THE COHOMOLOGY OF DIFFERENTIABLE TRANSFORMATION GROUPS.* [1]

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1. Introduction. If a Lie group G acts differentiably on a manifold $\mathfrak{M}$ then various spaces of tensor field on $\mathfrak{M}$ become in a natural way modules for the Lie algebra $\mathfrak{G}$ of G and the cohomology of $\mathfrak{G}$ with these coefficient modules in certain cases carries interesting information about the action of G. In this paper we will discuss this situation, at first in a somewhat more abstract setup, and develop a method for computing these cohomology groups in certain cases. In particular we shall show that if G is compact and semi-simple then even though these modules are infinite dimensional the conclusions of the First and Second Whitehead Lemmas [10], [11] are valid; namely the first and second cohomology groups are trivial. As one consequence we will show that differentiable actions of compact, semi-simple Lie groups admit only trivial infinitesimal deformations (§ 11) a fact whose global analogue will be found in [7]. Our second and motivating application of these general cohomology results is to a question initiated by one of the authors in [9]. Namely if a Lie group G acts differentiably on the base space of a differentiable fiber bundle B over M can G be made to act differentiably on B so as to be equivariant with respect to the fiber projection and so that each operation of G on B is a bundle map. We show here that the answer is yes if G is compact and simply connected and if the structural group of B is a solvable Lie group, and moreover that the way of "lifting" the action of G to B is essentially unique.

2. Cohomology and invariant cohomology of Lie algebras. In the following L is a finite dimensional Lie algebra over a field F of characteristic zero. For the defintion of an L-module and a complete discussion of the cohomology of L with coefficients in an L-module we refer the reader to [2] or [3] (we shall use the notation of the latter). The notion of the invariant cohomology of L with coefficients in an L-module is implicit in a number of

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papers, however the lack of an exposition with the relevant facts that we shall need makes the following discussion desirable.

Recall that if M is an L-module then the space $C^p(L,M)$ of p-cochains of L with coefficients in M is defined to be M if $p=0$ and to be the space of alternating multilinear maps of L^p into M if p is a positive integer. Then $C^*(L,M)$ the cochain complex of L with coefficients in M is the graded vector space $\bigoplus_{p\geqq 0} C^p(L,M)$ with the differential d of degree $+1$ defined by $dm(X)=X\cdot m$ for $m\in M$ and

$$\begin{aligned}dc(X_1,\cdots,X_{p+1}) &= \sum_{i=1}^{p+1}(-1)^{i+1}X_i(c(X_1,\cdots,\hat{X}_i,\cdots,X_{p+1})) \\ &+\sum_{i<j}(-1)^{i+j}c([X_i,X_j],X_1,\cdots,\hat{X}_i,\cdots,\hat{X}_j,\cdots,X_{p+1})\end{aligned}\tag{1}$$

for $c\in C^p(L,M)$, $p>0$. We recall from [3] that $d^2=0$ so there are defined graded vector spaces $Z^*(L,M)$ and $B^*(L,M)$ of cocycles and coboundaries and their quotient $H^*(L,M)$ the cohomology space of L with coefficients in M. We recall also that $C^*(L,M)$ becomes a graded L-module with $M=C^0(L,M)$ as a submodule if we define

$$\begin{aligned}(Xc)(X_1,\cdots,X_p) &= X(c(X_1,\cdots,X_p))-\sum_{i=1}^{q}c(X_1,\cdots,[X,X_i],\cdots X_p) \\ &= X(c(X_1,\cdots,X_p))-\sum_{i=1}^{q}(-1)^{i+1}c([X,X_i],X_1,\cdots,\hat{X}_i,\cdots X_p)\end{aligned}\tag{2}$$

for $c\in C^p(L,M)$, $p>0$, and that

$$d(Xc)=X(dc)\tag{3}$$

for $X\in L$, $c\in C^*(L,M)$. It follows from (1) and (2) that for $c\in C^p(L,M)$

$$\begin{aligned}dc(X_1,\cdots,X_{p+1}) &= \sum_{i=1}^{p+1}(-1)^{i+1}(X_ic)(X_1,\cdots,\hat{X}_i,\cdots X_{p+1}) \\ &+\sum_{i<j}(-1)^{i+j+1}c([X_i,X_j],X_1,\cdots,\hat{X}_i,\cdots\hat{X}_j,\cdots X_{p+1})\end{aligned}$$

We define a cochain $c\in C^*(L,M)$ to be *invariant* if $Xc=0$ for all $X\in L$. It follows from (3) that the graded subspace $C_I^*(L,M)=\bigoplus_{p\geqq 0} C_I^p(L,M)$ of $C^*(L,M)$ consisting of invariant cochains is stable under d and so gives rise to the graded vector spaces $Z_I^*(L,M)$ and $B_I^*(L,M)$ of invariant cocycles and invariant coboundaries (N. B. $B_I^*(L,M)=dC_I^*(L,M)$ *not* the possibly larger $C_I^*(L,M)\cap B^*(L,M)$) and their quotient $H_I^*(L,M)$ which we call the *invariant cohomology* of L with coefficients in M. By the *natural homomorphism* of $H_I^*(L,M)$ into $H^*(L,M)$ we mean the homomorphism induced by the inclusion of $C_I^*(L,M)$ in $C^*(L,M)$. We note from (4) that $d_I = d\,|\,C_I^*(L,M)$ is given on $C^p(L,M)$ by

$$(5)\quad d_I c(X_1,\cdots,X_{p+1}) = -\sum_{i<j}(-1)^{i+j}c([X_i,X_j],X_1,\cdots X_i,\cdots X_j,\cdots X_{p+1})$$

An L-module M is called *trivial* if $Xm = 0$ for all $X \in L$, $m \in M$ (i.e. if $C_I^0(L,M) = C^0(L,M) = M$). Any vector space V over F can be considered a trivial L-module, in which case we write d_T for the differential on $C^*(L,V)$. Clearly from (1)

$$(6)\quad d_T c(X_1,\cdots,X_p) = \sum_{i<j}(-1)^{i+j}c([X_i,X_j],X_1,\cdots X_i\cdots X_j,\cdots X_{p+1}).$$

In particular we regard F as a trivial L-module and write $C^*(L,F) = C^*(L)$ and $H^*(L,F) = H^*(L)$. Since $C^1(L) = L^*$, the dual space of L, and since by (6) $d_T c(X,Y) = c([X,Y])$ for $c \in C^1(L)$ it follows that $Z^1(L) = [L,L]^0$, the annihilator of $[L,L]$. Since $B^1(L)$ is clearly zero (for $df(X) = 0$ for $f \in F$) it follows that $H^1(1) \cong Z^1(L) = [L,L]^0 \cong (L/[L,L])^*$. Thus $H^1(L)$ is trivial if and only if L is its own commutator subalgebra. Note also that H^* is an additive functor, i.e. $H^*(L, M \oplus N) \cong H^*(L,M) \oplus H^*(L,N)$, so it follows that if $H^p(L) = 0$ then $H^p(L,V) = 0$ for any trivial finite dimensional L-module V.

If M is any L-module then for each $X \in L$ there is an endomorphism i_X of $C^*(L,M)$ homogeneous of degree -1 defined by $i_X m = 0$ for $m \in M = C^0(L,M)$ and $(i_X c)(X_1,\cdots,X_{p-1}) = c(X,X_1,\cdots,X_{p-1})$ for $c \in C^p(L,M)$, $p > 0$. For later reference we recall from [3] that the module operations of L on $C^*(L,M)$ is related to d and the operations i_X by

$$(7)\qquad Xc = di_X c + i_X dc \qquad X \in L,\ c \in C^*(L,M).$$

3. An extension of the Whitehead lemmas. We will show in § 8 that if $\mathfrak{G}$ is the Lie algebra of a compact Lie group then for a certain class of $\mathfrak{G}$-modules M the natural homomorphism of $H_I^*(\mathfrak{G},M)$ into $H^*(\mathfrak{G},M)$ is an isomorphism onto. If $\mathfrak{G}$ is semi-simple it follows from the theorems we are above to prove that for such $\mathfrak{G}$-modules $H^1(\mathfrak{G},M)$ and $H^2(\mathfrak{G},M)$ are trivial, a fact that would also be a consequence of the Whitehead Lemmas ([10] and [11]) if M were finite dimensional. It is in this sense that this theorem extends the Whitehead lemmas.

THEOREM. *If L is a finite dimensional Lie algebra such that $H^1(L) = 0$ then $H_I^1(L,M) = 0$ for all L-modules M. If in addition $H^2(L) = 0$ then $H_I^2(L,M) = 0$ for all L-modules M.*

Proof. $C_I{}^0(L,M)=\{m\in M\,|\,Xm=0$ for all $X\in L\}$ so if $m\in C_I{}^0(L,M)$ then $dm(X)=Xm=0$ for all $X\in L$ hence $B_I{}^1(L,M)=0$ and $H_I{}^1(L,M)\cong Z_I{}^1(L,M)$. By (5) of § 2 if $c\in C_I{}^1(L,M)$ then $dc(X,Y)=c([X,Y])$ so $Z_I{}^1(L,M)=\{c\in C_I{}^1(L,M)\,|\,c$ is zero on $[L,L]\}$. But $H^1(L)=0$ is equivalent to $L=[L,L]$ and hence implies $H_I{}^1(L,M)\cong Z_I{}^1(L,M)=0$.

