gamma} = a(\cos(\alpha_2 s + \beta_2)\,\mathbf{i} + \sin(\alpha_2 s + \beta_2)\,\mathbf{j}) + (\alpha_1 s + \beta_1)\mathbf{k}$$



The differential equation for geodesics gives us the following general fact:

Proposition 8 *Through each point P on a surface and in each direction at P there passes a unique geodesic.*

Proof: We are solving a differential equation of the form

$$u'' = a(u, v, u', v'), \quad v'' = b(u, v, u', v')$$

or equivalently a first order system

$$\begin{aligned} u' &= p \\ v' &= q \\ p' &= a(u, v, p, q) \\ q' &= b(u, v, p, q) \end{aligned}$$

and the Cauchy existence theorem gives a unique solution with initial conditions (u, v, p, q), namely the point of origin and the direction. □

Example: Given a point $\mathbf{a}$ on the unit sphere and a tangential direction $\mathbf{b}$ the span of $\mathbf{a}, \mathbf{b}$ is a plane through the origin which meets the sphere in a great circle through $\mathbf{a}$ with tangent $\mathbf{b}$. Thus *every* geodesic is a great circle.

There is one case – a surface of revolution – where the geodesic equations can be "solved", or anyway, reduced to a single integration. We have

$$E = 1 + f'(u)^2, \quad F = 0, \quad G = f(u)^2$$

and the equations become

$$\begin{aligned} \frac{d}{ds}((1 + f'^2)u') &= f'(f''u'^2 + fv'^2) \\ \frac{d}{ds}(f^2 v') &= 0 \end{aligned}$$

We ignore the first equation – it is equivalent to a more obvious fact below. The second says that

$$f^2 v' = c \tag{4}$$

where c is a constant. Now use the fact that the curve is parametrized by arc length (this is an "integral" of the equations), and we get

$$(1 + f'^2)u'^2 + f^2 v'^2 = 1 \tag{5}$$

Substitute for v' from (4) in (5) to get

$$(1 + f'^2)u'^2 + \frac{c^2}{f^2} = 1$$

and then

$$s = \int f \sqrt{\frac{1 + f'^2}{f^2 - c^2}} du$$

which is "only" an integration. Having solved this by $u = h(s)$, v can be determined by a further integration from (4):

$$v(s) = \int \frac{c}{f(h(t))^2} dt.$$

If we are only interested in the curve and not its arclength parametrization, then (4) and (5) give

$$(1 + f'(u)^2) \left(\frac{du}{dv} \right)^2 + f(u)^2 = \frac{f(u)^4}{c^2}$$

which reduces to the single integration

$$v = \int \frac{c}{f(u)} \sqrt{\frac{1 + f'(u)^2}{f(u)^2 - c^2}} du.$$

Another way of describing a geodesic is to introduce a refinement of the curvature κ of a curve on a surface. Let $\mathbf{N}$ be the normal to the surface, to avoid confusion with the normal to the curve. Then since $\mathbf{t}'$ is orthogonal to $\mathbf{t}$ we have

$$\mathbf{t}' = \kappa_n \mathbf{N} + \kappa_g \mathbf{N} \wedge \mathbf{t}$$

for functions κ_n, κ_g along the curve. Since $|\mathbf{t}'| = \kappa$ we have

$$\kappa^2 = \kappa_n^2 + \kappa_g^2.$$

Definition 18 *The normal curvature κ_n of a curve parametrized by arc length in a surface is defined by*

$$\kappa_n = \mathbf{t}' \cdot \mathbf{N}$$

where $\mathbf{N}$ is the normal to the surface.

The geodesic curvature κ_g is defined by

$$\kappa_g = \mathbf{t}' \cdot (\mathbf{N} \wedge \mathbf{t}).$$

Clearly with this definition, a curve is a geodesic if and only if its geodesic curvature vanishes. The only point of introducing it is that the geodesic curvature of any curve in the surface depends on the first fundamental form only, and so will be the same for isometric surfaces. The proof is an exercise – you can use the calculation in Proposition (7) to help.