Now let $c\in Z_I{}^2(L,M)$ and let V be the finite dimensional subspace of M spanned by $\{c(X,Y)\,|\,X,Y\in L\}$. We consider V as a trivial L-module. Assuming $H^2(L)=0$ it follows that $H^2(L,V)=0$. Now $c\in C^2(L,V)$ and comparing (5) and (6) of § 2 we see that $d_Tc=-\,d_Ic=0$ so $c\in Z^2(L,V)=B^2(L,V)$ and hence $C=d_T\theta$ for some $\theta\in C^1(L,V)\subset C^1(L,M)$. It will suffice to prove that $\theta\in C_I{}^1(L,M)$, for then comparing (5) and (6) of § 2 again $d_I(-\theta)=-\,d_I\theta=d_T\theta=c$ so $c\in B_I{}^2(L,M)$ and we will have shown $Z_I{}^2(L,M)=B_I{}^2(L,M)$. Now if $X,X_1,X_2\in L$ then from (2) of § 2

$$\begin{aligned}0=(Xc)(X_1,X_2)&=X(c(X_1,X_2))-c([X,X_1],X_2)-c(X_1,[X,X_2])\\&=X(\theta([X_1,X_2]))-\theta([[X,X_1],X_2])-\theta([X_1,[X,X_2]])\end{aligned}$$

and by the Jacobi identity and the linearity of θ it follows that

$$0=X(\theta([X_1,X_2]))-\theta([X,[X_1,X_2]])$$

so, referring to (2) of § 2 again, $(X\theta)([X_1,X_2])=0$. Thus $X\theta$ vanishes on $[L,L]=L$ so θ is invariant. q. e. d.

4. Topological *G*-Modules. In this and succeeding sections G will denote a compact Lie group, G_0 its identity component, and $\mathfrak{G}$ its Lie algebra of left invariant vector fields. By a *topological G-module* we shall mean a complete, metrizable, locally convex, real topological vector space (a Frechét space) M together with a fixed homomorphism T of G into the group of automorphisms of M such that for each $m\in M$ the map $g\to T(g)m$ of G into M is continuous. We denote the space of continuous linear functionals on M by M^* and we write $\langle m,l\rangle$ for $l(m)$ if $(l,m)\in M^*\times M$ (which of the many possible topologies to put on M^* is irrelevant for our purposes and we shall always regard M^* as untopologized). In general we will drop explicit reference to T and simply write gm instead of $T(g)m$. Note that since M is a Frechét space and for each $m\in M$ the orbit $\{gm\,|\,g\in G\}$ is compact, and hence bounded, it follows from the principle of uniform boundedness that

4.1. Theorem. *Given a neighborhood V of zero in the topological G-module M there is a neighborhood U of zero in M such that $gu\in V$ for all $(g,u)\in G\times U$. Equivalently if $\{m_r\}$ is a sequence in M converging to m then $\{gm_r\}$ converges to gm uniformly for $g\in G$.*

Noting that $g_r m_r - gm = g_r(m_r - m) + (g_r m - gm)$ it follows from 4. 1 that if $g_r \to g$ then $g_r m_r \to gm$.

4. 2. COROLLARY. *If M is a topological G module then $(g, m) \to gm$ is a continuous map of $G \times M$ into M.*

Now let M be a topological G-module and M^X the linear space of functions from a compact Hausdorff space X into M. We give M^X the topology of uniform convergence, i. e. a typical neighborhood of zero is $\{f \in M^X | f(X) \subseteq U\}$ where U is some neighborhood of zero in M. By a step function from X to M we mean an element $f \in M^X$ whose range is a finite subset $\{m_1, \cdot \cdot \cdot, m_r\}$ of M such that each $f^{-1}(m_i)$ is Borel measurable. Since a continuous function from X to M is uniformly continuous it follows easily that the space $C(X, M)$ of continuous maps of X to M is included in the closure of the space $S(X, M)$ of step functions. Given a Radon measure μ on X and $f \in S(X, M)$ we define $\int f d\mu = \sum_{m \in M} \mu(f^{-1}(m))m$. Then $f \to \int f d\mu$ is a continuous linear map from $S(X, M)$ into M and so (because M is complete) extends to a continuous linear map of the closure of $S(X, M)$ into M. Henceforth we shall only be concerned with $\int f d\mu$ when f is continuous and we shall need the following obvious facts.

(a) $f_\alpha \to f$ uniformly on $X \Rightarrow \int f_\alpha d\mu \to \int f d\mu$

(b) $\int f d\mu$ is bilinear in f and μ

(c) If $\mu(X) = 1$ then $\int f d\mu \in$ closed convex hull of $f(X)$.

(d) If T is a continuous linear map of M into a complete topological vector space then $\int (Tf) d\mu = T \int f d\mu$.

The space $C(G)$ of continuous real valued functions on G is a Banach space in the supremum norm and becomes a topological G-module under the operations given by $(gf)(x) = f(g^{-1}x)$. If $f \in C(G)$ we define $f * m$, for m an element of a topological G-module M, to be the integral of the continuous map $g \to f(g)gm$ of G into M with respect to normalized Haar measure on G. If $l \in M^*$ then by (d) above $\langle f * m, l \rangle = \int f(g) \langle gm, l \rangle d \cdot g$. From this and the invariance of Haar measure it follows that $\langle (gf) * m, l \rangle = \langle g(f * m), l \rangle$ and since M^* separates points of M it follows that $(gf) * m = g(f * m)$.

4. 3. THEOREM. *If M is a topological G-module then*

(1) *$(f, m) \to f * m$ is a continuous bilinear map of $C(G) \times M$ into M*

(2) *If $f \in C(G)$ is positive and $\int f(g) dg = 1$ then $f * m \in$ closed convex hull of $\{gm | f(g) \neq 0\}$.*

(3) *If $m \in M$ then $f \to f * m$ is an equivariant continuous linear map of $C(G)$ into M and m is in the closure of its range.*

Proof. Statement (1) is immediate from (a) and (b) above and state- (2) follows from (c). The equivariance of $f \to f * m$ (i. e. the fact that $(gf) * m = g(f * m)$ was shown above so it remains only to show that if W is a closed convex neighborhood of m in M then there is an $f \in C(G)$ such that $f * m \in W$. Let U be a neighborhood of the identity in G such that $gm \in W$ for $g \in U$ and let f be a continuous positive function of integral one on G with support in U. Then by (2) $f * m \in W$. q. e. d.

5. Almost invariant vectors. An element m_0 of a topological G-module M is called *almost invariant* if $\{gm_0 | g \in G\}$ spans a finite dimensional subspace of M. We denote the set of almost invariant vectors in M by M_0. Clearly M_0 is a subspace of M invariant under G, and in fact M_0 is just the linear span of the finite limensional G-invariant subspaces of M. An almost invariant vector in $C(G)$ is called an almost invariant function on G. From the bilinearity of $(f, m) \to f * m$ and the equivariance of $f \to f * m$ it follows that if f_0 is an almost invariant function and m any element of a topological G-module M then $f_0 * m \in M_0$. Now the Peter-Weyl theorem [1, p. 203] says that the almost invariant functions are dense in $C(G)$. Since we known that m is adherent to $\{f * m | f \in C(G)\}$ and that $f \to f * m$ is continuous we have the following essentially known result.

5. 1. THEOREM. *In any topological G-module M the subspace M_0 of almost invariant vectors is dense.*

5. 2. THEOREM. *If M is any topological G-module then the subspace M_0 of almost invariant vectors has a $\mathfrak{G}$-module stducture defined by $Xm = \lim_{t \to 0} 1/t((\operatorname{Exp} tX)m - m)$ for $X \in \mathfrak{G}$, $m \in M_0$.*

Proof. If V is any finite dimensional invariant subspace of M_0 $(g, v) \to gv$ is an analytic map of $G \times V \to V$ (this follows for example from [1, Prop 1, p. 128]) so that the indicated limit exists for $m \in V$. That this gives a $\mathfrak{G}$-module on V follows from [1, Theorem 2, p. 113]. Since M_0 is the linear span of its finite dimensional invariant subspaces the theorem follows. q. e. d.

6. Differentiable G-modules.

6. 1. LEMMA. *Let M be a topological G-module and $\{m_r\}$ a sequence in M converging to m. Given $X \in \mathfrak{G}$ and $l \in M^*$ suppose there exists a sequence*

$\{\omega_k\}$ in M converging to ω such that $d/dt\langle(\operatorname{Exp} tX)m_r, l\rangle = \langle(\operatorname{Exp} tX)\omega_r, l\rangle$. *Then* $d/dt\langle(\operatorname{Exp} tX)m, l\rangle = \langle(\operatorname{Exp} tX)\omega, l\rangle$.

Proof. By 4.1 gm_r converges to gm uniformly in g, so since l is linear and continuous and hence uniformly continuous, $\langle gm_r, l\rangle$ converges to $\langle gm, l\rangle$ uniformly in g and *a fortiori* $\langle(\operatorname{Exp} tX)m_k, l\rangle$ converges to $\langle(\operatorname{Exp} tX)m, l\rangle$ uniformly in t. Similarly $\langle(\operatorname{Exp} tX)\omega_k, l\rangle$ converges to $\langle(\operatorname{Exp} tX)\omega, l\rangle$ uniformly in t. By a theorem of elementary calculus this validates differentiation "under the limit sign" and gives the desired result. q. e. d.

6.2. Theorem. *Let M be a topological G-module $X \in \mathfrak{G}$, and suppose there is a function $\tilde{X}: M \to M$ such that*

$$d/dt\langle(\operatorname{Exp} tX)m, l\rangle = \langle(\operatorname{Exp} tX)\tilde{X}m, l\rangle$$

for all $m \in M$ and all l belonging to a separating family $S \subset M^*$. Then $\tilde{X}$ is a continuous linear map and for $m \in M_0$ $Xm = \lim_{t\to 0} 1/t((\operatorname{Exp} TX)m - m)$.

Proof. The linearity of $\tilde{X}$ is immediate from the linearity of differentiation and the fact that S is separating. The preceding lemma gives immediately that $\tilde{X}$ has a closed graph and, since M is a Frechét space $\tilde{X}$ is continuous. If $m \in M_0$ then by 5.2 $1/t((\operatorname{Exp} tX)m - m)$ converges strongly and *a fortiori*, weakly to a limit say m'. By definition of $\tilde{X}$ $\langle m', l\rangle = \langle\tilde{X}m, l\rangle$ for $l \in S$ and again since S is separating $m' = \tilde{X}m$. q. e. d.

6.3. Theorem. *Let M be a topological G-module, $X \in \mathfrak{G}$, and suppose that the linear map*

$$m \to \lim_{t\to 0} 1/t((\operatorname{Exp} tX)m - m)$$

of M_0 into itself (see 5.2) *is continuous. Then this map extends uniquely to a continuous linear map $\tilde{X}$ of M into itself and $d/dt\langle(\operatorname{Exp} tX)m, l\rangle = \langle(\operatorname{Exp} tX)\tilde{X}m, l\rangle$ for all $m \in M$, $l \in M^*$.*

Proof. The existence and uniqueness of $\tilde{X}$ follows from the completeness of M and the denseness of M_0 in M. Given $l \in M^*$, $m \in M_0$, and t_0 a real number $l_0: m \to \langle(\operatorname{Exp} t_0X)m, l\rangle$ is an element of M^* and

$$\frac{d}{dt}\bigg|_{t=t_0}\langle(\operatorname{Exp} tX)m_0, l\rangle = \frac{d}{dt}\bigg|_{t=0}\langle(\operatorname{Exp}(t+t_0)X)m_0, l\rangle$$

$$\frac{d}{dt}\bigg|_{t=0}\langle(\operatorname{Exp} tX)m_0, l_0\rangle = \langle\tilde{X}m_0, l_0\rangle = \langle(\operatorname{Exp} t_0X)\tilde{X}m_0, l\rangle.$$

Thus $\langle(\operatorname{Exp} tX)m, l\rangle$ is differentiable for $m \in M_0$ and $l \in M^*$ and its derivative is $\langle(\operatorname{Exp} tX)\tilde{X}m, l\rangle$. If m is an arbitrary element of M choose a sequence $\{m_r\}$ in M_0 converging to m. Then $\tilde{X}m_r \to \tilde{X}m$ and for any $l \in M^*$

$d/dt\langle(\operatorname{Exp} tX)m_r, l\rangle = \langle(\operatorname{Exp} tX)\tilde{X}m_r, l\rangle$ so by 6.1 $d/dt\langle(\operatorname{Exp} tX)m, l\rangle = \langle(\operatorname{Exp} tX)\tilde{X}m, l\rangle$. q. e. d.

6.4. THEOREM. *If M is a topological G-module then the following are equivalent conditions.*

(1) *For each $X \in \mathfrak{G}$ there is a function $\tilde{X}: M \to M$ such that for all l in a separating subset of M^* $\frac{d}{dt}\langle(\operatorname{Exp} tX)m, l\rangle = \langle(\operatorname{Exp} tX)Xm, l\rangle$ for all $m \in M$.*

(2) *For each $X \in \mathfrak{G}$ the map $X^0: m \to \lim_{t\to 0} 1/t((\operatorname{Exp} tX)m - m)$ of M_0 into itself* (see 5.2) *is continuous.*

(3) *For each $X \in \mathfrak{G}$ and $m \in M$ $t \to (\operatorname{Exp} tX)m$ has a weak derivative, X'_m, at $t = 0$.*

(4) *For each $X \in \mathfrak{G}$ there is a map $X^*: M \to M$ such that*

$$\frac{d}{dt}\langle(\operatorname{Exp} tX)m, l\rangle = \langle(\operatorname{Exp} tX)X^*m, l\rangle.$$

Moreover if any one, and hence all, of these conditions are satisfied then the maps X, X', and X^* are all equal and are the unique continuous linear map of M into intself which extends X^0.

Proof. It is clear that (4) implies (1) and that if $\tilde{X}$ and X^* exist that they are equal. From 6.2 it follows that (1) implies (2) and that if $\tilde{X}$ exists, it is the unique continuous linear extension of X^0. From 6.3 it follows that (2) implies (3) and that X^* is the unique continuous linear extension of X^0. It remains to show that (3) implies (4) and that $X' = X^*$. Assuming (3) holds let $l \in M^*$ and t_0 a real number and define $l_0 \in M^*$ by $m \to \langle(\operatorname{Exp} t_0X)m, l\rangle$. Then for any $m \in M$

$$\left.\frac{d}{dt}\right|_{t=t_0}\langle(\operatorname{Exp} tX)m, l\rangle = \left.\frac{d}{dt}\right|_{t=0}\langle(\operatorname{Exp}((t+t_0)X)m, l\rangle$$
$$\left.\frac{d}{dt}\right|_{t=0}\langle(\operatorname{Exp} tX)m, l_0\rangle = \langle X'm, l_0\rangle = \langle(\operatorname{Exp} t_0X)X'm, l\rangle.$$

This shows both that (4) holds and that $X^*m \equiv X'm$.

q. e. d.

6.5. *Definition.* A topological G-module is called differentiable if it satisfies any one and hence all of the properties (1)-(4) of 6.4.

Remark. It is easily shown by example that the maps $t \to (\operatorname{Exp} tX)m$ need not be strongly differentiable for all m in a differentiable G-module.

6.6. THEOREM. *If M is a differentiable G-module then M has a $\mathfrak{G}$-module structure which is characterized by the identity*

$$\langle Xm, l\rangle = \frac{d}{dt}\Big|_{t=0} \langle (\operatorname{Exp} tX)m, l\rangle$$

for $X \in \mathfrak{G}$, $m \in M$, $l \in M^*$. Moreover each of the module operations of $\mathfrak{G}$ on M is continuous.

Proof. Immediate from 5.1, 5.2, and 6.4.

6.7. *Definition.* If M is a differentiable G-module then the $\mathfrak{G}$-module structure for M described in 6.6 is called the derived $\mathfrak{G}$-module.

Henceforth differentiable G-modules will be regarded without explicit mention as $\mathfrak{G}$-modules, the derived $\mathfrak{G}$-module always being understood.

6.8. THEOREM. *If M is a differentiable G-module then:*

(1) $\frac{d}{dt}\langle (\operatorname{Exp} tX)m, l\rangle = \langle (\operatorname{Exp} tX)Xm, l\rangle X \in \mathfrak{G},\ m \in M,\ l \in M^*$

(2) $gXg^{-1}m = (ad(g)X)m,\ X \in \mathfrak{G},\ g \in G,\ m \in M$

(3) *If $m \in M$ then $Xm = 0$ for all $X \in \mathfrak{G}$ if and only if $gm = m$ for all $g \in G_0$.*

Proof. (1) follows easily from 6.4 and the definition of the derived $\mathfrak{G}$ module structure. To prove (2) we note that if $l \in M^*$ and we write $\tilde{l} = l \circ g$ then

$$\begin{aligned}\langle (ad(g)X)m, l\rangle &= \frac{d}{dt}\Big|_{t=0} \langle \operatorname{Exp} t(ad(g)X)m, l\rangle \\ &= \frac{d}{dt}\Big|_{t=0} \langle g(\operatorname{Exp} tX)g^{-1}m, l\rangle \\ &= \frac{d}{dt}\Big|_{t=0} \langle (\operatorname{Exp} tX)(g^{-1}m), \tilde{l}\rangle \\ &= \langle Xg^{-1}m, \tilde{l}\rangle = \langle gXg^{-1}m, l\rangle.\end{aligned}$$

If $gm = m$ for all $g \in G_0$ then $(\operatorname{Exp} tX)m = m$ for all $X \in \mathfrak{G}$ and all real t hence for each $X \in \mathfrak{G}\, t \to (\operatorname{Exp} tX)$ has strong and therefore weak derivative zero at $t = 0$ and by Definition $Xm = 0$. Conversely if $Xm = 0$ for a given $X \in \mathfrak{G}$ then by (1) $\frac{d}{dt}\langle \operatorname{Exp} tX, l\rangle \equiv 0$ so $\langle (\operatorname{Exp} tX)m, l\rangle \equiv \langle (\operatorname{Exp} 0X)m, l\rangle = \langle m, l\rangle$, $(\operatorname{Exp} tX)m \equiv m$. Since G is compact every element of G_0 is of the form $\operatorname{Exp} X$ and (3) follows. q.e.d.

7. The differentiable structure of $C^*(\mathfrak{G}, M)$. In this section we will show that if M is a differentiable G-module then $C^*(\mathfrak{G}, M)$ is in a natural way a differentiable G-module also. Moreover we shall show that the derived $\mathfrak{G}$-module structure for $C^*(\mathfrak{G}, M)$ coincides with the other "natural" $\mathfrak{G}$-module structure that it has *qua* cochain complex of the $\mathfrak{G}$-module M (§ 1, equation (2)). Finally we shall show that the differential d is continuous on $C^*(\mathfrak{G}, M)$ and commutes with each operation of G.

Since $C^*(\mathfrak{G}, M) = \bigoplus_p C^p(\mathfrak{G}, M)$ and $C^p(\mathfrak{G}, M) = 0$ for $p > \dim \mathfrak{G}$, to define a differentiable G-module structure on $C^*(\mathfrak{G}, M)$ it suffices to define one on each $C^p(\mathfrak{G}, M)$, $p > 0$. For a typical neighborhood of zero in $C^p(\mathfrak{G}, M)$ we take $\{c \in C^p(\mathfrak{G}, M) \mid c(X_1, \cdots, X_p) \in U \text{ if } X_1, \cdots, X_p \in B\}$ where B is a compact subset of $\mathfrak{G}$ and U a neighborhood of zero in M. Thus $c_\alpha \to c$ means that for each compact subset B of M c_α converges uniformly to c on B^p. The metrizeability, completeness, and local convexity of $C^p(\mathfrak{G}, M)$ follow directly from the corresponding properties of M. The operations of G on $C^p(\mathfrak{G}, M)$ are defined by $(gc)(X_1, \cdots, X_p) = g(c(ad(g^{-1})X_1, \cdots, ad(g^{-1})X_p))$. It is obvious that each such operation is an automorphism of $C^p(\mathfrak{G}, M)$ and that $(g_1g_2)c = g_1(g_2c)$. That $g \to gc$ is a continuous map of G into $C^p(\mathfrak{G}, M)$ for any $c \in C^p(\mathfrak{G}, M)$ is a straightforward argument which we leave to the reader. Given $X_1, \cdots, X_p$ in $\mathfrak{G}$ and a $l \in M^* c \to \langle c(X_1, \cdots, X_p), l \rangle$ is an element of $C^p(\mathfrak{G}, M))^*$ and the collection of such continuous linear functionals on $C^p(\mathfrak{G}, M)$ is clearly separating. Now put

$$g(t) = c(ad(\operatorname{Exp} - tX)X_1, \cdots, ad(\operatorname{Exp} - tX)X_p).$$

From the well-known fact that

$$\lim_{t \to t_0} \frac{1}{t - t_0} (ad(\operatorname{Exp} tX)Y - ad(\operatorname{Exp} t_0X)Y) = ad(\operatorname{Exp} t_0X)[X, Y]$$

it follows from the multilinearity of c that $g(t)$ is strongly differentiable at t_0 and that

$$g'(t_0) = -\sum_{i=1}^{p} c(ad(\operatorname{Exp} - t_0X)X_1, \cdots, ad(\operatorname{Exp} - t_0X)[X, X_i], \\ \cdots, ad(\operatorname{Exp} - t_0X)X_p).$$

(We have used the fact that a multilinear map of a finite dimensional space into a topological vector space is automatically continuous.) Putting $\lambda(t) = (\operatorname{Exp} tX) g(t)$ we have

$$\frac{1}{1 - t_0}(\lambda(t) - \lambda(t)) = \frac{1}{t - t_0}((\operatorname{Exp} tX) g(t_0) - g(t_0)) \\ + (\operatorname{Exp} tX)(\frac{1}{t - t_0}(g(t) - g(t_0))$$

By the joint continuity of $G \times C^p(\mathfrak{G}, M) \to C^p(\mathfrak{G}, M)$ the second term converges to $(\operatorname{Exp} t_0X)g'(t_0)$ as t converges to t_0 while by (1) of 6.8 the first term converges weakly to $(\operatorname{Exp} t_0X)Xg(0)$. Thus if $l \in M^*$ then

$$\frac{d}{dt}\bigg|_{t=t_0}\langle((\operatorname{Exp} tX)c)(X_1, \cdots, X_p), l\rangle = \frac{d}{dt}\bigg|_{t=t_0}\langle\lambda(t), l\rangle$$
$$= \langle((\operatorname{Exp} t_0X)(Xc))(X_1, \cdots, X_p), l\rangle$$

where $Xc \in C^p(\mathfrak{G}, M)$ is defined in (2) of §1. It follows that condition (1) of 6.2 is satisfied by the topological G-module $C^p(\mathfrak{G}, M)$ and hence that $C^p(\mathfrak{G}, M)$ is a differentiable G-module, the derived $\mathfrak{G}$-module structure moreover being that given in (2) of §1. If $\{c_n\}$ is a sequence in $C^p(\mathfrak{G}, M)$ converging to c then it is clear from (1) of §1 that for any $X_1, \cdots, X_{p+1}$ $dc_\alpha(X_1, \cdots, X_{p+1})$ converges in M to $dc(X_1, \cdots, X_p)$, from which it is clear that the map $d\colon C^p(\mathfrak{G}, M) \to C^{p+1}(\mathfrak{G}, M)$ has a closed graph and so, the domain and range being Frechét spaces, d is continuous. A similar argument shows that for each $X \in \mathfrak{G}$ the linear map $i_X\colon C^p(\mathfrak{G}, M) \to C^{p-1}(\mathfrak{G}, M)$ is continuous.

From (2) of (6.8) it follows that $Xgc = g(ad(g^{-1})X)$ for all $g \in G$, $X \in \mathfrak{G}$ and $c \in C^*(\mathfrak{G}, M)$. Using this and equation (1) of §1 a straightforward computation gives $g(dc) = d(gc)$.

8. The main theorem.

8.1. LEMMA. *If M is a differentiable G-module then each operation of G_0 on $C^*(\mathfrak{G}, M)$ is chain homotopic to the identity, i.e. for each $g \in G_0$ there is a linear map $\lambda(g)$ of $C^*(\mathfrak{G}, M)$ into itself such $gc - c = d\lambda(g)c + \lambda(g)dc$ for all $c \in C^*(\mathfrak{G}, M)$. Moreover, the map $g \to \lambda(g)$ can be cosen so that $g \to \lambda(g)c$ is continuous for each $c \in C^*(\mathfrak{G}, M)$.*

Proof. Since G_0 is compact and connected we can find, for each $g \in G_0$, $X \in \mathfrak{G}$ such that $g = \exp X$. We define $\lambda(g) \cdot c = \int_0^1 (\operatorname{Exp} tX)i_Xc \cdot dt$. Since $C^*(\mathfrak{G}, M)$ is a differentiable G-module from 6.8, (1) for any $l \in (C^*(\mathfrak{G}, M))^*$

$$\langle gc - c, l\rangle = \int_0^1 \frac{d}{dt}\langle(\operatorname{Exp} tX)c, l\rangle dt$$
$$= \int_0^1 \langle(\operatorname{Exp} tX)Xc, l\rangle dt$$
$$= \langle\int_0^1 ((\operatorname{Exp} tX)Xc\, dt), l\rangle$$

and recalling that $Xc = di_X c + i_X dc$ (§ 1, (7))

$$\langle gc - c, l\rangle = \langle \int_0^1 (\operatorname{Exp} tX)\, di_X c + \lambda(g)\, dc, l\rangle.$$

Since d commutes with $\operatorname{Exp} tX$ and integration we obtain

$$\langle gc - c, l\rangle = \langle d\lambda(g)c + \lambda(g)\, dc, l\rangle$$

This equation holds for all $l \in (C^*(\mathfrak{G}, M))^*$ and hence $\lambda(g)$ is a chain homotopy.

Now suppose that U is a small enough neighborhood of the identity so that the exponential map has a continuous inverse, f. Define then

$$\lambda_0(g)c = \int_0^1 (\operatorname{Exp} tf(g))\, i_{f(g)} c\, dt.$$

$\lambda_0(g)$ has the required properties and $g \to \lambda_0(g)c$ is continuous on U. Now choose $g_1, \cdots, g_n$ in G_0 so that g_iU cover G_0, and choose $\lambda(g_i)$ satisfying the lemma. It is easily seen that if we define $\lambda_i(g)$ for $g = g_i u \in g_iU$ by

$$\lambda_i(g) = \lambda(g_i) + \lambda_0(u) + \lambda(g_i)\, d\lambda_0(u) + d\lambda(g_0)\lambda_0(u)$$

then $gc - c = (d\lambda_i(g) + \lambda_i(g)w)c$ and $g \to \lambda_i(g)c$ is continuous on g_iU for each $c \in C^*(\mathfrak{G}, M)$. If we put $\phi_i(g)\lambda_i(g)c = 0$, $\{\phi_i\}$ a partition of unity refining the covering $\{g_iU\}$, for $g \notin g_iU$ then clearly $g \to \phi_i(g)\lambda_i(g)c$ is continuous on G_0, so if we put $\lambda(g)c = \sum_{i=1}^{n} \phi_i(g)\lambda_i(g)c$ then $g \to \lambda(g)c$ is continuous on U, and satisfies the requirements of the lemma.

8.2. Lemma. *Let μ_0 be the normalized Haar measure on G_0. If M is a differentiable G-module, define $A : C^*(\mathfrak{G}, M) \to C^*(\mathfrak{G}, M)$ by $Ac = \int gc\, d\mu_0(g)$. Then there is a linear map λ of $C^*(\mathfrak{G}, M)$ into itself such that $Ac - c = d\lambda c + \lambda dc$ for all $c \in C^*(\mathfrak{G}, M)$.*

Proof. Define $\lambda c = \int (g)c\, d\mu_0(g)$ is chosen as in Lemma 8.1.

8.3. Lemma. *Let M be a differentiable G-module and let $A : C^*(\mathfrak{G}, M) \to C^*(\mathfrak{G}, M)$ be the linear operator defined in Lemma 8.2. Then A has the following properties:*

(1) *It is a projection of $C^*(\mathfrak{G}, M)$ on $C_I^*(\mathfrak{G}, M)$.*

(2) *It commutes with d.*

(3) *If $z \in Z^*(\mathfrak{G}, M)$ then $Az - z \in B^*(\mathfrak{G}, M)$ i.e. Az is cohomologous to z.*

Proof. Property (1) follows directly from the invariance of Haar measure and statement (3) of 6.8 (applied to $C^*(\mathfrak{G}, M)$ instead of M). Statement (2) follows from the fact that d commutes with the operations of G on $C^*(\mathfrak{G}, M)$ and with the integral. Finally (3) is an immediate consequence of Lemma 8.2. q. e. d.

8.4. THE MAIN THEOREM. *If M is a derived $\mathfrak{G}$-module then the natural homomorphism $i^*: H_I^*(\mathfrak{G}, M) \to H^*(\mathfrak{G}, M)$ induced by the inclusion map of $C_I^*(\mathfrak{G}, M)$ in $C^*(\mathfrak{G}, M)$ is an isomorphism onto. In other words every cohomology class of $H^*(\mathfrak{G}, M)$ contains an invariant cocycle, and two invariant cocycles which differ by a coboundary differ by the coboundary of an invariant cochain.*

Proof. Immediate from 8.3.

8.5. COROLLARY. *If $\mathfrak{G}$ is semi-simple then $H^1(\mathfrak{G}, M) = H^2(\mathfrak{G}, M) = 0$ for all derived $\mathfrak{G}$-modules M.*

Proof. Immediate from 8.4 and the theorem of § 3.

9. The differentiability of tensor modules. Let $\mathfrak{M}$ be a differentiable G-space, i. e. $\mathfrak{M}$ is a differentiable ($= C^\infty$) manifold and there is given a differentiable map $(g, p) \to gp$ of $G \times \mathfrak{M}$ into $\mathfrak{M}$ (the *action* of G on $\mathfrak{M}$) such that $ep \equiv p$ and $(gg^1)p \equiv g(g^1p)$. For each $X \in \mathfrak{G}$ there is a differentiable vector field X^* on $\mathfrak{M}$ defined by X^*_p is the tangent to $t \to (\operatorname{Exp} - tX)p$ at $t = 0$. The map $X \to X^*$ is a homomorphism of $\mathfrak{G}$ into the Lie algebra of differentiable vector fields on $\mathfrak{M}$ which is called the *infinitesimal generator* of the action of G *on* $\mathfrak{M}$. Let $\mathfrak{T}$ be the space of all differentiable tensor fields on $\mathfrak{M}$ of any fixed (mixed) rank and symmetry type with the usual "C^∞-topology" (i. e. convergence means uniform convergence of each component and of each partial derivative of any order of a component on any compact subset of a coordinate neighborhood). It is well-known that $\mathfrak{T}$ is a complete locally convex space and in fact a Montel space. Moreover if $\mathfrak{M}$ is second countable, as we henceforth assume, then $\mathfrak{T}$ is metrizable. Each diffeomorphism ϕ of $\mathfrak{M}$ induces an automorphism of $\mathfrak{T}$ in a well-known way and we will write this automorphism as ϕ also. Moreover if $\{\phi_n\}$ is a sequence of diffeomorphism of $\mathfrak{M}$ converging to a diffeomorphism ϕ in the C^∞-topology then it is clear that $\phi_n T \to \phi T$ for any $T \in \mathfrak{T}$. By a theorem of Montgomery [4] if we write g^* for the operation $p \to gp$ of an element of G on $\mathfrak{M}$ then $g \to g^*$ is continuous from G into the group of diffeomorphisms of $\mathfrak{M}$ with the C^∞-topology. It follows from these last two facts that

$\mathcal{T}$ is a topological G-module if we define $gT = g^*T$. We shall call such a topological G-module a *module* of tensor fields (associated with the differentiable G-space $\mathfrak{M}$) and shall as usual write gT instead of g^*T. It follows easily from Theorem II of [5] that a module of tensor fields $\mathcal{T}$ is always differentiable and that for $X \in \mathfrak{G}$ and $T \in \mathcal{T}$ XT is just the usual Lie derivative of T with respect to the vector field X^*. We can now forget about the topology on $\mathcal{T}$ and even the action of G on $\mathcal{T}$. All that is important for the applications we shall make is summed up in

THEOREM. *Let $\mathfrak{M}$ be a differentiable G-space and $X \to X^*$ the infinitesimal generator of the action of G on $\mathfrak{M}$. Let $\mathcal{T}$ be the space of all differentiable tensor fields on $\mathfrak{M}$ of a fixed rank and symmetry type, and for $X \in \mathfrak{G}$ and $T \in \mathcal{T}$ let XT be the Lie derivative of T with respect to X^*. Then this makes $\mathcal{T}$ into a derived $\mathfrak{G}$-module so that* (8.4 and 8.5) $H^*(\mathfrak{G}, \mathcal{T}) \cong H_I^*(\mathfrak{G}, \mathcal{T})$ *and if $\mathfrak{G}$ is semi-simple* $H^1(\mathfrak{G}, \mathcal{T}) = H^2(\mathfrak{G}, \mathcal{T}) = 0$.

10. $H^*(G, R) \cong H^*(\mathfrak{G})$ as a special case. Consider G as a differentiable G-space under the map $(g, p) \to pg^{-1}$. Then for each $X \in \mathcal{T}$ the associated vector field X^* on G is just X itself. Let $\mathcal{T}$ be the tensor module of all differentiable real valued functions on G. If ω is a p-form on G then ω defines an element c of $C^p(\mathfrak{G}, \mathcal{T})$ by $c(X_1, \cdots, X_p)(x) = \omega_x((X_1)_x, \cdots, (X_p)_x)$. Conversely given $c \in C^p(\mathfrak{G}, \mathcal{T})$ define a p-form ω on G as follows: given $x \in G$ and tangent vectors $Y_1, \cdots, Y_p$ at x let $\omega_x(Y_1, \cdots, Y_p) = c(X_1, \cdots, X_p)(x)$ where X_i is the element of $\mathfrak{G}$ satisfying $(X_i)_x = Y_i$. It is readily verified that ω is differentiable and that these two maps are mutually inverse linear isomorphisms between $C^*(\mathfrak{G}, \mathcal{T})$ and the space $\Omega_0(G)$ of differentiable forms on G. Moreover it is a well-known fact (a proof can be found in [15]) that under this correspondence the differential on $C^*(\mathfrak{G}, \mathcal{T})$ corresponds to the exterior derivative on $\Omega^*(G)$. Thus by deRham's Theorem $H^*(G, R) \cong H^*(\mathfrak{G}, \mathcal{T})$ and so by the theorem of § 9 $H^*(G, R) \cong H_I^*(\mathfrak{G}, \mathcal{T})$. For $c \in C^p(\mathfrak{G}, \mathcal{T})$ we have

$$(gc)(X_1, \cdots, X_p)(x) = c(ad(g^{-1})X_1, \cdots, ad(g^{-1})X_p)(xg)$$

and if $X \in \mathfrak{G}$ then by left invariance $(ad(g^{-1})X)_{xg} = dRg(X_x)$ (where Rg is right translation by g). It follows that $c \in C_I^*(\mathfrak{G}, \mathcal{T})$ if and only if the corresponding form ω is right invariant. This gives the well-known identification of $H^*(G, R)$ with the cohomology of right invariant forms (form of Maurer-Cartan). On the other hand we see that each $c \in C^p(\mathfrak{G})$ corresponds to a unique $c' \in C_I{}^p(\mathfrak{G}, \mathcal{T})$ such that $c(X_1, \cdots, X_p) = c'(X_1, \cdots, X_p)(e)$; namely $c'(X_1, \cdots, X_p)(g) = c(ad(g)X_1, \cdots, ad(g)X_p)$. This sets up a linear isomorphism between $C^*(\mathfrak{G})$ and $C_I^*(\mathfrak{G}, \mathfrak{T})$ and referring to equations (5) and (6) of § 1 we see that d_T corresponds to $-d_I$ under this isomorphism. It follows that $H_I^*(\mathfrak{G}, \mathfrak{T}) \cong H^*(\mathfrak{G})$ and this gives the well-known result $H^*(G, R) \cong H^*(\mathfrak{G})$. In [2], where the cohomology theory of Lie algebras was first made explicit, there is a detailed account of theorems of this general nature.

11. Infinitesimal deformation of differentiable G-spaces. Let $\mathfrak{M}$ be a differentiable G-space, $\Phi\colon G \times \mathfrak{M} \to \mathfrak{M}$ the action of G on $\mathfrak{M}$ and $X \to X^*$ the infinitesimal geenrator of Φ. Suppose that for each $t \in [0,1] = I$ there is given an action Φ_t of G on $\mathfrak{M}$ such that $\Phi_0 = \Phi$ and such that $(g, p, t) \to \Phi_t(g, p)$ is a differentiable map of $G \times \mathfrak{M} \times I$ into $\mathfrak{M}$. Such a family Φ_t will be called a *deformation* of Φ and we write $X \to X_t^*$ for the infinitesimal generator of Φ_t. It is easily seen that $(X, p, t) \to (X_t^*)_p$ is a differentiable map of $\mathfrak{G} \times \mathfrak{M} \times I$ into the tangent bundle of $\mathfrak{M}$ and it follows that for each $X \in \mathfrak{G}$ there is a differentiable vector field DX on $\mathfrak{M}$ such that $(DX)_p = \left.\dfrac{d}{dt}\right|_{t=0} (X_t^*)_p$. Clearly D is a linear map of $\mathfrak{G}$ into the Lie algebra $\mathfrak{V}$ of differentiable vector fields on $\mathfrak{M}$ and since $[X, Y]_t^* = [X_t^*, Y_t^*]$ it follows that $D([X, Y]) = [DX, Y^*] + [X^*, DY]$. We call D the infinitesimal deformation of Φ associated with Φ_t and in general a linear map D': $\mathfrak{G} \to \mathfrak{V}$ satisfying the above identity is called an infinitesimal deformation of Φ. Now if Φ_t is a deformation of $\mathfrak{M}$ (i.e. $(p, t) \to \Phi_t(p)$ is a differentiable map of $\mathfrak{M} \times I \to \mathfrak{M}$ and for each $t \in I$ ϕ_t is a diffeomorphism of $\mathfrak{M}$) the vector field Z defined by $Z_p =$ tangent to $t \to \Phi_t(p)$ at $t = 0$ is called the infinitesimal deformation of $\mathfrak{M}$ associated with ϕ_t. From ϕ_t we can construct a deformation Φ_t of Φ by $\Phi_t(g, p) = \phi_t(\Phi(g, \phi_t^{-1}(p)))$. A deformation of Φ that can be defined in this way is called *trivial*. It is easily seen that the associated infinitesimal deformation D of Φ is given by $DX = [Z, X^*] = ad(Z)X^*$ where Z as above is the infinitesimal deformation of $\mathfrak{M}$ associated with ϕ_t. In general if $Z' \in \mathfrak{V} D'\colon X \to ad(Z')X^*$ is an infinitesimal deformation of Φ and we call such infinitesimal deformations of Φ *trivial* (at least if $\mathfrak{M}$ is compact Z' is the infinitesimal deformation of $\mathfrak{M}$ associated with some deformation ϕ_t of $\mathfrak{M}$ so in this case D' is the infinitesimal deformation of Φ associated with a trivial deformation of Φ). Now consider $\mathfrak{V}$ as a tensor module associated to the G-space $\mathfrak{M}$ and recall that the Lie derivative of $Y \in \mathfrak{V}$ with respect to $Z \in \mathfrak{V}$ is just $[Z, Y]$ (see, for example [5] Lemma c). It follows that $\mathfrak{V}$ is a derived $\mathfrak{G}$ module under the operations $XY = [X^*, Y]$. Now an element of $C^1(\mathfrak{G}, \mathfrak{V})$ is just a linear map $c\colon \mathfrak{G} \to \mathfrak{V}$ and $c \in Z'(\mathfrak{G}, \mathfrak{V})$

if and only if $0 \equiv dc(X,Y) = Xc(Y) - Yc(X) - c([X,Y])$ i.e. if and only if $c([X,Y]) = [X^*, c(Y)] + [c(X), Y^*]$. Thus $Z'(\mathfrak{G}, \mathfrak{V})$ is just the space of infinitesimal deformations of Φ. On the other hand $c \in D'(\mathfrak{G}, \mathfrak{V})$ if and only if for some $Z \in \mathfrak{V} c(X) \equiv dZ(X) = X(Z) = [-Z, X^*]$ i.e. if and only if c is a trivial infinitesimal deformation of Φ. Thus $H^1(\mathfrak{G}, \mathfrak{V}) = 0$ means every infinitesimal deformation of Φ is trivial. Since $\mathfrak{V}$ is a tensor module it follows from the Theorem of § 9 that

11.1. THEOREM. *If G is a compact semi-simple Lie group then every infinitesimal deformation of the action of G on a differentiable G-space is trivial.*

In [7] the authors prove a global form of this theorem; namely that if G is any compact Lie group (not necessarily semi-simple) then every deformation of the action of G on a compact differentiable G-space is trivial. It does not seem that either of these theorems implies the other in any obvious way.

Now let $\mathfrak{L}$ be a finite dimensional subalgebra of the Lie algebra $\mathfrak{V}$ of all differentiable vector fields on a compact differentiable manifold $\mathfrak{M}$. If $\mathfrak{V}$ is a compact semi-simple Lie algebra (i.e. the Killing form is negative definite) then the simply connected Lie group G with Lie algebra $\mathfrak{G}$ isomorphic to $\mathfrak{L}$ is compact and semi-simple. By Corollary 2 of Theorem XVIII of [6] an isomorphism of $\mathfrak{G}$ onto $\mathfrak{L}$ is the infinitesimal generator of an action of G on $\mathfrak{M}$. Applying 11.1

11.2. THEOREM. *If $\mathfrak{L}$ is a compact semi-simple sub-algebra of the Lie algebra $\mathfrak{V}$ of differentiable vector fields on a compact differentiable manifold, then every derivation of $\mathfrak{L}$ into $\mathfrak{V}$ is the restriction to $\mathfrak{L}$ of an inner derivation of $\mathfrak{V}$.*

12. Lifting of group actions. In this section we assume that our compact Lie group G is connected and denote by $\tilde{G}$ its simply connected covering group. We shall identify the Lie algebras of G and $\tilde{G}$ under the isomorphism given by the differential of the covering homomorphism. We note that a differentiable G-space $\mathfrak{M}$ is in a natural way a differentiable $\tilde{G}$-space and that the homomorphisms of $\mathfrak{G}$ into the Lie algebra $\mathfrak{V}$ of differentiable vector fields on $\mathfrak{M}$ which are the infinitesimal generators of the actions of G and $\tilde{G}$ on $\mathfrak{M}$ are the same.

In general a homomorphism $X \to X^*$ of $\mathfrak{G}$ into $\mathfrak{V}$ is not the infinitesimal generator of an action of G or even $\tilde{G}$ on $\mathfrak{M}$ if $\mathfrak{M}$ is not compact. However it is shown in [6, Theorem III, p. 95] that if each of the vector fields X^* generates a global one-parameter group of diffeomorphism of $\mathfrak{M}$ then $X \to X^*$

is the infinitesimal generator of a unique action of $\tilde{G}$ on $\mathfrak{M}$. Now if X^* does not generate a global one parameter group of diffeomorphism of $\mathfrak{M}$ then (for example, see [6, p. 84 and p. 73]) there is an integral curve σ of X^* defined on an intevral $[0,a)$ or $(a,0]$ such that $\lim_{t\to a}\sigma(t)=\infty$ (i. e. for each compact subset K of $\mathfrak{M}$ $\sigma(t)\notin K$ for t sufficiently close to a). Suppose now Y is a vector field on a differentiable manifold $\mathfrak{N}$, X a vector field on $\mathfrak{M}$ and $f\colon \mathfrak{N}\to\mathfrak{M}$ is a differentiable map such that $df(Y_p)\equiv X_{f(p)}$ (under these circumstances we say, following Chevalley [1, p. 84] that Y and X are f-related). Then if σ is an integral curve of Y $f\circ\sigma$ is an integral curve of X and it follows from the above remark that if X generates a global one parameter group of diffeomorphisms of $\mathfrak{M}$ and if f is proper then Y generates a global one parameter group of diffeomorphisms of $\mathfrak{N}$.

Now let $X\to X^*$ be the infinitesimal generator of the action of G on a differentiable G-space $\mathfrak{M}$ and let π be the projection of a differentiable fiber bundle B over $\mathfrak{M}$, having compact fiber. Then π is proper and it follows that there is a one-to-one correspondence between actions of $\tilde{G}$ on B for which π is equivariant and homomorphisms $\tau\colon X\to X^\tau$ of $\mathfrak{G}$ into the Lie algebra of differentiable vector fields on $\mathfrak{M}$ such that X^τ and X^* are π-related for all $X\in\mathfrak{G}$. We now specialize further and assume that B is a principle-bundle with structural group a compact connected Lie group H and we write $(h,b)\to hb$ for the action of H on B (this conflicts with the more customary usage in which the structural group acts on the right, but it is only necessary to define hb to be bh^{-1}).

For a diffeomorphism of B to be a bundle map, i. e. equivariant with respect to the action of H, means just that it commutes with each operation of H; hence if $Z\to Z^*$ is the homomorphism of the Lie algebra $\mathfrak{H}$ of H into the differentiable vector fields on B which generates the action of H, then a one parameter group of diffeomorphism of B consists of bundle maps if and only if its infinitesimal generator Y satisfies $[Y,Z^*]=0$ for all $Z\in\mathfrak{H}$. Thus

12. 1. THEOREM. *Let $X\to X^*$ be the infinitesimal generator of the action of G on a differentiable G-space $\mathfrak{M}$. Let H be a compact connected Lie group with Lie algsbra $\mathfrak{H}$, B a differentiable principle H-bundle over $\mathfrak{M}$ with projection π and $Z\to Z^*$ the infinitesimal generator of the action of H on B. Then there is a one-to-one correspondence between actions of $\tilde{G}$ on B equivariant with respect to π such that each operation of $\tilde{G}$ on B is a bundle equivalence, and homomorphisms $\tau\colon X\to X^\tau$ of $\mathfrak{G}$ into the Lie algebra of differentiable vector fields on B such that*

(1) *X^τ and X^* are π-related for all $X\in\mathfrak{G}$*

(2) *$[X^\tau,Z^*]=0$ for all $X\in\mathfrak{G}$, $Z\in\mathfrak{H}$.*

A homomorphism $\tau: X \to X^\tau$ satisfying (1) and (2) will be called a lifting of $\mathfrak{G}$ to B. A linear map $\tau: X \to X^\tau$ of $\mathfrak{G}$ into the Lie algebra of differentiable vector fields on B which satisfies these conditions will be called a *pseudo-lifting* of $\mathfrak{G}$ to B. To construct a pseudo-lifting of $\mathfrak{G}$ to B it is only necessary to choose an H-invariant Riemannian-metric for B and let $X_b{}^\tau$ be the unique vector at b orthogonal to the fibre which projects onto $X^*{}_{\pi(b)}$. It is easily checked that X^τ is differentiable and by construction it is π-related to X^*. That X^τ is H-invariant and hence commutes with Z^* for all $Z \in \mathfrak{H}$ follows from the invariance of the metric. Liftings of $\mathfrak{G}$ to B on the other hand need not always exist and one of the principle results of this section is that they in fact do exist (and are essentially unique) if H is a torus and G is semi-simple. In general we have the relation $Z^*{}_{hb} = dh(ad(h^{-1})Z)^*{}_b$ for $Z \in \mathfrak{H}$, $h \in H$. We now make a final simplifying assumption, namely that H is a torus so that it follows from the above relation that Z^* is an H-invariant vector field on B for all $Z \in \mathfrak{H}$. A vector field Y on B is called *vertical* if $\delta\pi(Y) = 0$ (i.e. Y is π-related to zero). Clearly for each $p \in \mathfrak{M}$ $Z \to Z^*|\pi^{-1}(p)$ is an isomorphism of $\mathfrak{H}$ with the space of vertical H-invariant vector fields on $\pi^{-1}(p)$. It follows that every vector field Y on B which is vertical and H-invariant is of the form $b \to (f(\pi(b)))^*{}_b$ for a uniquely determined function $f: \mathfrak{M} \to \mathfrak{H}$, moreover Y is differentiable if and only if f is differentiable (the latter meaning of course that $l \circ f$ is differentiable for every linear functional l on $\mathfrak{H}$). It follows that we may identify the space M of differentiable maps of $\mathfrak{M}$ into $\mathfrak{H}$ with the space of all vertical vector fields on B which are H-invariant (or, equivalently, which commute with Z^* for all $Z \in \mathfrak{H}$). We note that since $\mathfrak{H}$ is abelian any two elements of M, considered as vertical vector fields, commute. If Y is a vector field on $\mathfrak{M}$ and $m \in M$ then Ym is a well defined element of M; namely its value $(Ym)(p)$ at $p \in \mathfrak{M}$ is the unique element of $\mathfrak{H}$ such that $l((Ym)(p)) = Yp(l \circ m)$ for each linear functional l on $\mathfrak{H}$. Moreover by considering local product representations of B it is easily seen that if Y' is a vector field on B π-related to Y then $Ym = [Y', m]$. Clearly M becomes a $\mathfrak{G}$-module if we define $Xm = X^*{}_m$ for $X \in \mathfrak{G}$, $m \in M$. Moreover from the previous remark $Xm = [X^\tau, m]$ if $\tau: X \to X^\tau$ is any pseudo-lifting of $\mathfrak{G}$ to B. If $\mathfrak{F}$ is the tensor module of differentiable real valued functions on $\mathfrak{M}$ and $X_1, \cdots, X_n$ is a basis for $\mathfrak{H}$ then $(f_1, \cdots, f_n) \to f_1X_1 + \cdots + f_nX_n$ (where the latter means the elment of M whose value at p is $\sum f_i(p)X_i$) is an isomorphism (as $\mathfrak{G}$-modules) of the n-fold direct sum of $\mathfrak{F}$ with itself and M. It follows from the theorem of § 9 that $H^*(\mathfrak{G}, M) \cong H_I(\mathfrak{G}, M)$ and that if G is semi-simple then $H^1(\mathfrak{G}, M) = H^2(\mathfrak{G}, M) = 0$. We collect these results as

12.2. THEOREM. *With the assumptions of* 12.1 *and the additional assumption that H is a torus let M be the linear space of differentiable maps of $\mathfrak{M}$ into $\mathfrak{H}$. If we identify $m \in M$ with the vector field on B whose value at b is $(m(\pi(b))^*)_b$ then this gives a linear isomorphism of M with the linear space of differentialble vertical vector fields on B which commute with Z^* for all $Z \in \mathfrak{H}$. Considered as vertical vector fields on B any two elements of M commute. Moreover M is a $\mathfrak{G}$-module satisfying $H^*(\mathfrak{G}, M) = H^*{}_I(\mathfrak{G}, M)$ (and hence $H^1(\mathfrak{G}, M) = H^2(\mathfrak{G}, M) = 0$ if G is semi-simple) the module structure being characterized by the relation $Xm = [X^\tau, m]$* if *$\tau\colon X \to X^\tau$ is any pseudo-lifting of $\mathfrak{G}$ to B.*

Continuing now with the same assumptions, with each pseudo-lifting $\tau\colon X \to X^\tau$ of $\mathfrak{G}$ to B we associate a map c_τ of $\mathfrak{G} \times \mathfrak{G}$ into vector fields on B by $c_\tau(X, Y) = [X^\tau, Y^\tau] - [X, Y]^\tau$. Clearly c_τ is a measure of how much τ fails to be a lifting, i. e. $c_\tau \equiv 0$ if and only if τ is a lifting. Now $[X, Y]^\tau$ is π-related to $[X, Y]^*$ and by [1, p. 85] $[X^\tau, Y^\tau]$ is π-related to $[X^*, Y^*] = [X, Y]^*$. It follows that $c_\tau(X, Y)$ is π-related to 0, i. e. is vertical. Since $[X, Y]^\tau$ and $[X^\tau, Y^\tau]$ commute with Z^* for all $Z \in \mathfrak{H}$ ($[X, Y]^\tau$ by definition of a pseudo-lifting, $[X^\tau, Y^\tau]$ by the same plus the Jacobi identity) so does $c_\tau(X, Y)$, hence we can identify $c_\tau(X, Y)$ with an element of M. Moreover c_τ is clearly bilinear and skew-symmetric and hence an element of $C^2(\mathfrak{G}, M)$ If X_1, X_2, X_3 belong to $\mathfrak{G}$ then

$$
\begin{aligned}
[X_1{}^\tau, [X_2{}^\tau, X_3{}^\tau]] &= [X_1{}^\tau, [X_2, X_3]^\tau + c_\tau(X_2, X_3)] \\
&= [X_1, [X_2, X_3]]^\tau + c_\tau(X_1, [X_2, X_3]) + X_1 c_\tau(X_2, X_3)
\end{aligned}
$$

Now $[X_1{}^\tau, [X_2{}^\tau, X_3{}^\tau]]$ satisfies the Jacobi identity, i. e. its sum with its two cyclic permutations is zero. Since τ is linear so does $[X_1, [X_2, X_3]]^\tau$. Writing out the cyclic permutations of the above equation and summing we get an equation which gives $dc_\tau(X_1, X_2, X_3) = 0$, hence $c_\tau \in Z^2(\mathfrak{G}, M)$. We call c_τ the *error cocycle* of the pseudo-lifting τ.

Now let $\gamma \in C^1(\mathfrak{G}, M)$, i. e. γ is a linear map of $\mathfrak{G}$ into M. Then it is clear that $\sigma\colon X \to X^\sigma = X^\tau + \gamma(X)$ is another pseudo-lifting of $\mathfrak{G}$ to B and that conversely every pseudo lifting of $\mathfrak{G}$ to B is of this form for a unique $\gamma \in C^1(\mathfrak{G}, M)$. We call γ the difference cochain of σ and τ. Recalling from 12.2 that $[\gamma(X), \gamma(Y)] = 0$ for $X, Y \in \mathfrak{G}$

$$
\begin{aligned}
c_\sigma(X, Y) &= [X^\tau + \gamma(X), Y^\tau + \gamma(Y)] - [X, Y]^\tau - \gamma([X, Y]) \\
&= [X^\tau, Y^\tau] - [X, Y]^\tau - \gamma([X, Y] + X\gamma(Y) - Y\gamma(X) \\
&= c_\tau(X, Y) + d\gamma(X, Y)
\end{aligned}
$$

or in words, the difference of the error cocycles c_σ and c_τ is just the coboundary of the difference cochain $\gamma = \sigma - \tau$. Thus the set of error cocycles associated with pseudo-liftings of $\mathfrak{G}$ to B is an entire cohomology class $\omega \in H^2(\mathfrak{G}, M)$ which we call the *obstruction to a lifting of* $\mathfrak{G}$ *to* B. By what we have seen above a lifting of $\mathfrak{G}$ to B exists if and only if ω contains the zero element, thus we have the only apparently tautological statement

12.3. THEOREM. *If B is a principle torus bundle over a differentiable G-space $\mathfrak{M}$, then a lifting of $\mathfrak{G}$ to B exists if and only if the obstruction to a lifting of $\mathfrak{G}$ to B vanishes.*

Since $\omega \in H^2(\mathfrak{G}, M)$ which by 12.2 is zero if G is semi-simple.

12.4. COROLLARY. *If G is a compact semi-simple Lie group and B is a principle torus bundle over a differentiable G-space $\mathfrak{M}$ then there is a lifting of $\mathfrak{G}$ to B.*

We now consider the uniqueness problem for liftings of $\mathfrak{G}$ to B. Returning to our general situation let σ and τ be two liftings of $\mathfrak{G}$ to B. Their difference cochain γ is clearly a one-cocycle. Conversely $\gamma \in Z^1(\mathfrak{G}, M)$ and τ is a lifting of $\mathfrak{G}$ to B then so is $\sigma = \tau + \gamma$. Assume now that the obstruction cocycle ω is zero so that a lifting τ of $\mathfrak{G}$ to B exists and assume also that $H^1(\mathfrak{G}, M) = 0$ so that by the above remark every lifting σ of $\mathfrak{G}$ to B is of the form $\tau + dm$ for some $m \in M = C^0(\mathfrak{G}, M)$. More explicitly $X^\sigma = X^\tau + Xm = X^\tau + [X^\tau, m]$.

The most general bundle equivalence of B is of the form $b \to f(\pi(b))b$ when f is a differentiable map of $\mathfrak{M}$ into H. Since $\mathrm{Exp}\colon \mathfrak{H} \to H$ is the universal covering map of H it follows that every such f can be written in the form $\mathrm{Exp} \circ m$ for some $m \in M$. Moreover by taking a local product representation of B it is easily seen that if h is the bundle equivalence given by $b \to (\mathrm{Exp}(m(\pi(b)))b$ then $dh(X^\tau{}_{h^{-1}(b)}) = (X^\tau + Xm)_b$ i.e. $X^\sigma = dh \circ X^\tau \circ h^{-1}$. It follows that if $(g, b) \to gb$ is the action of $\tilde{G}$ on B generated by X^τ then the action generated by X^σ is $(gb) \to hgh^{-1}(\mathrm{b})$. Since $H^1(\mathfrak{G}, M) = 0$ if G is semi-simple combining these remarks with 12.1 and 12.4 we have

12.5. THEOREM. *If G is a semi-simple compact Lie group, $\mathfrak{M}$ a differentiable G-space, and B a torus bundle over $\mathfrak{M}$, then there is a differentiable action $(g, b) \to gb$ of $\tilde{G}$ on B which is equivariant with respect to the projection of B on $\mathfrak{M}$ and is such that each operation of $\tilde{G}$ on B is a bundle map. Moreover this action is essentially unique in the sense that every other such* *action of* $\tilde{G}$ *on* B *is of the form* $(g, b) \rightarrow hgh^{-1}b$ *where* h *is a differentiable bundle equivalence of* B.

The latter theorem can be significantly generalized as follows

12.6. THEOREM. *Let* G *be a simply connected compact Lie group,* H *a solvable, connected Lie group,* $\mathfrak{M}$ *a differentiable* G*-space and* B *a differentiable principle* H*-bundle over* $\mathfrak{M}$. *Then there is a differentiable action of* G *on* B *such that the projection of* B *on* $\mathfrak{M}$ *is equivariant and each operation of* G *on* B *is a bundle map. Moreover this action is essentially unique in the sense that any other action of* G *on* B *with these properties is related by conjugation with a bundle equivalence as in* 12.5.

Proof. By induction on $\dim H$. If $\dim H = 1$ then either H is a circle group and the theorem is a consequence of 12.5 or else H is isomorphic to the additive group of real numbers. In the latter case H is solid so that [8, p. 55] B is a product bundle so that existence of the required type of action of G on B is obvious. Uniqueness can be proved just as in 12.5. Now suppose $\dim H > 1$ and that the theorem holds for all H of smaller dimension. Then H has a closed normal subgroup N such that both N and H/N are conected solvable Lie groups of dimension smaller than that of H. Now B is a principle N-bundle over the orbit space B/N and B/N is a principle H/N bundle over $\mathfrak{M}$ and the composition of the projections $B \rightarrow B/N \rightarrow \mathfrak{M}$ is just the bundle projection $B \rightarrow \mathfrak{M}$. By the induction hypothesis we can lift the action of G—first to B/N and then to B. Moreover any action of the apropriate sort on B induces one on B/N. Since the liftings from $\mathfrak{M}$ to B/N and from B/N to B are essentially unique by the induction hypothesis, the same follows easily for the lifting from $\mathfrak{M}$ to B.

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REFERENCES.

[1] C. Chevalley, *Theory of Lie Groups*, Princeton University Press, Princeton, 1946.

[2] C. Chevalley and S. Eilenberg, "Cohomology theory of Lie groups and Lie algebras," *Transactions of the American Mathematical Society*, vol. 63 (1948), pp. 85-124.

[3] G. Hochschild and J-P. Serre, "Cohomology of Lie algebras," *Annals of Mathematics*, vol. 57 (1953), p. 591.

[4] D. Montgomery, "Topological groups of differentiable transformations," *Annals of Mathematics*, vol. 46 (1945), pp. 382-387.

[5] R. Palais, "A definition of the exterior derivative in terms of Lie derivatives," *Proceedings of the American Mathematical Society*, vol. 5 (1954), pp. 902-908.

[6] ———, "A global formulation of the Lie theory of transformation groups," *Memoirs of the American Mathematical Society*, No. 22 (1957).

[7] R. Palais and T. Stewart, "Deformations of compact, differentiable transformation groups," *American Journay of Mathematics*. To appear.

[8] N. Steenrod, *The Topology of Fiber Bundles*, Princeton University Press, Princeton, 1951.

[9] T. Stewart, "Lifting the action of a group in a fiber bundle," *Bulletin of the American Mathematical Society*, vol. 66 (1960), pp. 129-132.

[10] J. H. C. Whitehead, "Certain equations in the algebra of semi-simple infinitesimal groups," *Quarterly Journal of Mathematics* (1937), p. 220.

[11] ———, "On the decomposition of an infinitesimal group," *Proceedings of the Cambridge Philosophical Society*, vol. 32 (1936), p. 229.